

Improved Riemann solvers for an accurate resolution of 1D and 2D shock profiles with application to hydraulic jumps

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Abstract

From the early stages of CFD, the computation of shocks using Finite Volume methods has been a very challenging task as they often prompt the generation of numerical anomalies. Such anomalies lead to an incorrect and unstable representation of the discrete shock profile that may eventually ruin the whole solution. The two most widespread anomalies are the slowly-moving shock anomaly and the carbuncle, which are deeply addressed in the literature in the framework of homogeneous problems, such as Euler equations. In this work, the presence of the aforementioned anomalies is studied in the framework of the 1D and 2D SWE and novel solvers that effectively reduce both anomalies, even in cases where source terms dominate the solution, are presented. Such solvers are based on the augmented Roe (ARoe) family of Riemann solvers, which account for the source term as an extra wave in the eigenstructure of the system. The novel method proposed here is based on the ARoe solver in combination with: (a) an improved flux extrapolation method based on a previous work, which circumvents the slowly-moving shock anomaly and (b) a contact wave smearing technique that avoids the carbuncle. The resulting method is able to eliminate the slowly-moving shock anomaly for 1D steady cases with source term. When dealing with 2D cases, the novel method proves to handle complex shock structures composed of hydraulic jumps over irregular bathymetries, avoiding the presence of the aforementioned anomalies.

Keywords:

Riemann solver; ARoe; shallow water; source terms; shockwave anomalies; carbuncle;

1. Introduction

A wide variety of physical phenomena in the framework of fluid mechanics are modelled by nonlinear systems of hyperbolic conservation laws. Such systems admit discontinuous solutions in the form of shock waves and their resolution give rise to most of the computational challenges that prompt the development of novel numerical methods nowadays [1].

It has been widely reported in the literature that the computation of shock waves induce the generation of numerical anomalies in the solution, hereafter referred to as numerical shockwave anomalies. Two examples are the Carbuncle [2, 3] and the slowly-moving shock anomaly [4, 5, 6, 7], both leading to an incorrect representation of the discrete shock profile and even to a spurious behavior of the numerical solution around the shock wave that may eventually ruin the solution.

Some of the problems related to numerical shockwave anomalies were first identified by Cameron and Emery [8, 9], who proposed some improvements based on the addition of artificial viscosity. This idea seems the most preferred technique by the scientific community [8, 9, 4, 5, 7, 10, 11], though some authors have claimed that this approach may smear too much the shock profiles [12] and even converge to non-physical solutions [13, 14, 15].

Up to now, most of these studies have been carried out in the framework of Euler equations, but the growing needs for the computation of complex geophysical flows motivate their application to

18 the SWE. For such system of equations, numerical shockwave anomalies appear in the resolution of
19 hydraulic jumps [16]. Due to the non-linearity and non-monotonicity of the Hugoniot locus [16], the
20 slowly-moving shock anomaly is always present in the finite volume (FV) resolution of hydraulic jumps.
21 Furthermore, the carbuncle phenomenon has also been observed when computing jumps that undergo
22 a strong transition from high froude numbers [13].

23 A wide variety of approaches have been proposed by the scientific community as a cure to the
24 slowly-moving shock anomaly and the carbuncle. The focus has always been put on Euler equations
25 that are homogeneous and do not include extra terms that may sometimes be very difficult to handle.
26 When considering realistic applications in the framework of the SWE, the equations become non-
27 homogeneous. Source terms have to be considered to account for extra physical effects that are not
28 included in the pure mass and momentum conservation equations. It is well known that the presence of
29 source terms involves extra complexity in the design of numerical schemes, hence the extension of those
30 improved schemes to circumvent the carbuncle and the slowly moving shock anomaly is not trivial.

31 The idea proposed by Zaide and Roe [12, 6] to avoid the slowly moving shock anomaly was success-
32 fully extended to the SWE with source term in [16]. This approach is based on the extrapolation of
33 information in the shock cell from neighboring cells, thus avoiding extra diffusion. In [16], the authors
34 proposed a correction of the original flux extrapolation technique by enforcing the discrete equilibrium
35 between sources and fluxes at cell interfaces, given by the generalized Rankine Hugoniot (GRH) condi-
36 tion. Then, the corrected fluxes were upwinded using the augmented Roe (ARoe) solver. The resulting
37 method proved to reduce the spike in the numerical discharge caused by the aforementioned anomaly,
38 both in steady and transient cases, and the solution proved to be convergent to the exact (spike free)
39 solution.

40 In this work, the spike reducing method in [16] is revisited and enhanced. An improved correction
41 of the flux extrapolation function that ensures the exact equilibrium with independence of the grid
42 is proposed. The novel method, hereafter referred to as spike reducing (SR) method, eliminates the
43 spike of discharge with machine precision for steady cases with source term. Furthermore, the method
44 is extended to 2D by following a dimension-by-dimension approach using a Cartesian grid, which is
45 known to work well even when dealing with genuinely two dimensional shock profiles [6]. To the
46 knowledge of the authors, the exact cure to the steady slowly-moving shock anomaly for the 1D SWE
47 with source term is presented here for the first time. Moreover, the slowly-moving shock anomaly in
48 2D systems with source terms has not been previously addressed by the scientific community. The
49 potential instabilities of the proposed solver and their effect are also examined in this work.

50 The proposed method in 2D proves to perform well and shows no presence of the slowly moving
51 shock anomaly. However, as other traditional solvers, it suffers from the carbuncle. The SR method
52 is able to damp 1D carbuncles but when moving to 2D, shock profiles become unstable at high froude
53 numbers [6]. In order to circumvent this problem, two different numerical strategies based on the
54 addition of extra artificial viscosity in the shear waves are proposed. Such methods are designed for
55 the ARoe solver and are called shear wave correction (SWC) 1 and 2. The former is based on the
56 passive transport of the shear momentum by means of the numerical discharge in the normal direction
57 of the interface while the latter uses the HLLS [17] flux to advect the shear momentum. The HLLS
58 solver can be considered as an augmented version of the HLL solver [18], where S accounts for the
59 presence of source terms.

60 It is worth pointing out that the SWC techniques are intended to be simple solutions that can be
61 coupled with the SR method in order to construct a robust solver based on the ARoe approach. The
62 resulting solvers, hereafter referred to as ARoe SWC SR 1 and 2, aim at the accurate resolution of
63 hydraulic jumps over complex bathymetry in the SWE. The novelty of the proposed method is the
64 combination of: (a) a flux extrapolation/correction method that provides the exact solution for the
65 discharge in steady conditions under complex bathymetry, (b) a contact wave smearing method that
66 avoids the carbuncle, (c) an upwinding algorithm that considers the contribution of the source term

67 in the solution of the Riemann Problem (RP) ensuring the well-balanced property [20, 19] and (d) an
 68 entropy correction method, described in [20].

69 The article is organized as follows. In Section 2, the 2D SWE and Godunov's FV scheme are
 70 introduced. In Section 3, the ARoe and HLLS solvers are revisited. A description of the numerical
 71 shockwave anomalies in SWE, namely the carbuncle and the slowly-moving shock anomaly, is provided
 72 in Section 4. In Section 5, the novel solvers are detailed and a variety of numerical results, including
 73 1D and 2D cases, are presented in Section 6.

74 2. The Shallow Water Equations

75 The derivation of the differential formulation of the SWE can be done by applying the Reynolds
 76 Transport Theorem to the equation for the conservation of mass and momentum inside an integration
 77 volume of infinitesimal size. The SWE can be written in matrix form as follows

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}_1(\mathbf{W})}{\partial x_1} + \frac{\partial \mathbf{F}_2(\mathbf{W})}{\partial x_2} = \mathbf{T}, \quad (1)$$

78 with

$$\mathbf{W} = \begin{pmatrix} h \\ hu_1 \\ hu_2 \end{pmatrix}, \quad \mathbf{F}_1 = \begin{pmatrix} hu_1 \\ hu_1^2 + \frac{1}{2}gh^2 \\ hu_1u_2 \end{pmatrix}, \quad \mathbf{F}_2 = \begin{pmatrix} hu_2 \\ hu_2u_1 \\ hu_2^2 + \frac{1}{2}gh^2 \end{pmatrix}, \quad (2)$$

79 where g is the acceleration of gravity, h is the water depth, hu_1 is the depth averaged unitary discharge
 80 in the x_1 direction and hu_2 the depth averaged unitary discharge in the x_2 direction. x_1 and x_2 are
 81 the Cartesian directions. The source term considered here accounts for thrust exerted by variations
 82 on the bed elevation and is expressed as

$$\mathbf{T} = \left(0, \quad -gh \frac{\partial z}{\partial x_1}, \quad -gh \frac{\partial z}{\partial x_2} \right)^T, \quad (3)$$

83 where $z = z(x_1, x_2)$ is the bed elevation.

84 The methods herein described are based on the assumption that Equation (1) is a hyperbolic system
 85 of conservation laws. The definition of hyperbolic systems of conservation laws is presented below.

86 **Definition 1.** (*Hyperbolic system*). The system in (1) is said to be hyperbolic if the matrix $\mathcal{J}(\mathbf{W}, \mathbf{n}) \in$
 87 $\mathbb{R}^{3 \times 3}$ defined as

$$\mathcal{J}(\mathbf{W}, \mathbf{n}) = \frac{\partial \mathbf{F}_1}{\partial \mathbf{W}} n_1 + \frac{\partial \mathbf{F}_2}{\partial \mathbf{W}} n_2, \quad (4)$$

88 is diagonalizable with real eigenvalues for all $\mathbf{n} = (n_1, n_2) \in \mathbb{R}^2$ and for all $\mathbf{W} \in C$ with $C \subseteq \mathbb{R}^3$ the
 89 subset of physically relevant values of \mathbf{W} . If the 3 eigenvalues are distinct, then the system is said to
 90 be strictly hyperbolic [21].

91 To construct a Godunov's scheme, the computational domain is divided in 2D cells, defined by NE
 92 edges, where NE stands for number of edges. Vector $\mathbf{n}_k = (n_1, n_2)$ indicates the outward unit normal
 93 vector at edge k and l_k is the corresponding edge length. To construct the finite volume scheme, (1)
 94 is integrated inside a cell Ω_i using Gauss-Ostrogradsky theorem [21]

$$\frac{\partial}{\partial t} \int_{\Omega_i} \mathbf{W} d\Omega + \sum_{k=1}^{NE} \mathcal{F}_{\mathbf{n}_k} l_k = \int_{\Omega_i} \mathbf{T} d\Omega. \quad (5)$$

95 where $\mathcal{F}_{\mathbf{n}_k}$ is the normal flux to the cell interfaces, given by the projection of the flux matrix onto the
 96 edge normals

$$\mathcal{F}_{\mathbf{n}_k} = \mathbf{F}_1 n_1 + \mathbf{F}_2 n_2, \quad (6)$$

97 The SWE satisfy the rotational invariance property [21],

$$\mathbf{F}_1 n_1 + \mathbf{F}_2 n_2 = \mathbf{R}^{-1} \mathbf{F}(\mathbf{R}\mathbf{W}), \quad (7)$$

98 where $\mathbf{F} = \mathbf{F}_1$ and \mathbf{R} is the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & -n_1 & n_2 \end{pmatrix}, \quad (8)$$

99 that allows to define a new set of conserved variables $\mathbf{U} = \mathbf{R}\mathbf{W}$ and source term $\mathbf{S} = \mathbf{R}\mathbf{T}$ [20].

100 Thanks to the rotational invariance property, the 2D SWE can be written in an equivalent one-
 101 dimensional form using an equivalent form of the corresponding intercell flux for the approximate first
 102 order Godunov method as follows

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \sum_{k=1}^{NE} \mathbf{R}^{-1} \mathbf{F}_{i,k}^- \frac{\Delta t l_k}{A_i}, \quad (9)$$

103 where A_i is the cell area and $\mathbf{F}_{i,k}^-$ is the numerical flux, computed by solving a non-homogeneous RP
 104 for the x -split SWE at each cell edge. Note that the spatial direction of the RP will be locally defined
 105 in the direction of the cell edge normal.

106 2.1. The x -split SWE

107 A general 1D hyperbolic system of conservation laws with an arbitrary number of N_λ waves can
 108 be used to construct the following RP

$$\begin{cases} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S} \\ \mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_i & x < 0 \\ \mathbf{U}_{i+1} & x > 0 \end{cases} \end{cases} \quad (10)$$

109 where $\mathbf{U} = \mathbf{U}(x, t) \in \mathcal{C} \subset \mathbb{R}^{N_\lambda}$ is the vector of conserved variables with $x \in \Omega \subseteq \mathbb{R}$, $\mathbf{F}(\mathbf{U}) : \mathcal{C} \rightarrow \mathbb{R}^{N_\lambda}$
 110 is the vector of fluxes and \mathbf{S} the vector of sources.

111 The system in (10) is called x -split SWE when

$$\mathbf{U} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ -gh\partial_x z \\ 0 \end{pmatrix}, \quad (11)$$

112 with $u = u_1 n_1 + u_2 n_2$ the normal velocity and $v = -u_1 n_2 + u_2 n_1$ the tangential velocity. It is worth
 113 recalling that the x coordinate will be locally defined at each cell edge in the direction of the cell edge
 114 normal. When considering pure 1D flows, the discharge hu is also denoted by q .

115 3. Approximate augmented solvers for the x -split SWE

116 To preserve generality, the methods herein described will consider a hyperbolic system of conser-
 117 vation laws with an arbitrary number of equations and waves, denoted by N_λ , as defined in (10).

118 *3.1. The ARoe solver*

119 The ARoe solver is a complete linear solver which allows to approximate the RP in (10) by the
120 following constant coefficient linear RP

$$\begin{cases} \frac{\partial \hat{\mathbf{U}}}{\partial t} + \tilde{\mathbf{J}}_{i+\frac{1}{2}} \frac{\partial \hat{\mathbf{U}}}{\partial x} = \frac{1}{\Delta x} \bar{\mathbf{S}}_{i+1/2} \\ \hat{\mathbf{U}}(x, 0) = \begin{cases} \mathbf{U}_i & x < 0 \\ \mathbf{U}_{i+1} & x > 0 \end{cases} \end{cases} \quad (12)$$

121 where $\hat{\mathbf{U}}(x, t)$ is the approximate solution of (10) and $\tilde{\mathbf{J}}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}}(\mathbf{U}_i, \mathbf{U}_{i+1})$ is a constant matrix
122 defined as a function of left and right states (\mathbf{U}_i and \mathbf{U}_{i+1}) that represents an approximation of the
123 Jacobian at $x_{i+\frac{1}{2}}$. The source term \mathbf{S} in (10) is assumed to be a geometric source term, which generates
124 an steady contact wave at $x = 0$. The term $\bar{\mathbf{S}}_{i+1/2}$ stands for a suitable approximation of the integral of
125 the source term inside a control volume defined by $[-\Delta x/2, \Delta x/2]$. Imposing the consistency condition
126 [20], the following constraint is noticed

$$\delta \mathbf{F}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}} \delta \mathbf{U}_{i+\frac{1}{2}}. \quad (13)$$

127 Matrix $\tilde{\mathbf{J}}_{i+\frac{1}{2}}$ is considered to be diagonalizable with N_λ approximate real eigenvalues $\tilde{\lambda}_{i+\frac{1}{2}}^m$ and eigen-
128 vectors $\tilde{\mathbf{e}}^m$. With them, two approximate matrices, $\tilde{\mathbf{P}}_{i+\frac{1}{2}} = (\tilde{\mathbf{e}}^1, \dots, \tilde{\mathbf{e}}^{N_\lambda})_{i+\frac{1}{2}}$ and $\tilde{\mathbf{P}}_{i+\frac{1}{2}}^{-1}$ are con-
129 structed with the following property that they diagonalize the Jacobian $\tilde{\mathbf{J}}_{i+\frac{1}{2}} = (\tilde{\mathbf{P}} \tilde{\mathbf{\Lambda}} \tilde{\mathbf{P}}^{-1})_{i+\frac{1}{2}}$, with
130 $\tilde{\mathbf{\Lambda}}_{i+\frac{1}{2}} = \text{diag}(\tilde{\lambda}^1, \dots, \tilde{\lambda}^{N_\lambda})$

131 The system in (12) can be decoupled using $\tilde{\mathbf{P}}^{-1}$, leading to

$$\begin{cases} \frac{\partial \hat{\mathbf{W}}}{\partial t} + \tilde{\mathbf{\Lambda}}_{i+\frac{1}{2}} \frac{\partial \hat{\mathbf{W}}}{\partial x} = \bar{\mathbf{B}}_{i+\frac{1}{2}} \\ \hat{\mathbf{W}}(x, 0) = \begin{cases} \mathbf{W}_i = \tilde{\mathbf{P}}_{i+\frac{1}{2}}^{-1} \mathbf{U}_i & \text{if } x < 0 \\ \mathbf{W}_{i+1} = \tilde{\mathbf{P}}_{i+\frac{1}{2}}^{-1} \mathbf{U}_{i+1} & \text{if } x > 0 \end{cases} \end{cases} \quad (14)$$

132 with $\hat{\mathbf{W}} = \tilde{\mathbf{P}}_{i+\frac{1}{2}}^{-1} \hat{\mathbf{U}}$ the characteristic variables, $\hat{\mathbf{W}} = (\hat{w}^1, \dots, \hat{w}^{N_\lambda})$ and $\bar{\mathbf{B}}_{i+\frac{1}{2}} = \tilde{\mathbf{P}}_{i+\frac{1}{2}}^{-1} \bar{\mathbf{S}}_{i+\frac{1}{2}}$, the projec-
133 tion of the source term.

134 The derivation of the general solution $\hat{\mathbf{U}}(x, t)$ for a linear system is based on the expansion of the
135 solution as a linear combination of the vectors that compose the Jacobian's eigenvectors basis, using
136 the relation $\mathbf{U} = \tilde{\mathbf{P}} \mathbf{W}$, as follows

$$\hat{\mathbf{U}}(x, t) = \sum_{m=1}^{N_\lambda} \hat{w}^m(x, t) \tilde{\mathbf{e}}_{i+\frac{1}{2}}^m, \quad (15)$$

137 where the scalar values $\hat{w}^m(x, t)$ are the characteristic approximate solutions at the sought point and
138 represent the strength of each wave.

139 In the vicinity of $x = 0$, left and right states, denoted by \mathbf{U}_i^- and \mathbf{U}_{i+1}^+ , are defined as

$$\mathbf{U}_i^- = \lim_{x \rightarrow 0^-} \hat{\mathbf{U}}(x, t) \quad \mathbf{U}_{i+1}^+ = \lim_{x \rightarrow 0^+} \hat{\mathbf{U}}(x, t). \quad (16)$$

140 and their expressions can be derived using (15) [20].

$$\mathbf{U}_i^- = \mathbf{U}_i + \sum_{m=1}^I \left[\left(\alpha - \frac{\bar{\beta}}{\lambda} \right) \tilde{\mathbf{e}} \right]_{i+\frac{1}{2}}^m, \quad \mathbf{U}_{i+1}^+ = \mathbf{U}_{i+1} - \sum_{m=I+1}^{N_\lambda} \left[\left(\alpha - \frac{\bar{\beta}}{\lambda} \right) \tilde{\mathbf{e}} \right]_{i+\frac{1}{2}}^m \quad (17)$$

141 where the set of wave strengths and source strengths are defined as

$$\mathbf{A}_{i+\frac{1}{2}} = (\alpha^1, \dots, \alpha^{N_\lambda})_{i+\frac{1}{2}}^T = \left(\tilde{\mathbf{P}}^{-1} \delta \mathbf{U} \right)_{i+\frac{1}{2}}, \quad \bar{\mathbf{B}}_{i+\frac{1}{2}} = (\bar{\beta}^1, \dots, \bar{\beta}^{N_\lambda})_{i+\frac{1}{2}}^T = \left(\tilde{\mathbf{P}}^{-1} \bar{\mathbf{S}} \right)_{i+\frac{1}{2}}. \quad (18)$$

142 It is worth showing that the jump on the conserved variables between left and right states across the
143 interface is related to the integral of the source term as follows

$$\bar{\mathbf{S}}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}} (\mathbf{U}_{i+1}^+ - \mathbf{U}_i^-). \quad (19)$$

144 For the RP in (14), an approximate flux function $\hat{\mathbf{F}}(x, t)$, with a similar structure than $\hat{\mathbf{U}}(x, t)$, can
145 also be constructed. Intercell values for the fluxes are defined as

$$\mathbf{F}_i^- = \lim_{x \rightarrow 0^-} \hat{\mathbf{F}}(x, t), \quad \mathbf{F}_{i+1}^+ = \lim_{x \rightarrow 0^+} \hat{\mathbf{F}}(x, t). \quad (20)$$

146 The Rankine-Hugoniot condition across the stationary wave at $x = 0$ allows to relate the approx-
147 imate fluxes \mathbf{F}_i^- and \mathbf{F}_{i+1}^+ with the approximate solutions \mathbf{U}_i^- and \mathbf{U}_{i+1}^+ , which in combination with
148 Equation (19), the following relation among fluxes and conserved variables across the discontinuity is
149 obtained

$$\mathbf{F}_{i+1}^+ - \mathbf{F}_i^- = \tilde{\mathbf{J}}_{i+\frac{1}{2}} (\mathbf{U}_{i+1}^+ - \mathbf{U}_i^-). \quad (21)$$

150 Approximate fluxes on the left and right side of the t axis, \mathbf{F}_i^- and \mathbf{F}_{i+1}^+ , read

$$\mathbf{F}_i^- = \mathbf{F}_i + \sum_{m=1}^I \left[(\tilde{\lambda} \alpha - \bar{\beta}) \tilde{\mathbf{e}} \right]_{i+\frac{1}{2}}^m, \quad \mathbf{F}_{i+1}^+ = \mathbf{F}_{i+1} - \sum_{m=I+1}^{N_\lambda} \left[(\tilde{\lambda} \alpha - \bar{\beta}) \tilde{\mathbf{e}} \right]_{i+\frac{1}{2}}^m \quad (22)$$

151 Analogously, if defining $\delta \mathbf{F}_{i+1/2} = \tilde{\mathbf{P}}_{i+1/2} \mathbf{\Gamma}_{i+1/2}$, it is straightforward to obtain the following rela-
152 tion

$$\mathbf{\Gamma}_{i+1/2} = \tilde{\mathbf{\Lambda}}_{i+1/2} \tilde{\mathbf{A}}_{i+1/2} \quad (23)$$

153 with $\mathbf{\Gamma}_{i+1/2} = (\gamma^1, \dots, \gamma^{N_\lambda})_{i+1/2}$, that allows to rewrite (22) as

$$\mathbf{F}_{i+1/2}^- = \mathbf{F}_i + \sum_{m=1}^I [(\gamma - \bar{\beta}) \tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m, \quad \mathbf{F}_{i+1/2}^+ = \mathbf{F}_{i+1} - \sum_{m=I+1}^{N_\lambda} [(\gamma - \bar{\beta}) \tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m. \quad (24)$$

155 When applied to the x -split SWE in (10)-(11), the approximate Jacobian $\tilde{\mathbf{J}}$ for the homogeneous
156 part
157

$$\tilde{\mathbf{J}} = \begin{pmatrix} 0 & 1 & 0 \\ \tilde{c}^2 - \tilde{u}^2 & 2\tilde{u} & 0 \\ -\tilde{u}\tilde{v} & \tilde{v} & \tilde{u} \end{pmatrix}, \quad (25)$$

158 is constructed with the following Roe averaged variables [22]

$$\begin{aligned}\tilde{u}_{i+\frac{1}{2}} &= \frac{u_i\sqrt{h_i}+u_{i+1}\sqrt{h_{i+1}}}{\sqrt{h_i}+\sqrt{h_{i+1}}}, & \tilde{v}_{i+\frac{1}{2}} &= \frac{v_i\sqrt{h_i}+v_{i+1}\sqrt{h_{i+1}}}{\sqrt{h_i}+\sqrt{h_{i+1}}}, \\ \tilde{c}_{i+\frac{1}{2}} &= \sqrt{g\bar{h}_{i+1/2}},\end{aligned}\tag{26}$$

159 with $\bar{h}_{i+1/2} = (h_i + h_{i+1})/2$. The wave speeds are given by

$$\tilde{\lambda}_{i+\frac{1}{2}}^1 = (\tilde{u} - \tilde{c})_{i+\frac{1}{2}}, \quad \tilde{\lambda}_{i+\frac{1}{2}}^2 = \tilde{u}_{i+\frac{1}{2}}, \quad \tilde{\lambda}_{i+\frac{1}{2}}^3 = (\tilde{u} + \tilde{c})_{i+\frac{1}{2}}.\tag{27}$$

160 Concerning the numerical approximation of the integral of the source term at cell interfaces, $\bar{\mathbf{S}}_{i+\frac{1}{2}} =$
161 $(0, \bar{S}_2, 0)_{i+\frac{1}{2}}^T$, the differential approach will be used in this work

$$(\bar{S}_2)_{i+\frac{1}{2}} = -g\bar{h}_{i+1/2}(z_{i+1} - z_i).\tag{28}$$

162 3.2. The HLLS solver

163 The HLLS solver is an incomplete Riemann solver based on the assumption that the solution is
164 characterized by two real eigenvalues $\lambda^1(\mathbf{U}) \leq \lambda^2(\mathbf{U})$, corresponding to the wave speeds, plus an
165 extra wave of speed $\mathcal{S} = 0$ at $x = 0$ that accounts for the source term. Such approach leads to an
166 incomplete Riemann solver when applied to the SWE, which is characterized by 3 eigenvalues. Unlike
167 the ARoe scheme, the HLLS approach does not resolve the shear velocities, introducing a high amount
168 of numerical diffusion for such wave.

169 The HLLS solver is based on enforcing integral relations of (10) inside a control volume $[-x_L, x_R] \times$
170 $[0, \Delta t]$. The integral volume of $\mathbf{U}(x, \Delta t)$ is expressed as

$$\int_{-x_L}^{x_R} \mathbf{U}(x, \Delta t) dx = x_R \mathbf{U}_{i+1} + x_L \mathbf{U}_i + (\mathbf{F}_i - \mathbf{F}_{i+1})\Delta t + \bar{\mathbf{S}}_{i+\frac{1}{2}}\Delta t\tag{29}$$

171 The integral on the left hand side of (29) can be split considering a wave structure given by
172 $\lambda^1 \leq 0 \leq \lambda^2$, $-x_L < \lambda^1 \Delta t$ and $x_R > \lambda^2 \Delta t$

$$\int_{-x_L}^{x_R} \mathbf{U}(x, \Delta t) dx = \mathbf{U}_i(\lambda^1 \Delta t + x_L) + \mathbf{U}_{i+1}(x_L - \lambda^2 \Delta t) + \mathbf{U}_i^-(-\lambda^1 \Delta t) + \mathbf{U}_{i+1}^+(\lambda^2 \Delta t).\tag{30}$$

173 Substitution of (30) in (29) leads to

$$(\mathbf{U}_i - \mathbf{U}_i^-)\lambda^1 - (\mathbf{U}_{i+1} - \mathbf{U}_{i+1}^+)\lambda^2 + \mathbf{F}_{i+1} - \mathbf{F}_i = \bar{\mathbf{S}}_{i+\frac{1}{2}}\tag{31}$$

174 where an extra condition is required to obtain an expression for \mathbf{U}_i^- and \mathbf{U}_{i+1}^+ , due to the presence
175 of the source term. To this end, let us define first an approximate flux function $\hat{\mathbf{F}}(x, t)$ with a similar
176 structure than $\mathbf{U}(x, t)$. As in the ARoe solver, left and intercell numerical fluxes \mathbf{F}_i^- and \mathbf{F}_{i+1}^+ can be
177 defined using (20).

178 Rankine Hugoniot relations can be defined across the λ^1 and λ^2 waves and also across the steady
179 wave at $x = 0$. This allows to provide a relation between fluxes and conserved variables across such
180 waves. Moreover, the Roe's approach will be used in order to relate the intercell numerical fluxes
181 and conserved variables. This leads to the relation in Equation (21), where an approximation of the
182 Jacobian matrix, $\tilde{\mathbf{J}}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}}(\mathbf{U}_i, \mathbf{U}_{i+1})$, is used, according to Equations (12) - (13). The combination
183 of the aforementioned condition with the Rankine Hugoniot relation across the steady contact wave
184 yields

$$\bar{\mathbf{S}}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}} (\mathbf{U}_{i+1}^+ - \mathbf{U}_i^-) \quad (32)$$

185 which makes possible to write the jump of the conserved variables across the stationary wave at $x = 0$
 186 as

$$\mathbf{U}_{i+1}^+ - \mathbf{U}_i^- = \bar{\mathbf{H}}_{i+\frac{1}{2}}, \quad (33)$$

187 where

$$\bar{\mathbf{H}}_{i+\frac{1}{2}} = \tilde{\mathbf{J}}_{i+\frac{1}{2}}^{-1} \bar{\mathbf{S}}_{i+\frac{1}{2}}. \quad (34)$$

188 Combination of (31) and (33) leads to the following values for the intermediate states

$$\mathbf{U}_i^- = \frac{\mathbf{F}_i - \mathbf{F}_{i+1} + \lambda^2 \mathbf{U}_{i+1} - \lambda^1 \mathbf{U}_i + \bar{\mathbf{S}}_{i+\frac{1}{2}} - \lambda^2 \bar{\mathbf{H}}_{i+\frac{1}{2}}}{\lambda^2 - \lambda^1}, \quad (35)$$

$$\mathbf{U}_{i+1}^+ = \frac{\mathbf{F}_i - \mathbf{F}_{i+1} + \lambda^2 \mathbf{U}_{i+1} - \lambda^1 \mathbf{U}_i + \bar{\mathbf{S}}_{i+\frac{1}{2}} - \lambda^1 \bar{\mathbf{H}}_{i+\frac{1}{2}}}{\lambda^2 - \lambda^1}. \quad (36)$$

189 Expressions for left and right intercell fluxes can be straightforward derived from (35) and (36) by
 190 applying the corresponding RH condition

$$\mathbf{F}_i^- = \frac{\lambda^2 \mathbf{F}_i - \lambda^1 \mathbf{F}_{i+1} + \lambda^1 \lambda^2 (\mathbf{U}_{i+1} - \mathbf{U}_i) + \lambda^1 (\bar{\mathbf{S}}_{i+\frac{1}{2}} - \lambda^2 \bar{\mathbf{H}}_{i+\frac{1}{2}})}{\lambda^2 - \lambda^1}, \quad (37)$$

$$\mathbf{F}_{i+1}^+ = \frac{\lambda^2 \mathbf{F}_i - \lambda^1 \mathbf{F}_{i+1} + \lambda^1 \lambda^2 (\mathbf{U}_{i+1} - \mathbf{U}_i) + \lambda^2 (\bar{\mathbf{S}}_{i+\frac{1}{2}} - \lambda^1 \bar{\mathbf{H}}_{i+\frac{1}{2}})}{\lambda^2 - \lambda^1}. \quad (38)$$

191 Remark that the previous derivation was carried out under the assumption of $\lambda^1 \leq 0 \leq \lambda^2$, that
 192 is, a subcritical wave structure. When having supercritical and subcritical cases indistinctly, a general
 193 expression for the numerical fluxes must be used

$$\mathbf{F}_{i+\frac{1}{2}}^- = \begin{cases} \mathbf{F}_i & \text{if } \lambda^1 \geq 0 \\ \mathbf{F}_i^- & \text{if } \lambda^1 \leq 0 \leq \lambda^2 \\ \mathbf{F}_{i+1} - \bar{\mathbf{S}}_{i+\frac{1}{2}} & \text{if } \lambda^2 \leq 0 \end{cases}, \quad (39)$$

$$\mathbf{F}_{i+\frac{1}{2}}^+ = \begin{cases} \mathbf{F}_i + \bar{\mathbf{S}}_{i+\frac{1}{2}} & \text{if } \lambda^1 \geq 0 \\ \mathbf{F}_{i+1}^+ & \text{if } \lambda^1 \leq 0 \leq \lambda^2 \\ \mathbf{F}_{i+1} & \text{if } \lambda^2 \leq 0 \end{cases}. \quad (40)$$

194 When applied to the x -split SWE, the Jacobian matrix is computed as in (25) and the wave speeds
 195 are chosen as follows

$$\lambda^1 = \tilde{\lambda}_{i+\frac{1}{2}}^1 = (\tilde{u} - \tilde{c})_{i+\frac{1}{2}}, \quad \lambda^2 = \tilde{\lambda}_{i+\frac{1}{2}}^3 = (\tilde{u} + \tilde{c})_{i+\frac{1}{2}}. \quad (41)$$

196 Note that $\tilde{\lambda}_{i+\frac{1}{2}}^2$, which is associated to the contact wave, is not considered in the HLLS solver.

197 **4. Numerical shockwave anomalies**

198 Significant numerical anomalies arise in presence of shock waves. Such anomalies consist of spurious
 199 or even unstable shock profiles that may eventually ruin the solution. In the framework of the SWE,
 200 two kinds of anomalies are identified in the resolution of hydraulic jumps: the slowly-moving shock
 201 anomaly [4, 5] and the carbuncle [2, 3]. The slowly-moving shock anomaly is due to a non-linearity of
 202 the branch of the Hugoniot locus related to the shock wave of interest (hydraulic jumps in the SWE)
 203 and appears both in 1D and 2D cases with independence of the grid. On the other hand, the carbuncle
 204 can be regarded as a numerical instability in the discrete shock profile that commonly appears in the
 205 resolution of shock waves when using Cartesian grids due to a Cartesian-like representation of not
 206 purely grid-aligned shocks. We can also find it for particular unstructured triangular grids in a weaker
 207 form.

208 In this work, the aforementioned anomalies are studied in the framework of the SWE and some
 209 numerical approaches to circumvent the spurious effects they produce will be presented. The study
 210 herein described focuses on the resolution of hydraulic jumps, which occur when a supercritical flow
 211 suddenly changes to subcritical conditions, generating a steep free surface elevation where intense
 212 mixing takes place and a large amount of mechanical energy is dissipated. Mathematically, hydraulic
 213 jumps are modelled as pure discontinuities and are defined as:

214 **Definition 2.** (*Hydraulic jump*). *Let the following discontinuous solution*

$$\mathbf{U}(x, t) = \begin{cases} (h, hu, hv)_L & x < 0 \\ (h, hu, hv)_R & x > 0 \end{cases} \quad (42)$$

215 *be a weak solution of the SWE system in (10)–(11), where $(h, hu)_L$ and $(h, hu)_R$ are two different*
 216 *states laying on a Hugoniot curve and satisfying the entropy condition $\lambda^m(\mathbf{U}_L) > \mathcal{S} > \lambda^m(\mathbf{U}_R)$, with*
 217 *\mathcal{S} the speed of the jump, that undergoes a flow transition as $Fr_L < 1 < Fr_R$ or $Fr_R < 1 < Fr_L$.*
 218 *Solution in (42) is termed as hydraulic jump if and only if $\lambda^m(\mathbf{U}_L) > 0 > \lambda^m(\mathbf{U}_R)$.*

219 The Hugoniot locus provides an analytical relation between the conserved variables across dis-
 220 continuous waves and may become a very useful tool to analyze the origin of numerical shockwave
 221 anomalies. It allows to express the right hand side unitary discharge in x in terms of the right hand
 222 side water depth, for a given left hand side state, by imposing the Rankine Hugoniot conditions

$$(hu)_R - (hu)_L = \mathcal{S}(h_R - h_L) \quad (43)$$

$$\left(\frac{1}{2}gh^2 + hu^2\right)_R - \left(\frac{1}{2}gh^2 + hu^2\right)_L = \mathcal{S}((hu)_R - (hu)_L) \quad (44)$$

$$(huv)_R - (huv)_L = \mathcal{S}((hv)_R - (hv)_L) \quad (45)$$

223 Considering that $(h, hu, hv)_R$ is a given fixed point in the state space and using the parametrization
 224 $h_R = h_L + \xi$, Equations (43) and (44) can be combined to obtain

$$(hu)_R = (hu)_L + \xi \left(u_L \pm \sqrt{gh_L + \frac{1}{2}g\xi \left(3 + \frac{\xi}{h_L} \right)} \right) \quad (46)$$

225 which allows for the two possible solutions associated to the genuinely nonlinear characteristic fields.
 226 The right value of the shear unitary discharge, hv , is connected to the left value by means of a
 227 contact wave, which originates from a linearly degenerate characteristic field, and it does not involve
 228 changes in the hydrodynamics. Therefore, it can be calculated after computing $(hu)_R$ and h_R as follows

$$(hv)_R = \left(\frac{u_L + \mathcal{S}}{u_R + \mathcal{S}} \right) (hv)_L \quad (47)$$

229 where $u_R = (hu)_R/h_R$, $u_L = (hu)_L/h_L$ and \mathcal{S} is computed using (43) or (44).

230 4.1. The slowly moving shock anomaly

231 The slowly-moving shock problem is addressed in this section. In the framework of the SWE, it
 232 is associated to hydraulic jumps. The slowly-moving shock anomaly was first investigated by Roberts
 233 in [4], who defined it as numerical noise generated in the discrete shock transition layer which is
 234 transported downstream. Such noise is also referred to as post-shock oscillations. The slowly-moving
 235 shock problem is related to nonlinearities of the Hugoniot curves, which in the case of the SWE, are
 236 found in those branches of the Hugoniot locus related to hydraulic jump-type solutions. It is worth
 237 pointing out that even for non-linear systems, the slowly-moving shock problem does not appear if the
 238 Hugoniot curves are linear [6].

239 Generally, physical shockwaves have a finite width, determined by the physical dissipation processes
 240 taking place within the shock. This is the case of hydraulic jumps, whose width has to do with
 241 the turbulent transition between the supercritical, more energetic region and the subcritical region.
 242 Contrary to this, shocks are mathematically represented by pure discontinuities in hyperbolic systems.
 243 On the other hand, when considering the numerical resolution of shockwaves using the FV method,
 244 a numerical width, different from the physical width, is enforced by the grid size [6]. This leads to
 245 intermediate states which cannot be given a direct physical interpretation, as the shock width is not
 246 controlled by the physical dissipation mechanisms within the shock but only by the grid size. Such
 247 states cannot be removed even when refining the grid, hence numerical schemes must be designed in a
 248 particular way to overcome such flaw.

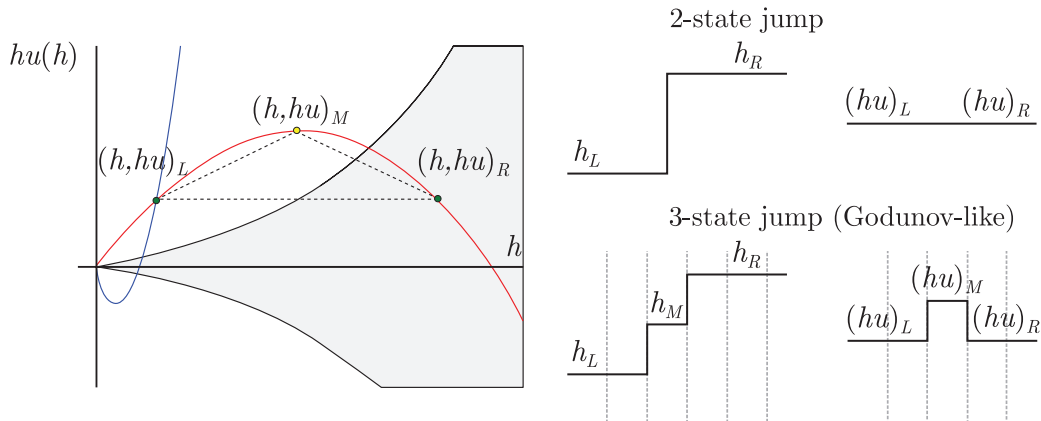


Figure 1: Hugoniot Locus and sketch of the analytical solutions for a 2-state and 3-state hydraulic jumps.

249 In order to illustrate the aforementioned ideas in the framework of the SWE, let us compare
 250 analytically the solution of an ideal steady hydraulic jump, also called 2-state jump (given by the
 251 states \mathbf{U}_L and \mathbf{U}_R), with the solution of a 3-state jump (given by the states \mathbf{U}_L , \mathbf{U}_M and \mathbf{U}_R), which
 252 resembles the discrete solution provided by Godunov's scheme. Recall that the discretization of the
 253 analytical shock leads to the presence of a third intermediate state. Both solutions are weak solutions
 254 of the SWE in (10)-(11) and they are both valid, as they satisfy the Rankine-Hugoniot conditions
 255 in (43)-(44). The representation of such solutions in the phase space are presented in Figure 1 (left),
 256 including the Hugoniot curves. Note that the region of the plot with a gray background is the subcritical
 257 flow region while the region with a white background is supercritical flow region. An sketch of both

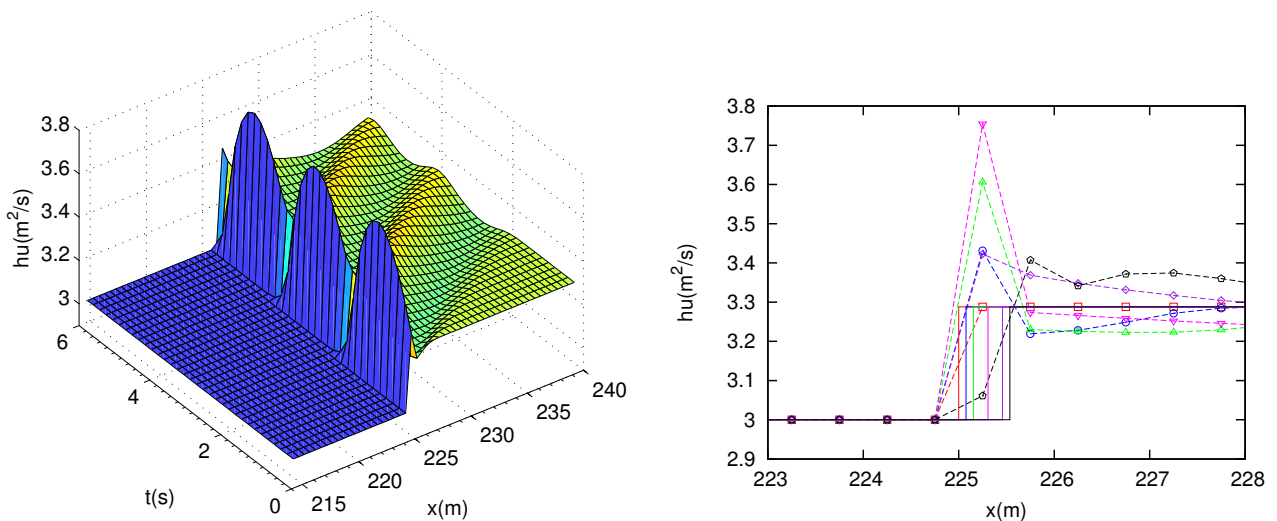


Figure 2: Case 4.1.1. Continuous representation in time of the numerical hu along the x axis (left) and numerical solution for hu at different times within the time interval $[0, 2]$ s using the ARoe scheme in [20].

solutions is depicted in Figure 1 (right). For the sake of simplicity, the third component of the vector of conserved quantities, hv , will be hereafter neglected.

It can be observed in Figure 1 that the 2-state jump is given by two states, \mathbf{U}_L and \mathbf{U}_R , lying on the intersection of the Hugoniot curve with the constant discharge curve (horizontal line). On the other hand, the 3-state jump is given by 3 states, the left and right states lay on the constant discharge curve while the intermediate state has a higher value of discharge due to the nonlinearity and nonmonotonicity of the Hugoniot curve. This solution is feasible as it lays on the Hugoniot curve. This is the origin of the slowly moving shockwave anomaly, which takes the form of a spurious spike in the numerical discharge. It is worth pointing out that the slowly moving shock anomaly is always present with independence of the choice for the Riemann solver.

4.1.1. Numerical simulation of an 1D slowly moving hydraulic jump

This test case is configured as a RP with initial data $h_L = 0.5$ m, $(hu)_L = 3$ m²/s, $h_R = 1.6$ m and $(hu)_R = 3.28787832816$ m²/s, which generates a moving shock wave with speed $\mathcal{S} = 0.26171$ m/s. The computational domain is $[0, 450]$ and the discontinuity for the RP is located at $x = 225$ m. The computational domain is divided in 900 cells of size $\Delta x = 0.5$ m and the CFL number is 0.8. The bed elevation is set to $z = 0$ m.

The solution is computed using the ARoe scheme in [20], which in this case is reduced to the traditional Roe method as the source term is nil. The solution is presented in Figure 2 as a surface plot in the space-time domain $[215, 240] \times [0, 6]$ (left) and is also depicted at 5 different times within the interval $[0, 2]$ (right).

It can be observed that the intermediate state for the discharge is not bounded by the left and right states, as expected. This spike in the numerical discharge generates a shedding of spurious oscillations in the subcritical region (downstream). It is worth noting that the solution provided by the HLLS coincide with that of the ARoe scheme if choosing the wave speeds as the Roe celerities.

4.1.2. Numerical simulation of an 2D slowly moving hydraulic jump

Here, we consider a supercritical flow hitting a circular obstacle and generating an slowly moving hydraulic jump (bow shock) upstream the obstacle. The computational domain is $\Omega = [0, 80] \times [0, 100]$. The solution is computed at $t = 80$ s setting CFL=0.4 and $\Delta x = 1.0$ m. The water depth and

286 discharge number at the inlet are set as $h_L = 0.8$ m and $hu_L = 9$ m²/s respectively. Trans-
 287 missive boundary conditions are set at the other boundaries. The solid body is defined as $\mathcal{W} =$
 288 $\{\mathbf{x} \mid (x - 80)^2 + (y - 50)^2 \leq 400, \mathbf{x} \in \Omega\}$ and the bed elevation is set to $z(x, y) = 0$. The bed elevation
 289 is given by

$$z(x, y) = \begin{cases} 0 & \text{if } x < 5 \\ 0.03(x - 5.0) & \text{if } x > 5 \end{cases} \quad (48)$$

290 The numerical solution for hu at $t = 80$ provided by the ARoe and HLLS solver is depicted in
 291 Figure 3. It can be observed that the spike is now present along the whole shock profile. However,
 292 the spurious shedding of oscillations in the downstream direction (positive x direction) which was
 293 present in 1D cases is not observed here. On the other hand, a spurious shedding of oscillations in
 294 the transverse direction is now observed. This is due to the combination of the spike and the non
 295 alignment of the shock profile with the grid. When the shock jumps from one y column of cells to
 296 another, such oscillation is shed in the transverse direction. If the spike was reduced or the numerical
 297 diffusion of the shear waves was increased, this undesired effect would be diminished. The change in
 298 the diffusion of the shear waves is observed in Figure 3. When using the ARoe scheme, shear waves
 299 are accurately captured and the oscillation is transported in the transverse direction. On the other
 300 hand, if the HLLS is used, this oscillation is highly damped.

301 In Figure 4, a space-time representation of $hu(x, y, t)$ at a fixed $y = 50$ m (left) and $x = 43$ m
 302 (right) is depicted. The solution is provided by the ARoe solver (top) and the HLLS solver (bottom).
 303 The presence of the spike is clearly noticed in all plots. The plots on the right hand side show the
 304 propagation of the transverse spurious waves that emanate from the center of the domain ($y = 50$). It
 305 is observed that when using the HLLS solver, such waves are damped and the shock speed also suffers
 306 some variations due to a very poor resolution of the contact waves.

307 4.1.3. Sensitivity of the slowly-moving shock anomaly to the mesh size

308 In this section, the role of the computational mesh in the resolution of slowly-moving hydraulic
 309 jumps is studied. The size of the mesh is closely related to the artificial (numerical) diffusion introduced
 310 by the solver, hence a comparison among different meshes will allow to evaluate the role of the diffusion
 311 in the resolution of slowly-moving shocks and its effect in the slowly-moving shock anomaly.

312 The case configuration in Section 4.1.2 is adopted here and three different meshes are considered:
 313 a coarse mesh composed of 40×50 cells, an intermediate mesh of 80×100 cells and a fine mesh of
 314 160×200 cells. As in Section 4.1.2, the solution is computed at $t = 80$ s using CFL=0.4. The numerical
 315 solution for hu provided by the HLLS solver is depicted in Figure 5 and a $x - t$ representation of hu
 316 at $y = 50$ m is provided in Figure 6, for each mesh.

317 The numerical results evidence that in presence of transcritical shocks (hydraulic jumps), the
 318 numerical diffusion barely helps in smearing the shock profile. It is well known that the numerical
 319 diffusion usually smears the solution, which is clearly noticeable in presence of discontinuities. However,
 320 when dealing with transcritical shocks, the diffusive effect is reduced as the wave celerities change sign
 321 across such kind of shocks (in this case, λ_1 changes from positive to negative across the shock), enforcing
 322 sharp discontinuities. Furthermore, it is worth pointing out that the spike is not reduced as the grid
 323 is refined. The magnitude of the spike only depends on the sub-cell shock position (as well as on the
 324 left and right states). Note that the frequency of spikes is higher as the grid is refined, since the time
 325 of residence of the shock inside a cell is reduced, which is observed in Figure 6.

326 4.2. The carbuncle

327 When computing 2D strong shocks, such as hydraulic jumps, a numerical instability in the discrete
 328 shock profile may appear. This is known as the carbuncle and was first observed in simulations of air
 329 flow around blunt bodies by Peery and Imlay [2]. The study of the Carbuncle was initially developed in

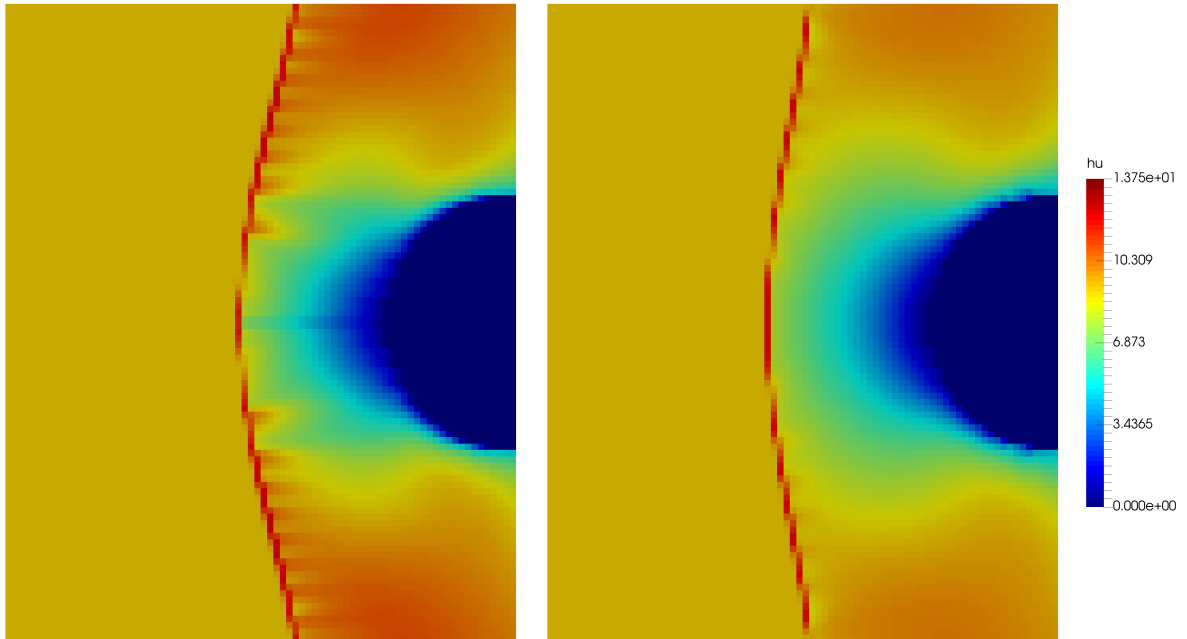


Figure 3: Case 4.1.2. Numerical solution for hu at $t = 80$ provided by the ARoe (left) and HLLS solver (right).

330 the framework of Euler equations. For such equations, the most common example is the simulation of
 331 a hypersonic flow around an infinite cylinder on a structured grid. The presence of the cylinder creates
 332 a detached bow shock around it, where the carbuncle is prone to appear. This example was discussed
 333 in detail by Quirk [23], Pandolfi and Ambrosio [24]. They observed that the main disturbance in the
 334 shock profile appears in the center of the domain, where the shock is better aligned with the grid.

335 At present, the explanation for the carbuncle is still not clear. Previous literature suggest that the
 336 occurrence of the carbuncle has to do with the Cartesian-like representation of purely 2D shock profiles.
 337 A soft curvature in the analytical shock profile is represented by a jump of the discrete shock profile in
 338 one cell column. Moreover, an stability analysis of the solution suggest that the position of the shock
 339 is unstable and might jump by up to two cells in any direction [25]. When such jump happens, a cross
 340 flow is triggered due to the presence of a discrete shock profile in the x direction, as depicted in Figure
 341 7 (left). The cross flow generates a recirculation downstream, which enhances the spreading of the
 342 shock profile and eventually originates a carbuncle-like structure, as depicted in Figure 7 (right).

343 A similar problem to the carbuncle behavior can be seen in one dimension, described as a one-
 344 dimensional carbuncle [26]. This phenomenon is closely related to the slowly-moving shock anomaly.
 345 Not all possible shocks given by the Hugoniot locus are numerically stable. When trying to compute
 346 an stationary shock corresponding to one of these unstable equilibria results in a shock that does not
 347 remain stationary. Either the shock moves to a stable location or it enters into a limit cycle [6].

348 Most strategies to suppress the carbuncle instability are based on the detection of strong shocks and
 349 addition of artificial viscosity on such regions. To this end, the flux approximation must be computed
 350 using an incomplete Riemann solver, which provides a poorer resolution of shear waves, damping the
 351 cross flow and eventually the recirculation downstream. A comparison between the numerical solution
 352 of a bow shock around a solid square body provided by the ARoe and the HLLS is sketched in Figure

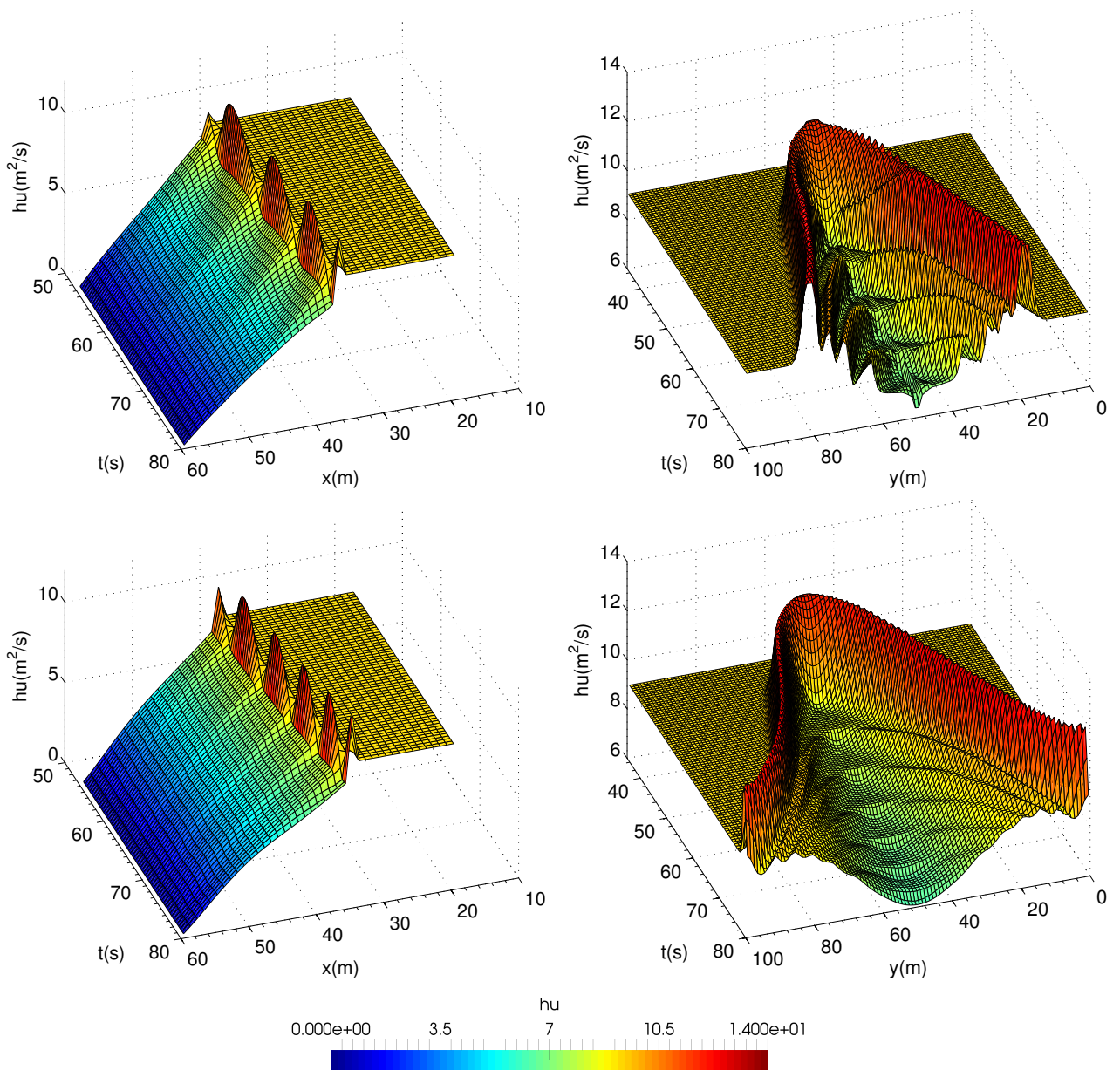


Figure 4: Case 4.1.2. Evolution in time of the numerical hu along the x direction (at $y = 50$) (left) and y direction (at $x = 43$) (right), provided by the ARoe (top) and HLLS solver (bottom).

353 7 (right).

354 4.2.1. Numerical simulation of a supercritical flow against a solid body

355 This test case considers a supercritical flow hitting a circular obstacle and generating a detached
 356 steady bow shock upstream the obstacle. Due to the grid alignment of the 2D shock in the center of the
 357 domain, the carbuncle is likely to appear. The computational domain is $\Omega = [0, 40] \times [0, 100]$. The solu-
 358 tion is computed at $t = 50$ s setting $CFL=0.4$ and $\Delta x = 0.5$ m. The water depth and unitary discharge
 359 at the inlet are set as $h_L = 1$ m and $hu_L = 30$ m²/s respectively. Transmissive boundary conditions are
 360 set at the other boundaries. The solid body is defined as $\mathcal{W} = \{\mathbf{x} \mid (x - 40)^2 + (y - 50)^2 \leq 400, \mathbf{x} \in \Omega\}$
 361 and the bed elevation is set to $z(x, y) = 0$ m. The solution for the unitary x -discharge is computed
 362 using the ARoe solver and the HLLS solver and is presented in Figure 8.

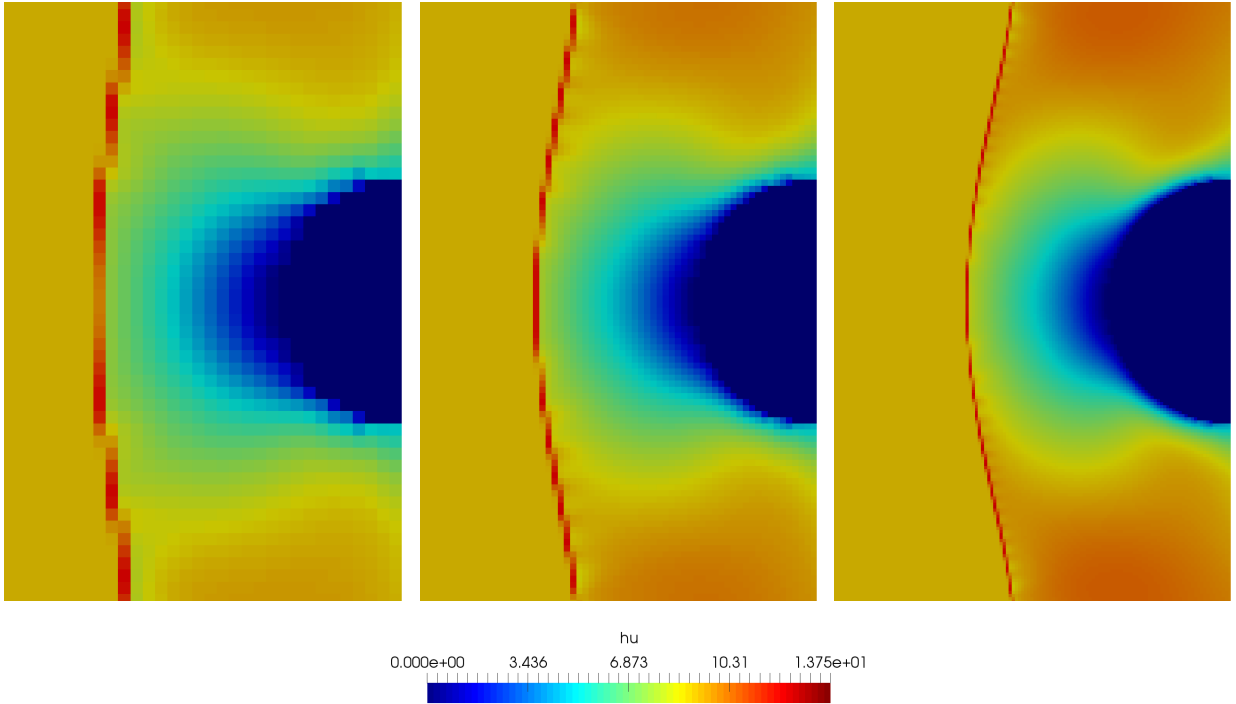


Figure 5: Case 4.1.3. Numerical solution for hu at $t = 80$ provided by the and HLLS solver using 40×50 cells (left), 80×100 cells (middle) and 160×200 cells (right).

363 Figure 8 shows that the choice of the Riemann solver is of great relevance for the carbuncle to
 364 appear. When choosing a complete solver that accurately captures the shear waves, namely the ARoe
 365 solver, the carbuncle instability is triggered, which ruins the solution. On the other hand, when ignoring
 366 shear waves in the internal structure of the numerical solution, that is, when using the HLLS solver,
 367 this instability is highly damped and the carbuncle does not appear. As a result, the shock profile is
 368 accurately captured.

369 It is also important to mention that the spurious spike in the numerical discharge due to the slowly
 370 moving shockwave anomaly is also visible in the results presented in Figure 8. This spike is present
 371 along the strongest transition region of the shock profile (leftmost part of the bow shock) and can be
 372 observed in the numerical solution provided by the HLLS method, as the ARoe solution does not show
 373 a clear shock profile due to the carbuncle. This motivates the design of improved solvers that are able
 374 to circumvent both the carbuncle and the slowly-moving shock anomaly at the same time.

375 5. Improved Riemann solvers

376 When designing numerical schemes for the computation of slowly-moving shocks, the addition of
 377 extra artificial viscosity seems to be the most preferred technique in the scientific community [8, 9, 4, 5,
 378 7, 10, 11]. To avoid extra diffusion, a different possibility is to use of a flux interpolation method, which
 379 prevents from the evaluation of the physical fluxes in the shock cells. This idea of flux interpolation
 380 was first presented by Zaide and Roe [12]. Unlike other traditional methods based on the addition of
 381 artificial viscosity, the method proposed by Zaide uses dissipation to control the shock structure rather
 382 than to approach the true viscous solution and therefore they do not expand the shock profile [6]. Such
 383 method was designed for the Euler equations and has been successfully extended to the 1D SWE with
 384 source term in [16].

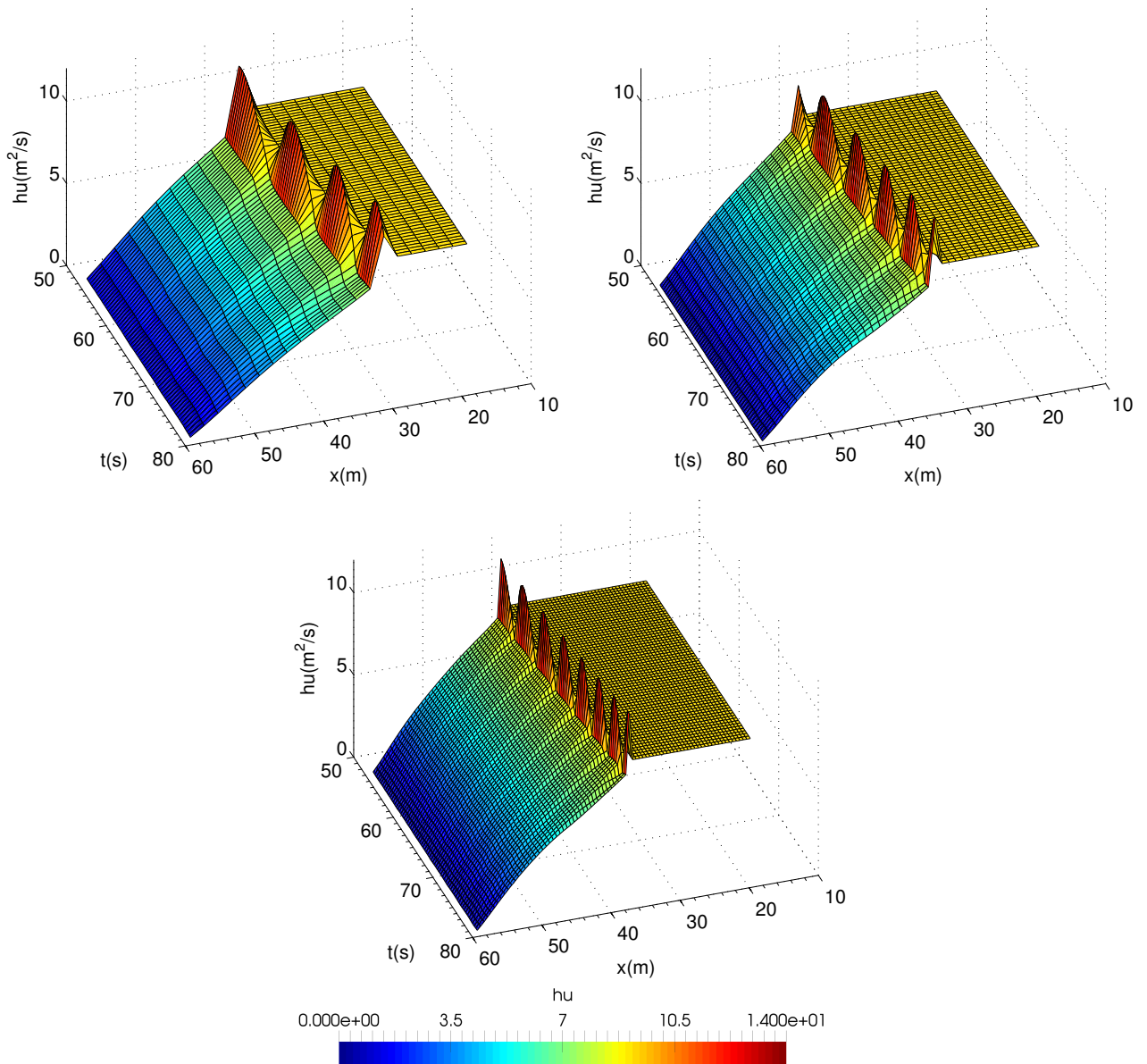


Figure 6: Case 4.1.3. Evolution in time of the numerical hu along the x direction (at $y = 50$) provided by the HLLS solver using 40×50 cells (top-left), 80×100 cells (top-right) and 160×200 cells (bottom).

385 In this section, the method in [16] is revisited and improved and novel approaches for the extension
 386 of this method to 2D are presented. Furthermore, the proposed methods are enhanced with carbuncle-
 387 suppressing strategies in order to prevent the numerical solution from both the slowly moving shock
 388 anomaly and the carbuncle.

389 5.1. An spike-reducing method in 1D: the Spike Reducing (SR) ARoe solver

390 This new method is of application to the 1D SWE, given by the first and second equation of the
 391 system in (10)-(11). Using standard notation, cells will be defined as

$$\Omega_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right], i = 1, \dots, N. \quad (49)$$

392 The extension of flux function A [6] to non-homogeneous equations is carried out by means of a

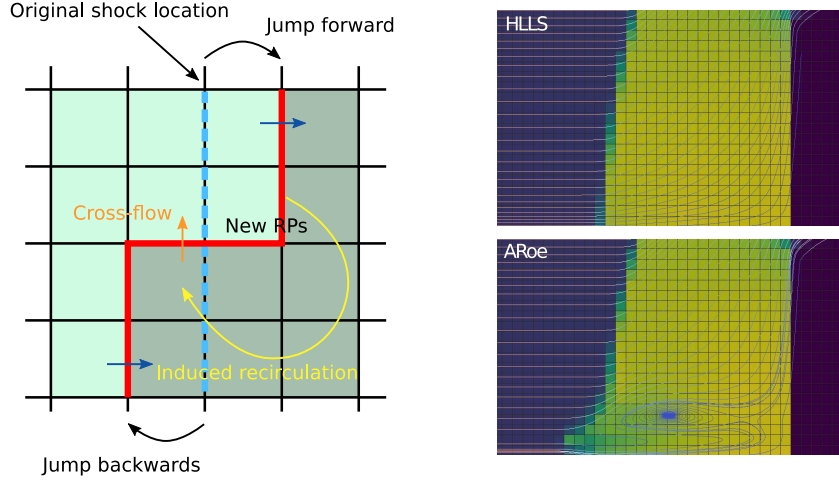


Figure 7: Sketch of the mechanism involved in the generation of the carbuncle (left) and numerical solution with/without carbuncle (right).

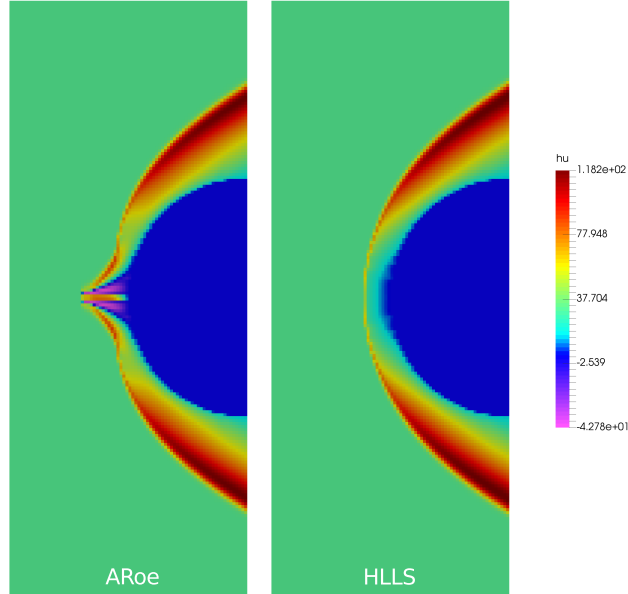


Figure 8: Case 4.2.1. Numerical hu provided by the ARoe solver (left) and by the HLLS solver (right).

393 suitable correction of the interpolation technique that ensures a an exact equilibrium between fluxes
 394 and source term [16]. Then, the corrected fluxes are upwinded using the ARoe solver in flux-splitting
 395 form. To that end, new cell-centered fluxes, $\check{\mathbf{F}}_i$, are computed first by means of extrapolation from
 396 neighboring cells. At every cell, the new flux is calculated as

$$\check{\mathbf{F}}_i = \frac{1}{2}(\mathbf{F}_{i+1} + \mathbf{F}_{i-1}) - \frac{1}{2}\tilde{\mathbf{J}}_{i-1,i+1}(\mathbf{U}_{i+1} - 2\mathbf{U}_i + \mathbf{U}_{i-1}), \quad (50)$$

397 with $\tilde{\mathbf{J}}_{i-1,i+1} = \tilde{\mathbf{J}}_{i-1,i+1}(\mathbf{U}_{i+1}, \mathbf{U}_{i-1})$ a 3-cell Roe's Jacobian matrix, introduced in [6]. Then, the flux
 398 extrapolation in (50), $\check{\mathbf{F}}_i$, in (50) must be corrected in the following way

$$\hat{\mathbf{F}}_i = \check{\mathbf{F}}_i + \varphi, \quad (51)$$

399 where φ is a correction term, so that $\hat{\mathbf{F}}_i$ satisfies the following properties

- 400 1. Left-equilibrium: Under steady state, $\hat{\mathbf{F}}_i$ must hold the generalized Rankine-Hugoniot (GRH) con-
 401 dition at $x_{i-1/2}$, given by $\hat{\mathbf{F}}_i - \mathbf{F}_{i-1} = \bar{\mathbf{S}}_{i-1/2}$.
 402 2. Right-equilibrium: Under steady state, $\hat{\mathbf{F}}_i$ must hold the GRH condition at $x_{i+1/2}$, given by $\mathbf{F}_{i+1} -$
 403 $\hat{\mathbf{F}}_i = \bar{\mathbf{S}}_{i+1/2}$.
 404 3. Consistence: When data is smooth, the novel flux function $\hat{\mathbf{F}}_i$ must converge to \mathbf{F}_i with at least
 405 first order of accuracy, $\hat{\mathbf{F}}_i = \mathbf{F}_i + \mathcal{O}(\Delta x)$.

406 The correction term is computed as follows

$$\varphi = \bar{\mathbf{S}}_{i-1/2} - (1 - x_{S,i}) \bar{\mathbf{S}}_{i-1,i+1}. \quad (52)$$

407 where $x_{S,i} = \frac{h_i - h_{i+1}}{h_{i-1} - h_{i+1}}$ is the position of the shock inside the cell and $\bar{\mathbf{S}}_{i-1,i+1}$ and $\bar{\mathbf{S}}_{i-1/2}$ are approached
 408 as (see Appendix A)

$$\bar{\mathbf{S}}_{i-1,i+1} = \begin{pmatrix} 0 \\ -g \frac{h_{i-1} + h_i}{2} (z_i - z_{i-1}) - g \frac{h_i + h_{i+1}}{2} (z_{i+1} - z_i) \end{pmatrix}, \quad (53)$$

409 and

$$\bar{\mathbf{S}}_{i-1/2} = \begin{pmatrix} 0 \\ -g \frac{h_{i-1} + h_i}{2} (z_i - z_{i-1}) \end{pmatrix}. \quad (54)$$

410 For the application of the technique presented here, it is required to find the cells where the shocks
 411 are contained. We propose to use Roe celerities, $\tilde{\lambda}^m$ to unequivocally locate such a cell. Let us consider
 412 the cells, Ω_i , as single items contained in the domain Ω such that $\Omega = \{\Omega_i \mid i \in [1, \dots, N]\}$. Considering
 413 the possibility of multiple hydraulic jumps within the domain, we denote the set of cells containing a
 414 positive-flow hydraulic jump as

$$\mathcal{D}^+ = \left\{ \Omega_i \mid \Omega_i \in \Omega \wedge \tilde{\lambda}_{i-1/2}^1 \cdot \tilde{\lambda}_{i+1/2}^1 < 0 \wedge h_{i-1} < h_{i+1} \right\} \quad (55)$$

415 and the set of cells containing a negative-flow hydraulic jump as \mathcal{D}^- , analogously calculated. We
 416 propose to use the novel flux function in 1D problems as follows

$$\hat{\mathbf{F}}_i = \begin{cases} \mathbf{F}_i & \text{if } \Omega_i \notin \mathcal{D}^+ \cup \mathcal{D}^- \\ \tilde{\mathbf{F}}_i - (1 - x_{S,i}) \bar{\mathbf{S}}_{i-1,i+1} + \bar{\mathbf{S}}_{i-1/2} & \text{if } \Omega_i \in \mathcal{D}^+ \cup \mathcal{D}^- \end{cases} \quad (56)$$

417 that is, only in those cells containing a hydraulic jump.

418 Finally, the expression for the numerical fluxes at cell interfaces is presented. Using definitions in
 419 Section 3.1, we can write the non-homogeneous version of the numerical flux in (24) to account for the
 420 contribution of the source term as

$$\mathbf{F}_{i+1/2}^- = \hat{\mathbf{F}}_i + \sum_{m=1}^I [(\hat{\gamma} - \beta) \tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m, \quad \mathbf{F}_{i+1/2}^+ = \hat{\mathbf{F}}_{i+1} - \sum_{m=I+1}^{N_\lambda} [(\hat{\gamma} - \beta) \tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m \quad (57)$$

421 where $\hat{\gamma}$ are the components of $\hat{\mathbf{\Gamma}}_{i+1/2} = \tilde{\mathbf{P}}_{i+1/2}^{-1} \delta \hat{\mathbf{F}}_{i+1/2}$, the projection of the jump in the extrapolated
 422 fluxes across cell interfaces, $\delta \hat{\mathbf{F}}_{i+1/2} = \hat{\mathbf{F}}_{i+1} - \hat{\mathbf{F}}_i$.

423

424 As a summary, it is worth listing the steps for the implementation of the SR approach:

- 425 • Step I: Detection of hydraulic jumps using Equation (55). Cells containing hydraulic jumps will
426 be stored in $\mathcal{D}^+ \cup \mathcal{D}^-$.
- 427 • Step II: Extrapolate the physical fluxes only in those cells containing jumps, $\Omega_i \in \mathcal{D}^+ \cup \mathcal{D}^-$,
428 according to Equation (56).
- 429 • Step III: Compute the numerical fluxes at cell interfaces using (57).

430 5.2. Extension to 2 dimensions of the SR ARoe solver

431 The 2D extension of the spike-reducing solver presented in the previous section is developed by
432 applying the 1D methodology to each direction independently. Let us consider the 2D system in (1),
433 with \mathbf{F}_1 and \mathbf{F}_2 the fluxes in the x_1 and x_2 direction. We will consider a Cartesian mesh with cell
434 indexes denoted by k in the x_1 direction and l in the x_2 direction. When considering a Cartesian mesh,
435 it is possible to define the interpolated fluxes as done in the 1D case

$$\check{\mathbf{F}}_{1k,l} = \frac{1}{2}(\mathbf{F}_{1k+1,l} + \mathbf{F}_{1k-1,l}) - \frac{1}{2}\tilde{\mathbf{J}}_{(k-1,k+1),l}(\mathbf{W}_{k+1,l} - 2\mathbf{W}_{k,l} + \mathbf{W}_{k-1,l}), \quad (58)$$

$$\check{\mathbf{F}}_{2k,l} = \frac{1}{2}(\mathbf{F}_{2k,l+1} + \mathbf{F}_{2k,l-1}) - \frac{1}{2}\tilde{\mathbf{J}}_{k,(l-1,l+1)}(\mathbf{W}_{k,l+1} - 2\mathbf{W}_{k,l} + \mathbf{W}_{k,l-1}). \quad (59)$$

436 where $(\cdot)_{k,l}$ stands for the cell average value inside cell $\Omega_{k,l}$.

437 In [6], the author outlines that in the stationary case, each intermediate shock state is adjacent to
438 at least two end states of the shock, but not necessarily aligned in the x or y direction. Therefore, the
439 interpolated flux in the x -direction should lie on a straight line in flux space with the interpolated flux
440 in the y -direction. Although a genuinely two-dimensional method would be preferable at a first glance,
441 requiring to use interpolated fluxes computed from information in both directions, the dimension-by-
442 dimension method proposed here is powerful enough to provide the sought results.

443 As in the 1D case, a shock-detection algorithm is required. A dimension-by-dimension shock
444 detection procedure is proposed. When dealing with oblique shocks in Cartesian meshes, a dimension-
445 by-dimension shock detection method may provide discrete shock profiles, such that two consecutive
446 cells containing the shock only share one vertex and no edges. This means that there would be a 2D
447 discontinuity of the shock profile, which can reduce the robustness of the method. Unlike in the 1D
448 case, the shock profile detected by the algorithm will now span a width of 3 cells in order to avoid this
449 problem. The set of cells defining a positive-flow hydraulic jump in the x_1 and x_2 directions are given
450 by

$$\mathcal{D}^{+,x_1} = \left\{ \Omega_{k-1,l} \cup \Omega_{k,l} \cup \Omega_{k+1,l} \mid \tilde{\lambda}_{k-1/2,l}^1 \cdot \tilde{\lambda}_{k+1/2,l}^1 < 0 \wedge h_{k-1,l} < h_{k+1,l} \right\} \quad (60)$$

451 and

$$\mathcal{D}^{+,x_2} = \left\{ \Omega_{k,l-1} \cup \Omega_{k,l} \cup \Omega_{k,l+1} \mid \tilde{\lambda}_{k,l-1/2}^1 \cdot \tilde{\lambda}_{k,l+1/2}^1 < 0 \wedge h_{k,l-1} < h_{k,l+1} \right\} \quad (61)$$

452 respectively. The set of cells defining a negative jump is defined equivalently. The sets of cells containing
453 x_1 and x_2 shocks are finally constructed as

$$\mathcal{D}^{x_1} = \mathcal{D}^{+,x_1} \cup \mathcal{D}^{-,x_1}, \quad \mathcal{D}^{x_2} = \mathcal{D}^{+,x_2} \cup \mathcal{D}^{-,x_2} \quad (62)$$

454 and is used to define the novel fluxes, $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{F}}_2$, as follows

$$\hat{\mathbf{F}}_{1k,l} = \begin{cases} \mathbf{F}_{1k,l} & \text{if } \Omega_{k,l} \notin \mathcal{D}^{x_1} \\ \check{\mathbf{F}}_{1k,l} - (1 - x_{S,k,l})\bar{\mathbf{T}}_{(k-1,k+1),l} + \bar{\mathbf{T}}_{k-1/2,l} & \text{if } \Omega_{k,l} \in \mathcal{D}^{x_1} \end{cases} \quad (63)$$

$$\hat{\mathbf{F}}_{2k,l} = \begin{cases} \mathbf{F}_{2k,l} & \text{if } \Omega_{k,l} \notin \mathcal{D}^{x_2} \\ \check{\mathbf{F}}_{2k,l} - (1 - y_{S,k,l})\bar{\mathbf{T}}_{k,(l-1,l+1)} + \bar{\mathbf{T}}_{k,l-1/2} & \text{if } \Omega_{k,l} \in \mathcal{D}^{x_2} \end{cases} \quad (64)$$

455 where

$$\bar{\mathbf{T}}_{(k-1,k+1),l} = \begin{pmatrix} 0 \\ -g \frac{h_{k-1,l} + h_{k,l}}{2} (z_{k,l} - z_{k-1,l}) \\ 0 \end{pmatrix} - g \frac{h_{k,l} + h_{k+1,l}}{2} (z_{k+1,l} - z_{k,l}) \quad (65)$$

456 and

$$\bar{\mathbf{T}}_{k-1/2,l} = \begin{pmatrix} 0 \\ -g \frac{h_{k-1,l} + h_{k,l}}{2} (z_{k,l} - z_{k-1,l}) \\ 0 \end{pmatrix}. \quad (66)$$

457 Note that $\bar{\mathbf{T}}_{k,(l-1,l+1)}$ and $\bar{\mathbf{T}}_{k,l-1/2}$ are computed analogously. The fluxes in (63) and (64) are used
458 to construct the normal flux and to define RPs at cell interfaces, as done in the 1D case.

459 After remapping the fluxes in the domain, using (63) and (64), the ARoe solver can be used to
460 upwind such corrected extrapolated fluxes and to construct the numerical fluxes. To this end, at an
461 arbitrary interface with direction (n_1, n_2) , the numerical flux in (9) is constructed using (24)

$$\mathbf{F}_{i+1/2}^- = \hat{\mathbf{F}}_i + \sum_{m=1}^I [(\hat{\gamma} - \beta)\tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m, \quad \mathbf{F}_{i+1/2}^+ = \hat{\mathbf{F}}_{i+1} - \sum_{m=I+1}^{N_\lambda} [(\hat{\gamma} - \beta)\tilde{\mathbf{e}}]_{i+\frac{1}{2}}^m \quad (67)$$

462 where $\hat{\gamma}$ are the components of $\hat{\mathbf{\Gamma}}_{i+1/2} = \tilde{\mathbf{P}}_{i+1/2}^{-1} \delta \hat{\mathbf{F}}_{i+1/2}$, the projection of the jump of the rotated
463 extrapolated fluxes across cell interface, $\delta \hat{\mathbf{F}}_{i+1/2} = \mathbf{R} \left(\hat{\mathbf{F}}_1 n_1 + \hat{\mathbf{F}}_2 n_2 \right)_{i+1} - \mathbf{R} \left(\hat{\mathbf{F}}_1 n_1 + \hat{\mathbf{F}}_2 n_2 \right)_i$. Note
464 that $i+1$ and i will have a correspondence with such pairs of k, l that give the location at each side
465 of the interface.

466 5.3. Circumventing the carbuncle: increasing the viscosity in contact waves

467 5.3.1. The shear wave correction 1 (SWC1) approach for the ARoe solver: the ARoe SWC1

468 This approach is based on the use of the internal solution for the normal discharge, hu , provided by
469 the ARoe solver for the x -split SWE, to approximate the numerical flux of the shear discharge, huv ,
470 which is the third component of \mathbf{F} in (11). According to previous studies [20], the internal solution for
471 the discharge, hereafter referred to as star solution and denoted by $(hu)^*$, is constant across the contact
472 discontinuity at $x = 0$ introduced by the source term, that is $(hu)_{i+1/2}^+ = (hu)_{i+1/2}^- = (hu)_{i+1/2}^*$. Such
473 solution stands for the intercell numerical discharge and can be used to construct the flux of the shear
474 wave using the upwind approach as follows

$$(huv)_{i+1/2}^* = \begin{cases} (hu)_{i+1/2}^* v_i & \text{if } (hu)_{i+1/2}^* \geq 0 \\ (hu)_{i+1/2}^* v_{i+1} & \text{if } (hu)_{i+1/2}^* < 0 \end{cases} \quad (68)$$

475 where $(huv)_{i+1/2}^* = (huv)_{i+1/2}^+ = (huv)_{i+1/2}^-$ is the intercell numerical discharge for the shear wave,
476 which is the third component of $\mathbf{F}_{i+1/2}^\pm$. The first and second components of $\mathbf{F}_{i+1/2}^\pm$ will be computed
477 using (24).

478 *5.3.2. The shear wave correction 2 (SWC2) approach for the ARoe solver: the ARoe SWC2*

479 This approach is based on the HLLS flux to approximate the numerical flux of the shear discharge,
 480 huv . The first and second components of $\mathbf{F}_{i+1/2}^\pm$ will be computed using (24) while the third component
 481 will be approached using (39) and (40) as follows

$$(huv)_{i+1/2}^* = \begin{cases} (huv)_i & \text{if } \tilde{\lambda}^1 \geq 0 \\ (huv)_i^* & \text{if } \tilde{\lambda}^1 \leq 0 \leq \tilde{\lambda}^3 \\ (huv)_{i+1} & \text{if } \tilde{\lambda}^3 \leq 0 \end{cases} . \quad (69)$$

482 where

$$(huv)_i^* = \frac{\tilde{\lambda}^3(huv)_i - \tilde{\lambda}^1(huv)_{i+1} + \tilde{\lambda}^1\tilde{\lambda}^3((hv)_{i+1} - (hv)_i) + \bar{S}_{2,i+\frac{1}{2}}\tilde{v}_{i+\frac{1}{2}}}{\tilde{\lambda}^3 - \tilde{\lambda}^1}, \quad (70)$$

483 *5.4. Combination of solvers*

484 The proposed ARoe SWC1 and ARoe SWC2 solvers can be easily combined with the SR approach
 485 to yield a spike-reducing carbuncle-free method. Then, the resulting solver from this combination can
 486 be either used in the whole computational domain or just in those cells containing the shocks. If
 487 choosing the first option, the diffusion of the SWC approaches can smear other relevant waves in the
 488 domain, specially when using the SWC2 approach. On the other hand, if such solver is only used in
 489 those cells containing shocks, the properties of the ARoe solver in accurately capturing shear waves
 490 can be maintained in the rest of the domain where no extra diffusion is required.

491 **6. Numerical results**

492 *6.1. Steady jump over a hump: assessment of the ARoe SR method*

493 In this test case, an steady flow over the following bed elevation profile

$$z(x) = \begin{cases} 0 & \text{if } x < 8 \\ 0.05(x - 8) & \text{if } 8 \leq x \leq 12 \\ 0.2 - 0.05(x - 12)^2 & \text{if } 12 \leq x \leq 14 \\ 0 & \text{if } x > 12 \end{cases} \quad (71)$$

494 is considered. This case includes a subcritical-supercritical transition and the formation of a hydraulic
 495 jump. The computational domain is $[0, 20]$ and the solution is computed at $t = 400$ s. The CFL number
 496 is set to 0.45 and the computational domain is discretized in 100 cells. The discharge is imposed as
 497 $hu = 0.6$ m²/s upstream and the water depth is imposed downstream to generate a hydraulic jump.
 498 Different values for h downstream, are chosen to generate the jump at different locations in order to
 499 evaluate the performance of the proposed SR method for different locations of the jump inside the cell.
 500 The complete configuration of boundary conditions is presented in Table 1, where the absolute shock
 501 position in the domain and the normalized shock position inside the cell, x_S , are also included.

502 This test case aims at evaluating the novel ARoe SR method and comparing it with the traditional
 503 ARoe solver and the spike-reducing approaches provided in [6], called flux function A, and in [16],
 504 hereafter referred to as old spike-reducing method. It is worth pointing out that the difference of the
 505 novel ARoe SR method with the latter resides in the approximation of the term $\bar{\mathbf{S}}_{i-1,i+1}$ in Equation
 506 (53). The old version of the method [16] does not use the composite trapezoid rule as the ARoe
 507 SR method does, which does not allow to exactly satisfy the GRH conditions on the left and right
 508 interfaces. This makes the method convergent to the exact solution (no spike solution) but not exact.
 509 On the other hand, the original method by Zaide [6] does not include any correction for the source
 510 term, hence the errors are significant.

Case	$q_{BC:left}(m^2/s)$	$h_{BC:right}(m)$	Shock position (m)	x_S
A	0.6	0.6185	13.298	0.01
B	0.6	0.6200	13.278	0.11
C	0.6	0.6220	13.252	0.24
D	0.6	0.6256	13.201	0.495
E	0.6	0.6280	13.166	0.67
F	0.6	0.6300	13.135	0.825
G	0.6	0.6320	13.102	0.99

Table 1: Section 6.1. Boundary conditions.

511 A comparison between the aforementioned methods is presented in Figure 9. In this plot, the
512 cell-averaged value of the numerical discharge in the cell containing the jump is depicted against the
513 normalized position of the jump inside the cell. This is done by carrying out several simulations using
514 the BC in Table 1. It is observed that the old spike reducing method presented in [16] provides a
515 much better result than the traditional ARoe solver and than the flux function A approach, which
516 was not intended to work for non-homogeneous problems. However, it still shows the presence of the
517 spike. On the other hand, when using the novel ARoe SR method, the spike is completely eliminated
518 with machine precision for the shock locations considered in this test case. At any location in Table
519 1, the cell-averaged numerical discharge is exactly equal to $0.6 \text{ m}^2/\text{s}$. A further investigation of the
520 performance of the ARoe SR solver is presented in subsection 6.1.1, where all shock locations in quasi-
521 steady conditions are carefully examined.

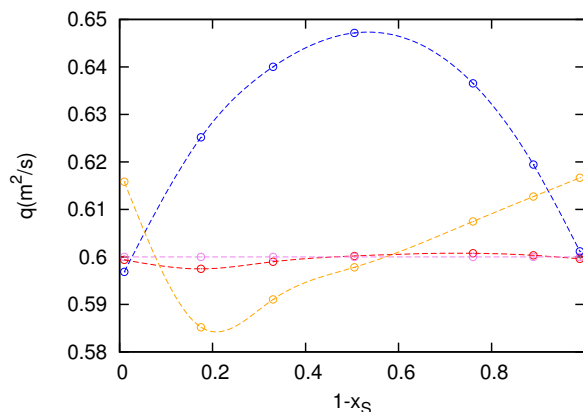


Figure 9: Section 6.1. Representation of the spike of discharge against the position of the shock within the cell for the traditional Roe flux ($- \circ -$), for the flux function A in [12] ($- \circ -$), for the old spike-reducing method in [16] ($- \circ -$) and for the ARoe SR solver ($- \circ -$), using 100 cells and CFL=0.45.

522 In Figure 10, the numerical solution for h and hu , when choosing $h_{BC:right} = 0.621 \text{ m}$, computed by
523 the traditional ARoe method, the flux function A in [6], the old spike reducing method in [16] and the
524 novel ARoe SR method is presented. The numerical results evidence that only when using the ARoe
525 SR method, the exact solution is obtained. Such method is able to get rid of the spike with machine
526 precision. In Figure 11, a piecewise constant representation of the same solution (dashed black line),
527 including the intermediate states provided by the solvers (continuous red line), is shown. It can be
528 seen that equilibrium is achieved when the left and right states of hu coincide with the values in the
529 adjacent cells, which occurs for all solvers. However, the spike is only eliminated when all intermediate
530 states are equal to the cell value. This only happens for the ARoe SR solver.

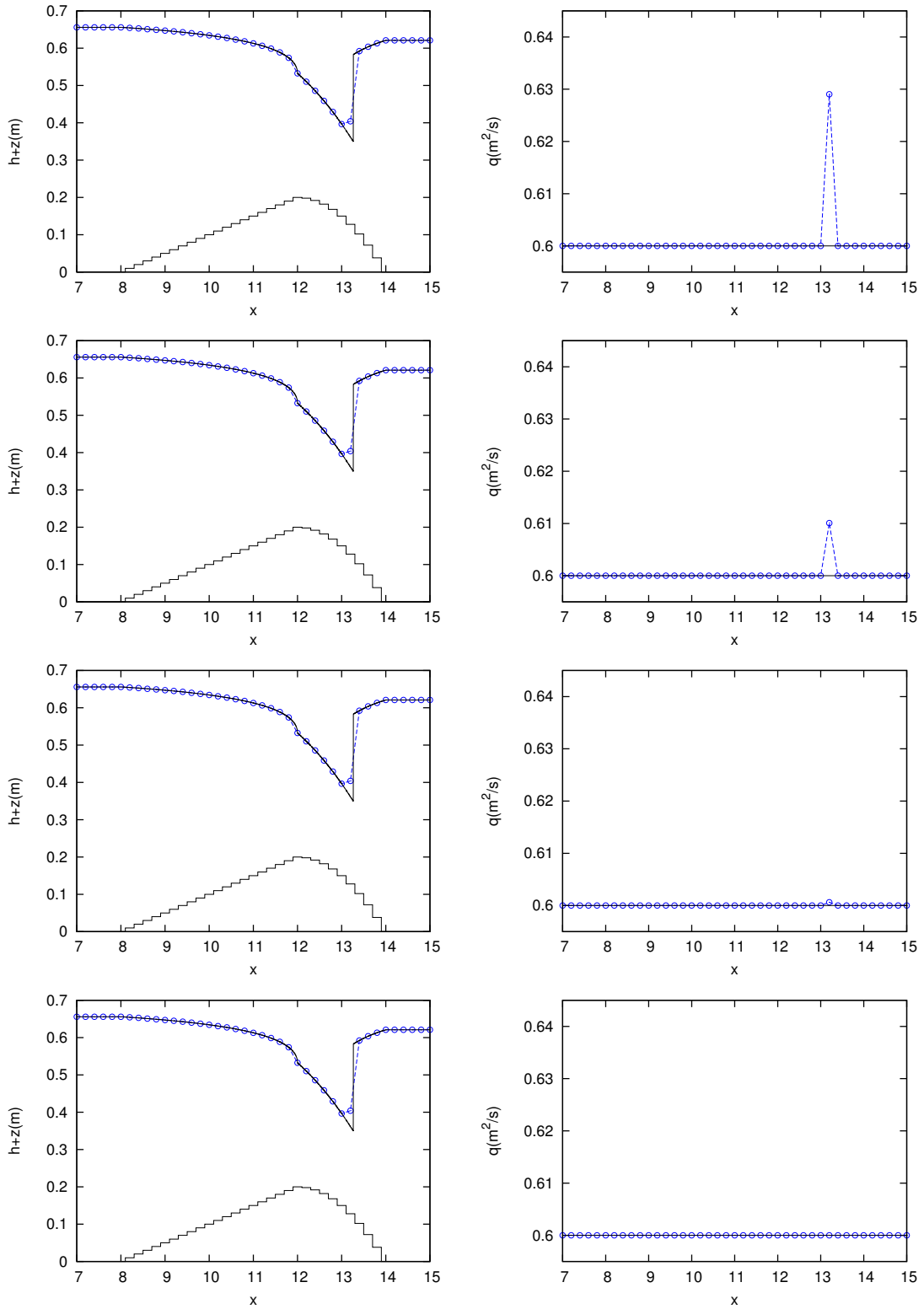


Figure 10: Section 6.1. Numerical solution provided by the traditional ARoe solver (top), the flux function A (middle-top), the old spike-reducing method (middle-bottom) and the novel SR method (bottom).

531 *6.1.1. Detecting small regions of instability for the ARoe SR solver*

532 It is worth noting that small regions of instability have been reported in these types of problems,

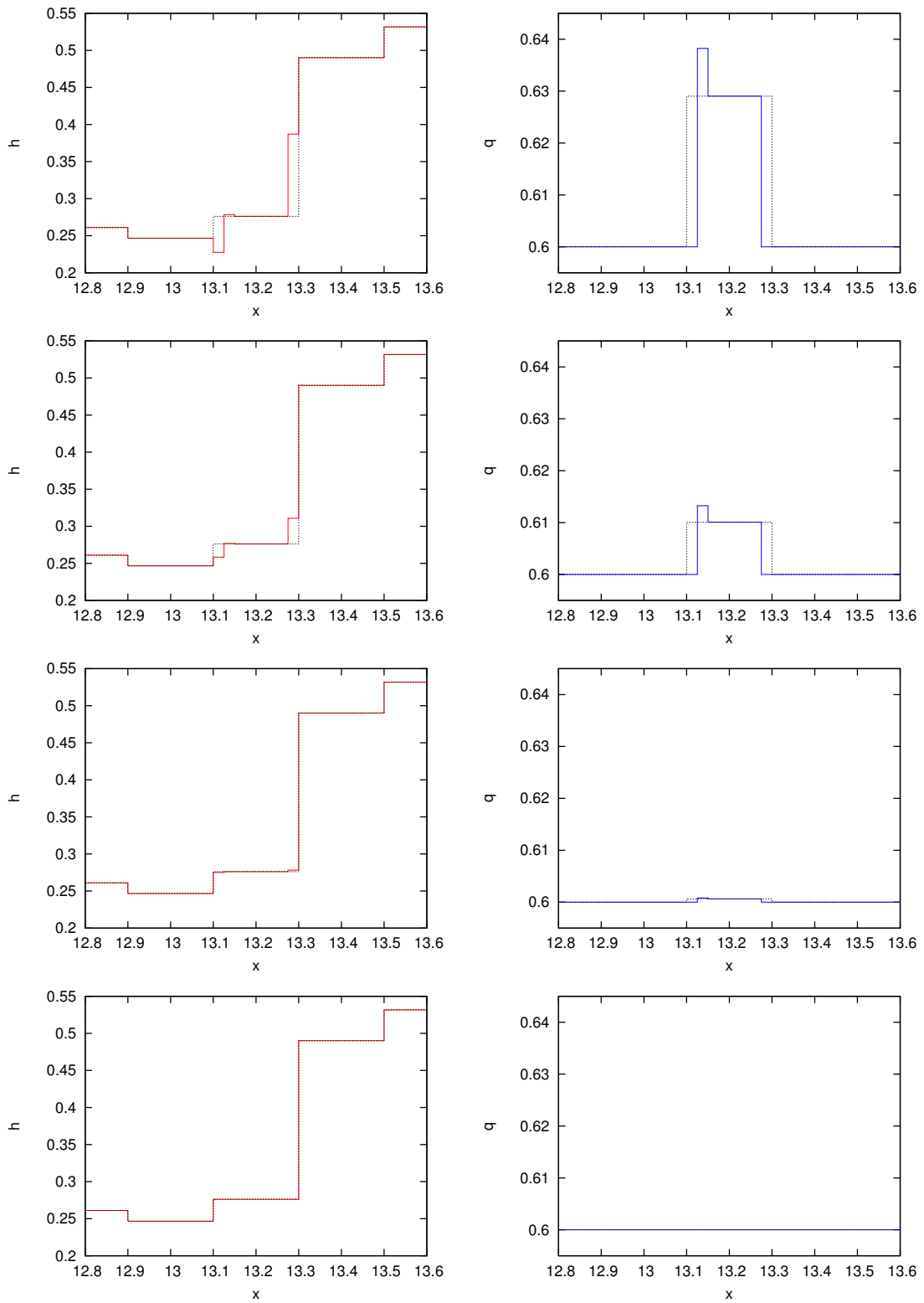


Figure 11: Section 6.1. Piecewise constant representation of the solution, including intermediate states, provided by the traditional ARoe solver (top), the flux function A (middle-top), the old spike-reducing method (middle-bottom) and the novel SR method (bottom).

533 so a further investigation on this must be carried out. In order to detect potential instabilities of the
534 methods, all shock locations, x_S , should be examined. Obtaining the solution for every location x_S
535 in the interval $[0, 1]$ would be very costly, since a very large number of steady simulations would be
536 required. Therefore, a different approach to examine all shock locations is proposed here. Departing
537 from the steady solution, the boundary condition downstream, $h_{BC:right}$, can be slowly changed in
538 time in order to sweep all shock locations inside the cell. Note that the rate of change must be low in
539 order to maintain quasi-steady conditions and detect the possible regions of instability.

540 Let us consider the test case in Section 6.1 and a grid of 100 cells. First, an steady solution is
541 obtained by imposing $hu = 0.6 \text{ m}^2/\text{s}$ upstream and $h_{BC:right} = 0.6320 \text{ m}$ downstream. Then, a new
542 simulation is configured using as initial condition the steady solution obtained with the aforementioned
543 configuration, maintaining the upstream boundary condition and imposing the following downstream
544 water depth

$$h_{BC:right}(t) = 0.6320 - 2.7 \cdot 10^{-4}t. \quad (72)$$

545 The simulation is run for $t = 100 \text{ s}$. Figure 12 shows the evolution in time of the numerical
546 discharge at cell $i = 67$ ($x = 13.2$) in the interval $t = [0, 100]$ computed by the traditional ARoe
547 solver and the novel SR solver. As expected, the traditional ARoe solver shows a clear evidence of the spike,
548 showing that the amplitude of such anomaly depends on the shock location. Note that the diagram
549 $hu - t$ resembles the representation $hu - x_S$ as the flow is quasi-steady. Concerning the SR solver,
550 it is observed that the method is not able to exactly eliminate the spike in the transition of the shock
551 across an interface. Such regions of instability are not displayed in Figure 9 since they are very narrow
552 and only the solution for 7 shock locations are represented in this figure. It is worth pointing out that
553 this spurious behavior of the ARoe SR solver is not observed when considering moving jumps in 1D
554 [16] or 2D shocks, as it will be shown in the numerical tests involving 2 spatial dimensions. Further
555 comments on this issue are presented in the conclusions section.

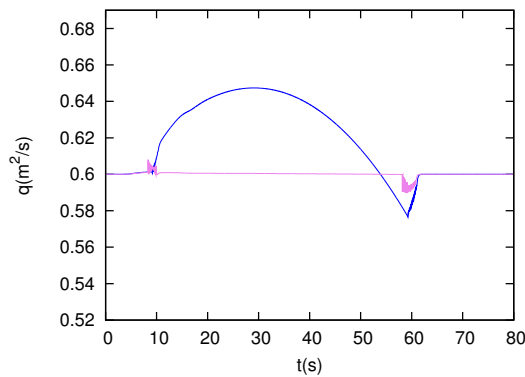


Figure 12: Section 6.1.1. Evolution in time of the numerical discharge at cell $i = 67$ ($x = 13.2$) computed by the traditional ARoe solver and the novel SR solver.

556 6.2. Steady supercritical flow against a solid wedge: the Mach reflection

557 We consider here the resolution of a Mach reflection (MR) pattern that arises from the reflection
558 of an oblique shock against a solid wall [27]. Oblique shocks normally appear when a supercritical
559 straight flow encounters a wedge that deflects it. The presence of the wedge involves a change in the
560 flow field and aligns the flow in its direction, θ . The region of influence of the wedge corresponds to
561 the region downstream the resulting oblique shock. The discontinuity between both regions is a shock
562 wave with an angle β . When the incident oblique shock, hereafter denoted by I, encounters a solid wall,
563 the MR, composed by waves M and R, may appear as depicted in Figure 13. This type of reflection

564 leads to the so-called 3-shock solution, that separate the incident flow, whose state is denoted by (0),
 565 from the regions (1), (2) and (3).

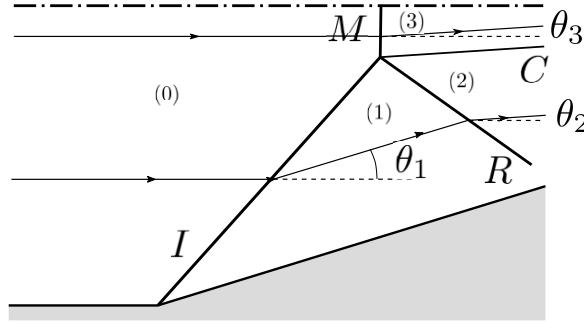


Figure 13: Section 6.2. Sketch of a MR wave pattern, including relevant angles and states.

566 The analytical 3-shock solution for a given incoming Froude number and deflection angle can be
 567 obtained using the so-called shock polar diagram. Such diagram is a representation of $h/h_0 = h/h_0(\theta)$,
 568 with θ in the x -axis and h/h_0 in the y -axis. The solution for the MR will be located on the intersection
 569 between the curves $h_1/h_0 = h_1/h_0(\theta)$ and $h_{2,3}/h_0 = h_{2,3}/h_0(\theta)$, where $h_{2,3}$ is the water depth in regions
 570 (2) and (3). Note that $h_{2,3}/h_0$ can be easily computed as $h_{2,3}/h_0 = h_1/h_0(\theta_1) \cdot h_{2,3}/h_1(\theta' - \theta_1)$, where
 571 θ' is the deflection angle with respect to θ_1 .

572 In this test case, we consider a supercritical flow aligned to the x -axis and confined in a straight
 573 channel with solid walls. The flow is defined by $Fr_0 = 4.2$ and $h_0 = 1$ m, and is deflected by a wedge of
 574 $\theta_1 = 23.3048^\circ$, generating an incident attached shock, I, which is eventually reflected by the top wall.
 575 The computational domain is given by $\Omega = [0, 100] \times [0, 55]$ and the solid domain is defined by the
 576 points $(15, 0)$, $(80, 28)$ and $(80, 0)$. Solid BCs are considered on the lateral walls, while a supercritical
 577 BC is considered at the inlet and a transmissive BC at the outlet. A flat bed is considered in order to
 578 compute the analytical solution. The mach-polar diagram and the analytical solution for the 3-shock
 579 structure is depicted in Figure 14.

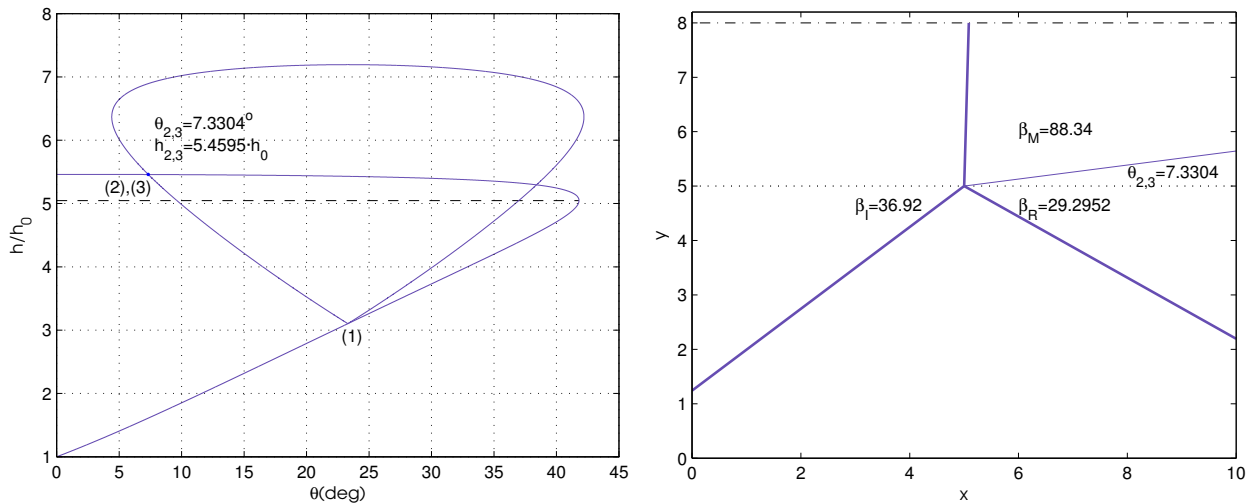


Figure 14: Section 6.2. Mach polar diagram including the analytical solution (left) and sketch of the analytical solution for the Mach stem (right)

580 This test case is used to assess the SWC approaches for the ARoe solver and has a two-folded
 581 aim: (a) the evaluation of such approaches for controlling the carbuncle and (b) the evaluation of
 582 the increased viscosity in the shear waves provided by such approaches. To this end, the solution is

583 computed at $t = 200$ s in a 400×220 grid using the ARoe scheme, HLLS scheme and their combinations
 584 with the SWC 1 and 2 techniques. The numerical x velocity magnitude, u , is depicted for all the
 585 aforementioned schemes in Figure 15, where the analytical shock structure has been overlapped in
 586 white. A detail of the solution around the 3-shock reflection region is depicted in Figure 16, where
 587 differences among the selected solvers can be easily observed.

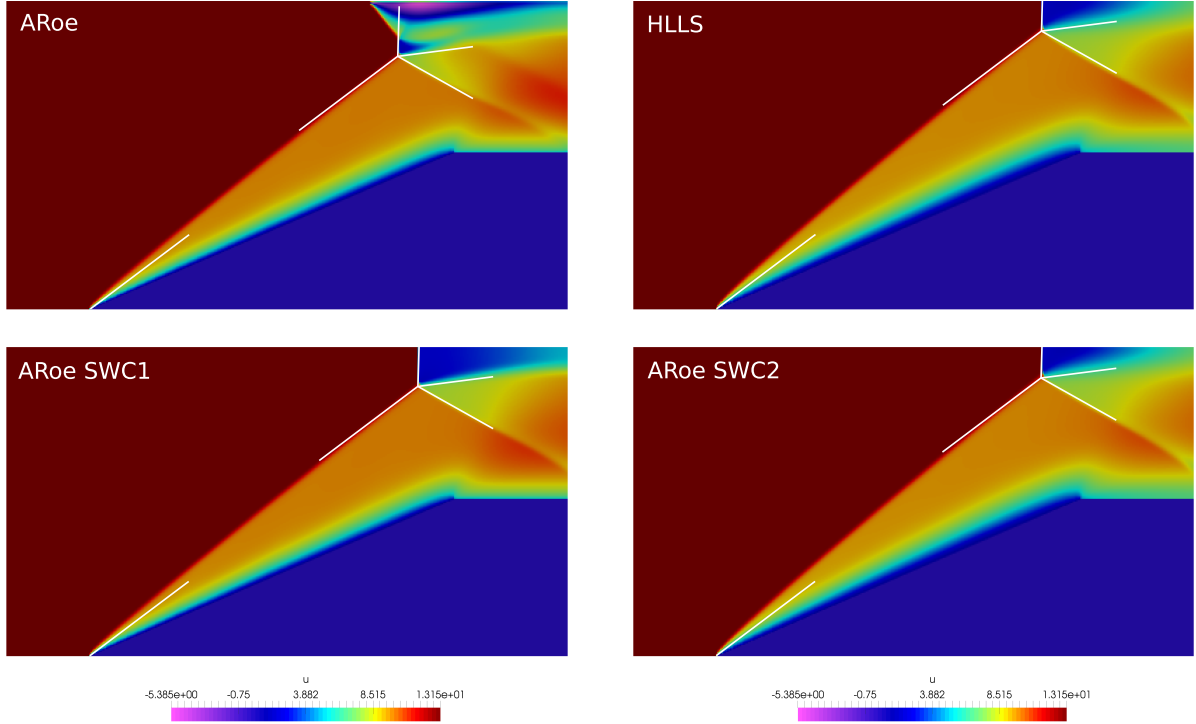


Figure 15: Section 6.2. Numerical solution for u computed by the ARoe solver (upper left), HLLS solver (upper right), ARoe SWC1 (lower left) and ARoe SWC2 methods (lower right).

588 In Figures 15 and 16, it is evidenced that the computation of the Mach stem completely fails when
 589 using the ARoe scheme, whereas the HLLS solver and the ARoe solver in combination with the SWC1
 590 and SWC2 approaches do capture the M wave without carbuncles. Concerning the numerical viscosity
 591 of the schemes, it can be observed that the best choice would be the ARoe SWC1, as it does not smear
 592 too much the contact wave, C, behind the M and R waves. The HLLS solver and the ARoe SWC2
 593 solver, which is based on the HLLS, resolve the contact wave in the same way and therefore yield the
 594 similar amount of viscosity for such wave. These solvers produce a higher numerical diffusion across
 595 shear waves since they only consider an eigenstructure of the system composed of 2 waves, neglecting
 596 the presence of the contact wave. On the other hand, the R wave is accurately captured by all schemes
 597 as it is associated to a genuinely nonlinear field, resolved by all schemes. It is also worth pointing
 598 out that the representation of the wedge using a Cartesian grid produces a significant boundary layer
 599 along the solid body. This boundary layer is developed at the beginning of the edge and that is why
 600 the angle of the computed I wave does not match the analytical angle at that point. The position of
 601 the triple point (where all waves intersect) varies from one solver to another as it depends on the size
 602 of the boundary layer and the position of the I wave, as well as on the boundary conditions.

6.3. Numerical simulation of a 2D slowly moving hydraulic jump

604 Here, we consider the test case in Section 4.1.2. It consists of the computation of a supercritical
 605 flow that hits a circular obstacle and generates an slowly moving hydraulic jump upstream the obstacle.

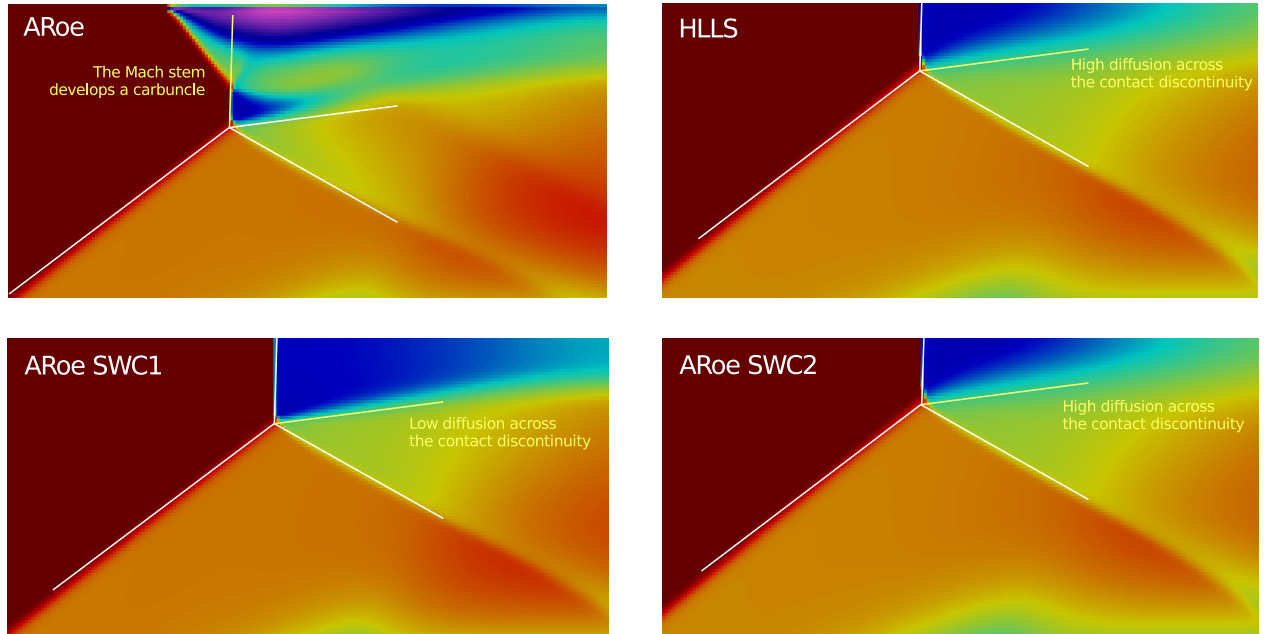


Figure 16: Section 6.2. Detail of the numerical solution for u computed by the ARoe solver (upper left), HLLS solver (upper right), ARoe SWC1 (lower left) and ARoe SWC2 methods (lower right), including comments of the features of the solution in yellow.

606 The solution is computed at $t = 80$ s setting CFL=0.4 and $\Delta x = 1.0$ m. The configuration of the case
 607 is given in Section 4.1.2

608 The numerical solution for hu at $t = 80$ s provided by the ARoe SWC1 SR and ARoe SWC2 SR
 609 solver is depicted in Figure 17. In Figure 18, a space-time representation of $hu(x, y, t)$ at a fixed $y = 50$
 610 m (left) and $x = 43$ m (right) is depicted. The solution is provided by the ARoe SWC1 SR solver (top)
 611 and the ARoe SWC2 SR solver (bottom).

612 Unlike the results for the ARoe and HLLS scheme in Section Section 4.1.2, the spike is not present
 613 now. The magnitude of the spurious waves, depicted in Figure 18, are greatly reduced if compared to
 614 the results for the ARoe and HLLS solver. As expected, the ARoe SWC2 SR solver proves to be more
 615 diffusive than the ARoe SWC1 SR solver.

616 6.4. Steady supercritical flow against a solid cylinder: circumventing the carbuncle and the spike

617 The same test case presented in Section 4.2.1 is considered here. It considers a supercritical flow
 618 against a solid cylinder, which creates a bow shock around it. The numerical solution is computed
 619 at $t = 50$ s using the ARoe, HLLS, ARoe SWC1, ARoe SWC2, ARoe SWC1 SR and ARoe SWC2
 620 SR solvers, setting CFL=0.4. The grid is composed of 72×180 cells, with $\Delta x = 0.5$. The numerical
 621 solution for hu in the (x, y) plane is depicted in Figure 19 and a longitudinal cross section of the
 622 solution for h and hu at $y = 50$ is presented in Figure 20.

623 As expected, it is observed that the HLLS method as well as the improved versions of the ARoe
 624 solver avoid the carbuncle. Concerning the presence of the spike in the numerical discharge, it is
 625 evidenced that the SR approach does completely eliminate the spurious discharge, showing no discontinuity
 626 across the hydraulic jump and allowing a smooth transition between the supercritical inlet and
 627 the stagnation point at the leftmost edge of the solid cylinder. Note that the ARoe SWC2 method
 628 does not show any spike either, but this is because of the particular location of the shock (close to a
 629 cell interface) when using this solver.

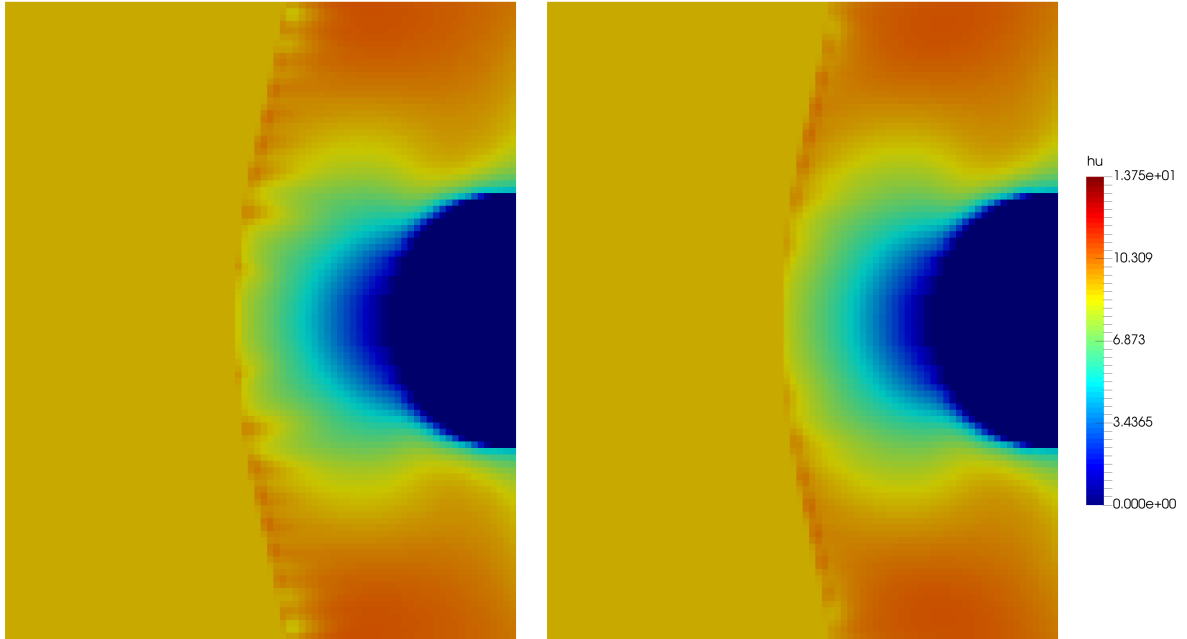


Figure 17: Section 6.3. Numerical solution for hu at $t = 80$ s provided by the ARoe SWC1 SR (left) and ARoe SWC2 SR solver (right).

630 6.4.1. Sensitivity to grid perturbations

631 In order to assess the sensitivity of the solvers to small perturbations of the grid, the previous test
 632 case (Section 6.4) is computed using a grid composed of 72×180 cells, that is, scaling the original
 633 grid using a scale factor of $10/9$, which yields $\Delta x = 0.5/0.9$. A comparison between the original and
 634 perturbed grid is provided in Figure 21.

635 When using the perturbed grid, as the cell size is slightly increased, cell interfaces will not be
 636 at the same position and therefore the relative position of the shock inside the cell, x_S , will have
 637 changed. In spite of this variation, the numerical solution should be robust enough to not show
 638 significant differences from one grid to another. A comparison of the numerical solution computed in
 639 the original and perturbed grids will allow to evaluate the robustness of the methods. A longitudinal
 640 cross sectional representation of h and hu , computed by all solvers, at $y = 50$, is presented in Figure
 641 22. It can be observed that the ARoe solver is again unstable, producing a carbuncle, while the other
 642 solvers provide a stable solution even without spikes when the SR approach is used. If comparing the
 643 solutions with those in Section 6.4, we notice that the solution provided by the HLLS solver, the ARoe
 644 SWC2, ARoe SWC2 SR and ARoe SWC1 SR does not change significantly from one case to the other,
 645 which evidences the robustness of those methods. A more significant variation is observed for the
 646 ARoe SWC1 solver, which now shows a greater spike and a region with lower discharge downstream
 647 the spike. Nevertheless, the ARoe SWC1 solver is robust enough to not produce any carbuncle.

648 6.4.2. Sensitivity to flow perturbations in time

649 In the previous tests, two different grids have been used to analyze the robustness of the method
 650 to variations in the sub-cell shock location induced by the configuration of the grid. However, in

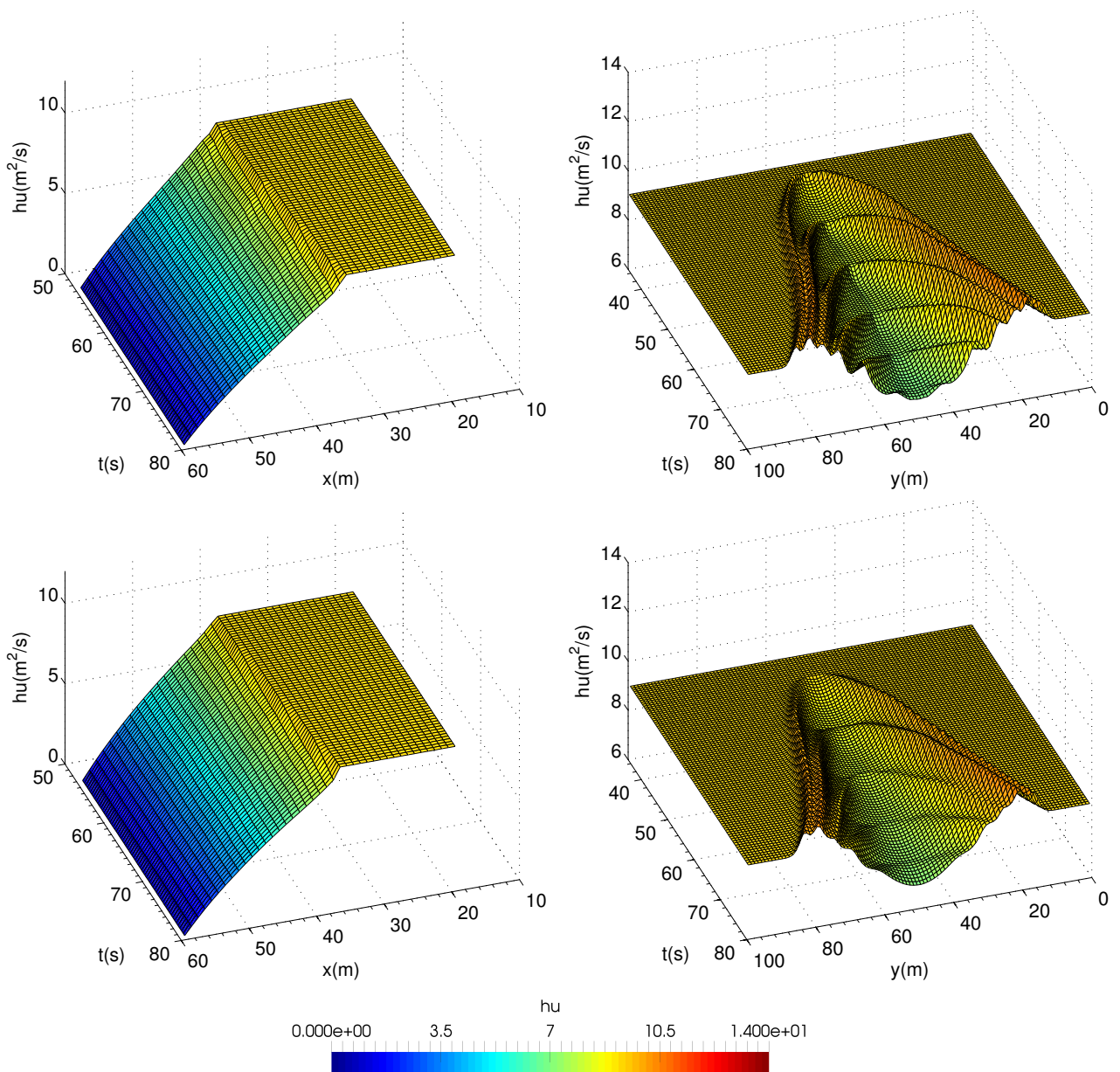


Figure 18: Section 6.3. Evolution in time of the numerical hu along the x direction (at $y = 50$ m) (left) and y direction (at $x = 43$ m) (right), provided by the ARoe SWC1 SR (top) and the ARoe SWC2 SR solver (bottom).

651 both cases the solution was driven to the steady state using the same boundary and initial conditions.
 652 In this section, the goal is to analyze the performance of the methods to overcome perturbations
 653 and variations in the flow conditions that produce a transition between two steady regimes (hence
 654 a continuous variation of the sub-cell shock position in time). The same configuration than in the
 655 previous cases is considered and the flow is driven to the steady state in the same way. Once the
 656 steady regime is achieved, the inlet discharge is modified in time (during 10 s) in order to produce a
 657 transition to a new steady state and assess the behavior of the solution during the transition. The
 658 unitary discharge at the inlet is imposed as follows

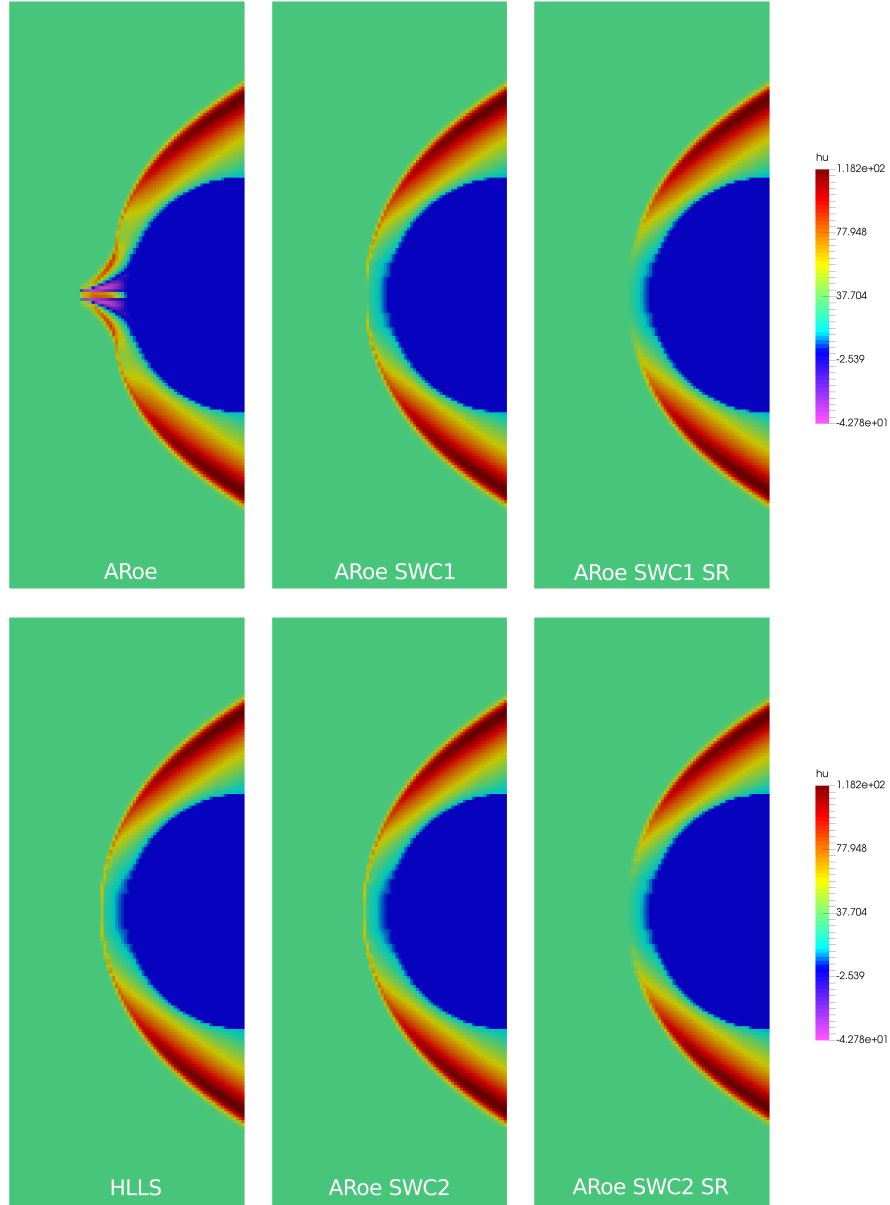


Figure 19: Section 6.4. Numerical hu at $t = 50$ s provided by the ARoe, HLLS, ARoe SWC1, ARoe SWC2, ARoe SWC1 SR and ARoe SWC2 SR solvers.

$$hu = \begin{cases} 30 & \text{if } t \leq 40 \\ 30 - (t - 40) & \text{if } 40 < t < 50 \\ 20 & \text{if } 50 < t \end{cases} \quad (73)$$

659 Note that $t = 40$ s is the time when the initial steady regime has been achieved, that the discharge
660 is modified using a linear function in the interval $t \in (30, 40)$ s and that the discharge generating the
661 new steady state is $hu = 20$ m²/s.

662 In order to evaluate the robustness of the methods to handle such flow variations and the efficiency
663 of the SR approach in eliminating the spike when the shock is moving, the time evolution of the cell
664 averaged solution in three consecutive cells is going to be analyzed. Such cells will be located just on

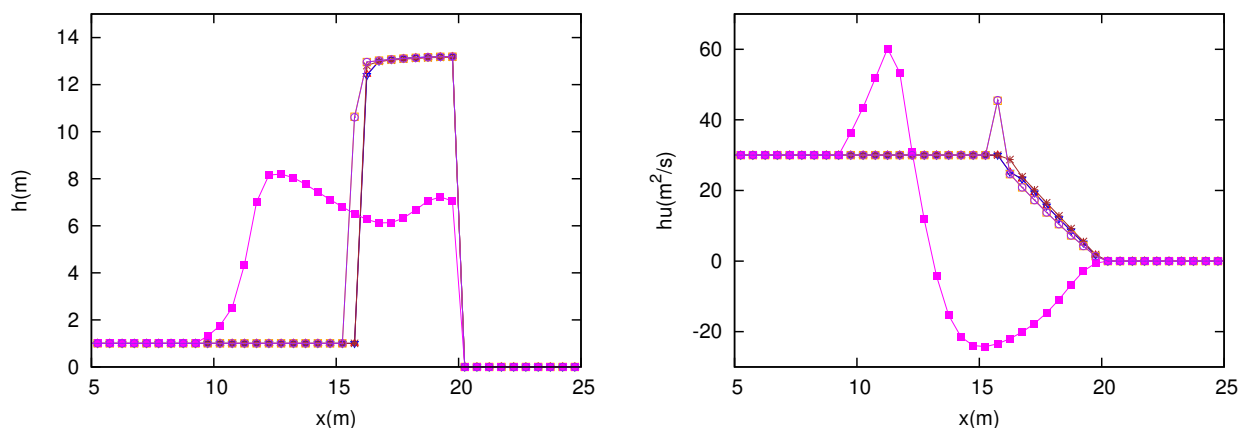


Figure 20: Section 6.4. Numerical h and hu at $y = 50$ and $t = 50$ s provided by the ARoe ($- \blacksquare -$), HLLS ($- \circ -$), ARoe SWC1 ($- \times -$), ARoe SWC2 ($- \square -$), ARoe SWC1 SR ($- \triangle -$) and ARoe SWC2 SR solvers ($- \nabla -$).

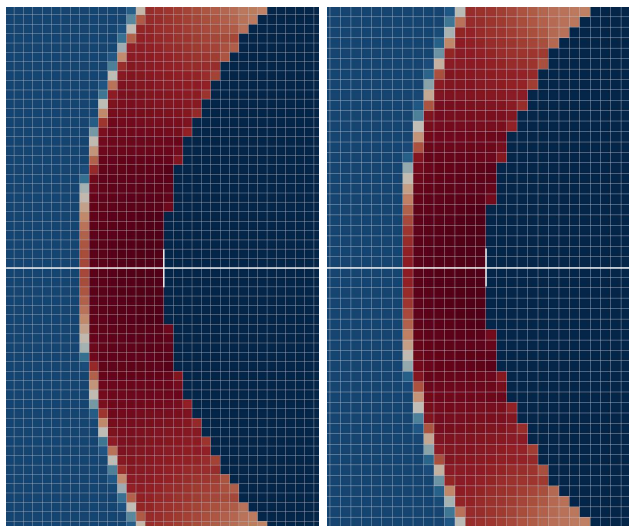


Figure 21: Section 6.4.1. Comparison between the original grid (left), composed of 80×200 cells, and the perturbed grid (right), composed of 72×180 cells.

665 the left hand side of the shock (for the position of the shock at $t = 40$ s) and correspond to cell numbers
 666 $i = 30, 31, 32$. As the inlet discharge is reduced from 30 to 20 m^2/s , the shock will travel upstream
 667 across those cells and the evolution of the variables in time at those cells can be tracked. The solution
 668 for the numerical hu inside those cells, computed by all the schemes, is depicted in Figure 23. The
 669 solution in the intermediate cell ($i = 31$) is depicted with a solid line while the solution in the other
 670 cells is depicted with a dashed line. Note that the SR property will only be satisfied if the solid line
 671 for hu is bounded by the dashed lines (when the shock is contained in the intermediate cell), which
 672 means that the discharge in the intermediate cell is bounded by the left and right discharges.

673 A close examination of Figure 23 allows to extract the following remarks:

- 674 • All the solvers, except the traditional ARoe solver, possess the sufficient robustness to handle flow
 675 fluctuations in time without developing carbuncles with this flow configuration.
- 676 • The transition between the steady state of reference ($t = 30$ s) and the new steady state ($t = 60$
 677 s) is properly achieved by all the solvers, except the ARoe solver, reaching again an steady state
 678 around $t = 60$ s.

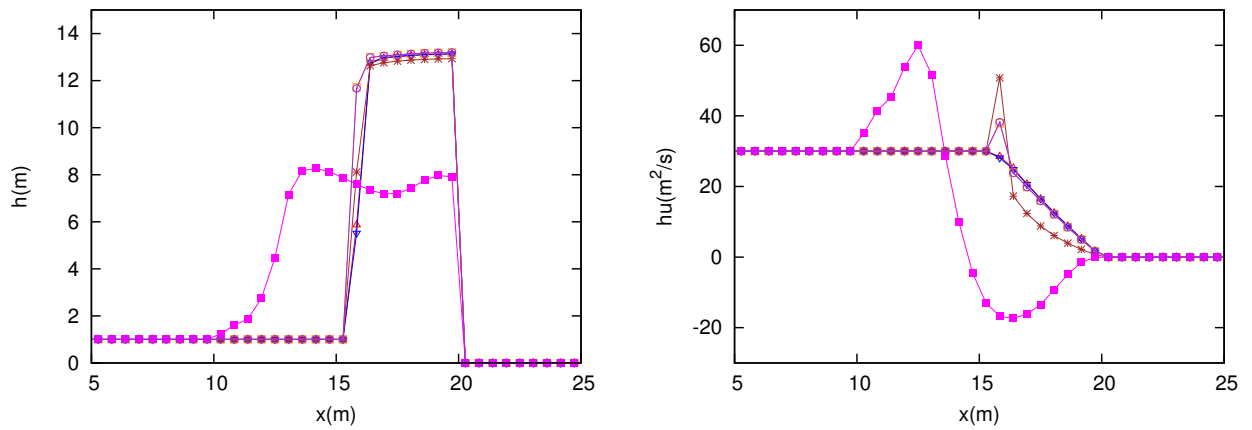


Figure 22: Section 6.4.1. Numerical h and hu at $y = 50$ and $t = 50$ s provided by the ARoe ($- \blacksquare -$), HLLS ($- \circ -$), ARoe SWC1 ($- \times -$), ARoe SWC2 ($- \square -$), ARoe SWC1 SR ($- \triangle -$) and ARoe SWC2 SR solvers ($- \nabla -$).

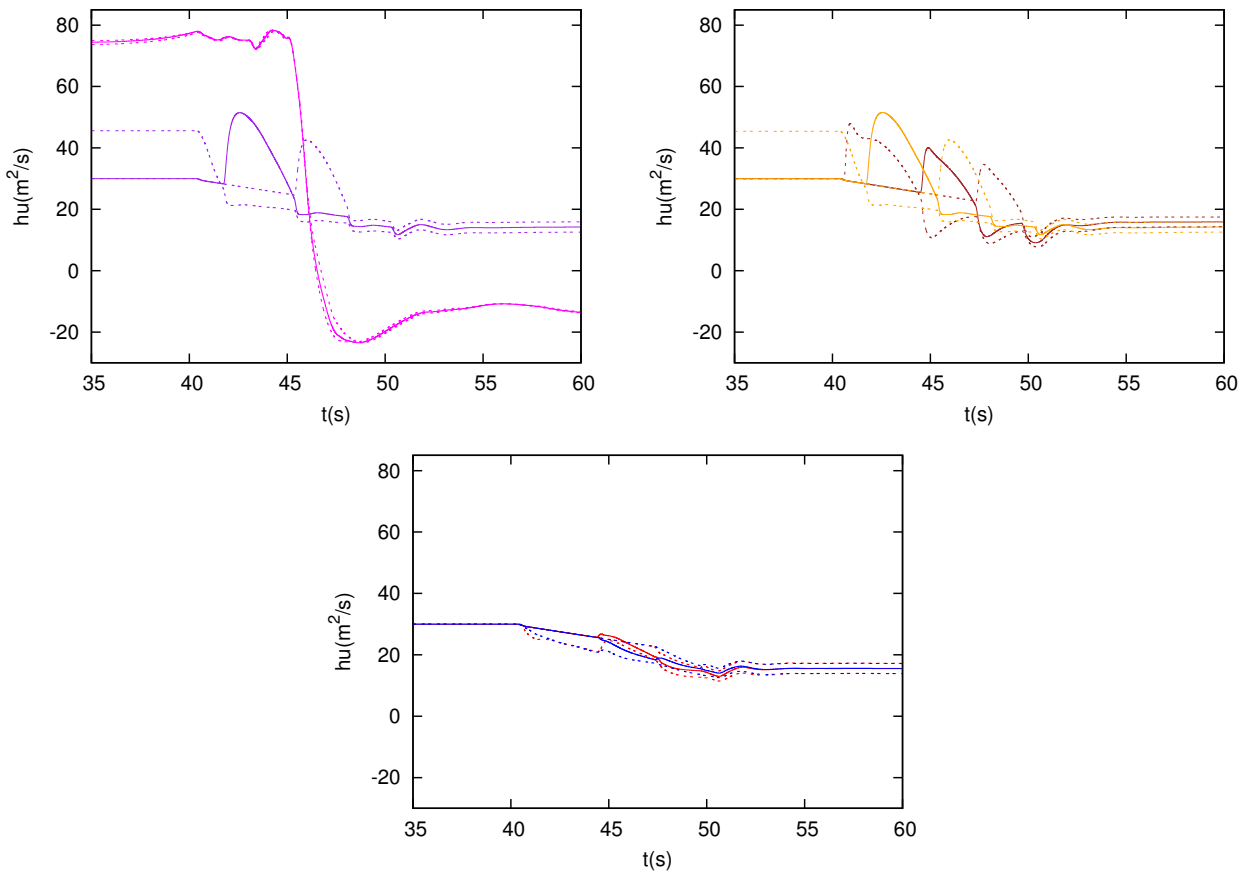


Figure 23: Section 6.4.2. Numerical hu inside cells $i = 30, 31, 32$ provided by the ARoe ($- \dashv -$) and HLLS ($- \dashv -$) solvers (top-left), ARoe SWC1 ($- \dashv -$) and ARoe SWC2 ($- \dashv -$) (top-right) and ARoe SWC1 SR ($- \dashv -$) and ARoe SWC2 SR solvers ($- \dashv -$) (bottom).

- 679 • The transition between those states is only smooth for the ARoe SWC1 SR and ARoe SWC2
680 SR solvers, which are able to reduce the slowly moving shock anomaly (spike). The other solvers
681 suffer from this anomaly, which is evidenced as an increasing-decreasing evolution of the discharge
682 in time, with a maximum value of discharge when the shock is located around the middle of the

683 cell. Note that this behavior is observed for the 3 cells as the shock propagates upstream.

- 684 • When using the ARoe SWC2 SR solver, the discharge in the intermediate cell (solid blue line) is
 685 completely bounded by the discharge in the left and right cells (dashed blue lines). The ARoe
 686 SWC1 SR solver also provides a numerical discharge in the intermediate cell that is virtually
 687 bounded by the neighboring states.
- 688 • Note that the regions of instability observed in Section 6.1.1, Figure 12, are not present in this
 689 case when using the SR approach.

690 *6.5. Steady supercritical flow against a solid cylinder with bed variation: circumventing the carbuncle*
 691 *and the spike*

692 This test case considers a supercritical flow over a non-flat bottom, hitting a circular obstacle. The
 693 computational domain is $\Omega = [0, 40] \times [0, 100]$ and the solution is computed at $t = 50$ s setting CFL=0.4
 694 and using two different grids with $\Delta x = 0.5$ and $\Delta x = 0.25$ m. The water depth and unitary discharge
 695 at the inlet are set as $h_L = 1$ m and $hu_L = 20$ m²/s respectively. Transmissive boundary conditions are
 696 set at the other boundaries. The solid body is defined as $\mathcal{W} = \{\mathbf{x} \mid (x - 40)^2 + (y - 50)^2 \leq 400, \mathbf{x} \in \Omega\}$
 697 and the bed elevation is given by

$$z(x, y) = \begin{cases} 0 & \text{if } x < 5 \\ 0.05(x - 5) + \sin(0.05\pi(x - 5)) \cos(0.05\pi y) & \text{if } x > 5 \end{cases} \quad (74)$$

698 which is depicted in Figure 24.

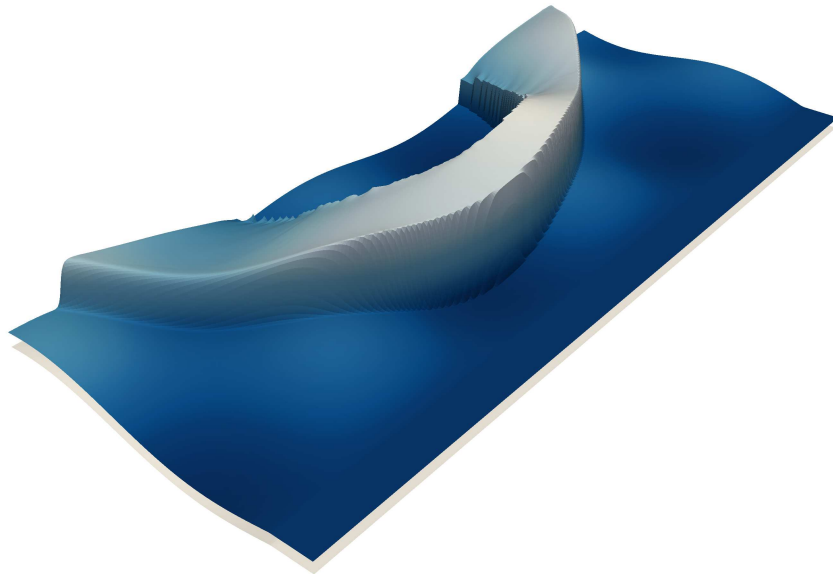


Figure 24: Section 6.5. Three dimensional representation of the solution for $h + z$ and z .

699 The numerical solution for hu , provided by the the ARoe, HLLS, ARoe SWC1, ARoe SWC2, ARoe
 700 SWC1 SR and ARoe SWC2 SR solvers, is depicted in Figure 25 and 26, using $\Delta x = 0.5$ and $\Delta x = 0.25$
 701 respectively. It is observed that all the proposed solvers are able to handle bed variations and converge
 702 to a carbuncle-free solution. In the case of using the SR approach, the spike in the numerical discharge
 703 is also eliminated.

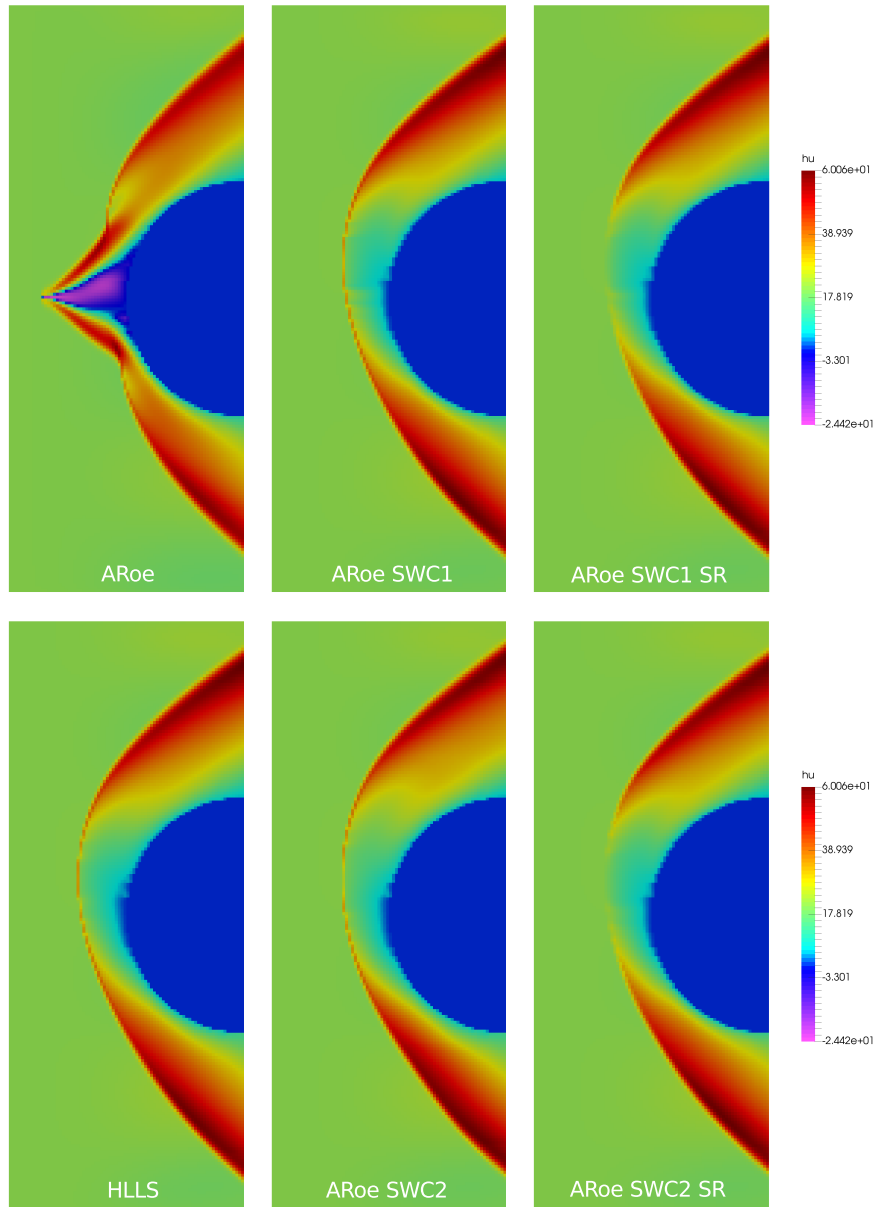


Figure 25: Section 6.5. Numerical hu at $t = 50$ s provided by the ARoe, HLLS, ARoe SWC1, ARoe SWC2, ARoe SWC1 SR and ARoe SWC2 SR solvers, computed using $\Delta x = 0.5$ m.

704 7. Conclusions

705 The study and prevention of numerical shockwave anomalies in the framework of the SWE is
 706 addressed in this work. Such anomalies are usually observed in the computation of shock waves and
 707 have been studied by the scientific community for more than two decades, typically in the framework of
 708 Euler equations. The most studied problems are the carbuncle and the slowly-moving shock anomaly,
 709 both leading to an incorrect and oscillating discrete shock profile that may eventually ruin the solution.

710 In this article, the aforementioned anomalies are addressed for the SWE with bed slope source term,
 711 both in one and two spatial dimensions. In a previous work, the authors provided a thorough study
 712 on slowly-moving shocks in the framework of the SWE and presented an spike reduction method [16],
 713 based on a flux interpolation idea by Zaide and Roe [6] coupled with the use of an augmented Riemann

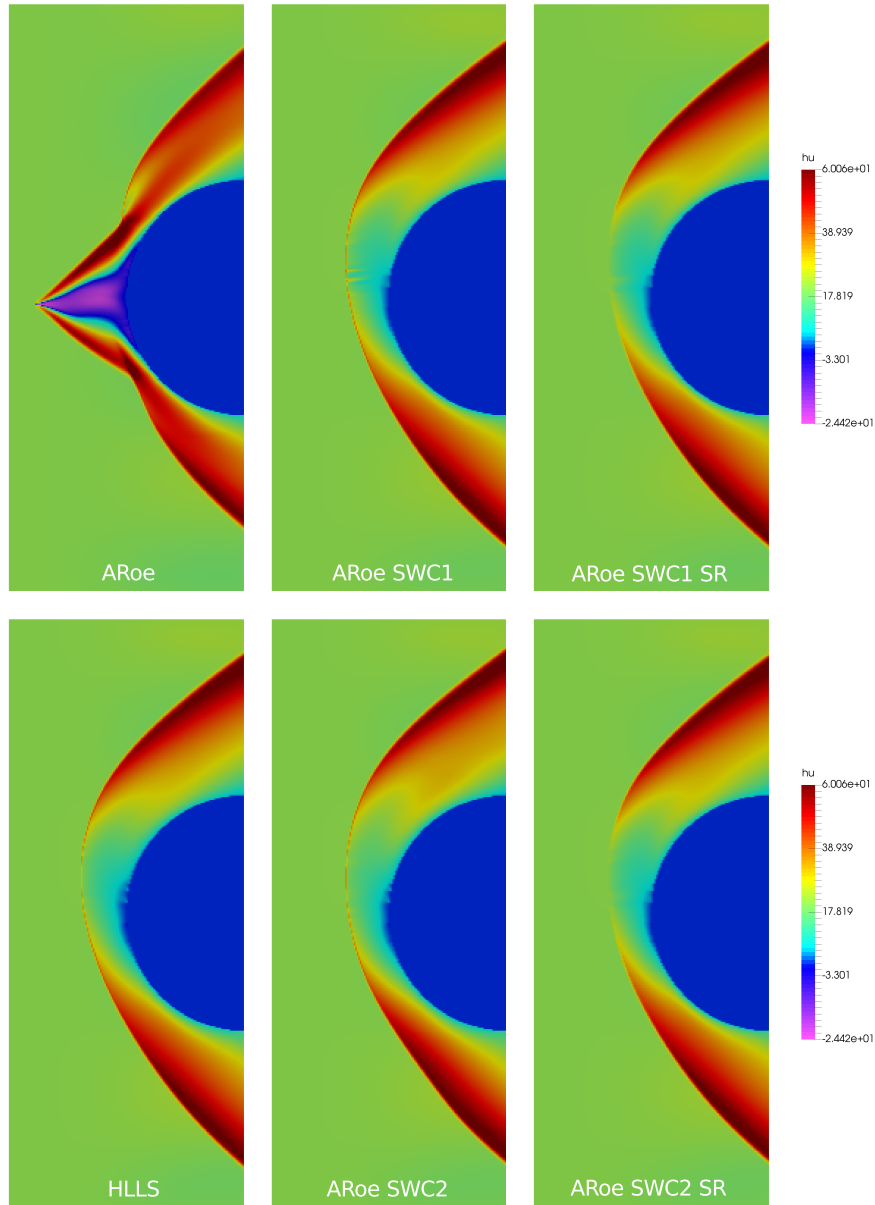


Figure 26: Section 6.5. Numerical hu at $t = 50$ s provided by the ARoe, HLLS, ARoe SWC1, ARoe SWC2, ARoe SWC1 SR and ARoe SWC2 SR solvers, computed using $\Delta x = 0.25$ m.

714 solver, the ARoe solver. The resulting scheme proved to reduce the spike in the numerical discharge
 715 without adding extra diffusion and allowed the convergence of the solution to the exact solution for
 716 the first time.

717 In this work, the spike reducing method in [16] is reconsidered and improved by using a more suitable
 718 source term discretization in the correction of the physical fluxes that allows the exact equilibrium
 719 between sources and fluxes at cell interfaces. The resulting scheme, called SR method, is able to
 720 eliminate the spike with machine precision in 1D steady cases, as it is evidenced in Test Case 6.1,
 721 except when the shock is located in the neighborhood of cell interfaces. In those cases, the spike is not
 722 exactly eliminated but is still reduced if compared to other traditional solutions. To the knowledge of
 723 the authors, this achievement is first reported here.

724 Furthermore, the SR method is successfully extended to two spatial dimensions by means of a
725 dimension-by-dimension approach in a Cartesian grid. Compared to other traditional solvers, there is
726 a significant improvement in the elimination of the spike line. This can be observed when comparing
727 Figures 4 and 18, where the spike is completely eliminated. It is also important to mention that
728 the small instability regions of the SR method reported in 1D quasi-steady cases are negligible when
729 considering 2D problems.

730 The slowly-moving shock anomaly in 2D involves a shedding of oscillations in the transverse direc-
731 tion, motivated by this anomaly and mainly by the Cartesian representation of the shock profile. The
732 numerical results evidence that such spurious waves are reduced when the SR method is applied. The
733 choice of a solver with high viscosity in the contact wave (e.g. the HLLS solver) also helps.

734 The proposed 2D SR method proves to perform well for the elimination of the slowly moving shock
735 anomaly, however, it suffers from the carbuncle. To address this problem, two different numerical
736 strategies based on the addition of extra viscosity in the shear waves are proposed. Such methods,
737 designed to be coupled with the ARoe solver, are called shear wave correction (SWC) 1 and 2. The
738 former is based on the passive transport of the shear momentum by means of the numerical discharge
739 in the normal direction of the interface while the latter uses the HLLS flux to transport the shear
740 momentum.

741 Such techniques are designed to be easily coupled with the ARoe solver and the SR method and
742 yield a robust solver that is able to handle the slowly-moving shock anomaly and the carbuncle at the
743 same time when computing both transient and steady hydraulic jumps over irregular topography. The
744 resulting solvers, called ARoe SWC SR 1 and 2, include: (a) a flux extrapolation/correction method
745 that provides the exact solution for the discharge in steady conditions under complex topography,
746 namely the SR method, (b) a contact wave smearing method that avoids the carbuncle, namely the
747 SWC 1 or 2, (c) an upwinding algorithm that considers the contribution of the source term in the
748 solution of the Riemann Problem (RP) ensuring the well-balanced property, namely a version of the
749 ARoe solver based on a flux-splitting technique, and (d) an entropy correction method.

750 The numerical results evidence that ARoe SWC SR 1 is less diffusive than the ARoe SWC SR
751 2, but both are able to circumvent the slowly-moving shock anomaly and the carbuncle in hydraulic
752 jumps at the same time with independence of the grid for steady and transient problems in 1D and
753 2D. A sensitivity analysis of the solvers to the mesh size (variations of the sub-cell shock position in
754 steady state) and to flow perturbations (transient variations of the sub-cell shock position) shows that
755 the SWC 1 approach, when it is not combined with the SR method, is less robust than the SWC 2
756 approach, as the latter is based on the HLLS flux which is more diffusive. When the SWC approaches
757 are combined with the SR method, both yield similar results and a greater robustness.

758 Taking all the results into account, we recommend the application of the ARoe SWC SR 1 rather
759 than the ARoe SWC SR 2, since the former offers a similar performance than the latter though it does
760 not involve the combination of the ARoe with the HLLS flux. The ARoe SWC SR 1 solver computes
761 the shear component of the momentum by using the normal discharge to the interface, an approach
762 that has proved to provide accurate results when considering contact discontinuities associated to
763 the transport of passive quantities [28]. Note that in the x -split SWE, the shear component of the
764 momentum is associated to a linearly degenerated field and can be regarded as a passive quantity.

765 It is worth saying that there are other sophisticated carbuncle-free solvers in the literature that
766 do not require to detect the presence of strong shocks [13, 14]. However, to be combined with the SR
767 technique proposed here, the SWC technique is more suitable and easier of implementation as the SR
768 method does need the detection of strong shocks.

769 **Appendix A. Derivation of the correction function φ**

770 The correction term φ has to be derived to satisfy the properties stated in Section 5.1. To this
 771 end, let us consider the expression for $\tilde{\mathbf{F}}_i$ in (50) and suppose that the intermediate state \mathbf{U}_i can be
 772 expressed as a linear combination of the left and right states (linear Hugoniot)

$$\mathbf{U}_i = x_{S,i}\mathbf{U}_{i-1} + (1 - x_{S,i})\mathbf{U}_{i+1}, \quad (\text{A.1})$$

773 where \mathbf{U}_{i-1} , \mathbf{U}_i and \mathbf{U}_{i+1} are any arbitrary left, middle and right states defining a hydraulic jump as
 774 depicted in Figure 1. Parameter $x_{S,i}$ accounts for the normalized position of the shock inside the cell,
 775 here approximated by

$$x_{S,i} = \frac{h_i - h_{i+1}}{h_{i-1} - h_{i+1}}. \quad (\text{A.2})$$

776 If inserting (A.1) in (50), we obtain

$$\tilde{\mathbf{F}}_i = \frac{1}{2}(\mathbf{F}_{i+1} + \mathbf{F}_{i-1}) - \left(\frac{1}{2} - x_{S,i}\right)\tilde{\mathbf{J}}_{i-1,i+1}(\mathbf{U}_{i+1} - \mathbf{U}_{i-1}), \quad (\text{A.3})$$

777 Considering now steady state conditions, we can substitute $\mathbf{F}_{i+1} = \mathbf{F}_{i-1} + \bar{\mathbf{S}}_{i-1,i+1}$ and $\tilde{\mathbf{J}}_{i-1,i+1}(\mathbf{U}_{i+1} -$
 778 $\mathbf{U}_{i-1}) = \bar{\mathbf{S}}_{i-1,i+1}$ in (A.3), yielding

$$\tilde{\mathbf{F}}_i = \mathbf{F}_{i-1} + (1 - x_{S,i})\bar{\mathbf{S}}_{i-1,i+1}, \quad (\text{A.4})$$

779 In order to satisfy the GRH condition at $x_{i-1/2}$, $\hat{\mathbf{F}}_i - \mathbf{F}_{i-1} = \bar{\mathbf{S}}_{i-1/2}$, the following equality must
 780 hold

$$(\mathbf{F}_{i-1} + (1 - x_{S,i})\bar{\mathbf{S}}_{i-1,i+1} + \varphi) - \mathbf{F}_{i-1} = \bar{\mathbf{S}}_{i-1/2}, \quad (\text{A.5})$$

781 hence, φ reads

$$\varphi = \bar{\mathbf{S}}_{i-1/2} - (1 - x_{S,i})\bar{\mathbf{S}}_{i-1,i+1}. \quad (\text{A.6})$$

782 If considering the GRH condition at $x_{i+1/2}$ and carry out an analogous derivation of φ , we obtain

$$\varphi = x_{S,i}\bar{\mathbf{S}}_{i-1,i+1} - \bar{\mathbf{S}}_{i+1/2}. \quad (\text{A.7})$$

783 From the equality of Equations (A.6) and (A.7), we obtain the following condition

$$\bar{\mathbf{S}}_{i-1,i+1} = \bar{\mathbf{S}}_{i-1/2} + \bar{\mathbf{S}}_{i+1/2}. \quad (\text{A.8})$$

784 that is to say, if the integrals at cell interfaces are computed using the trapezoidal rule, the centered
 785 integral should be computed using a composite trapezoidal rule. For instance

$$\bar{\mathbf{S}}_{i-1,i+1} = \begin{pmatrix} 0 \\ -g\frac{h_{i-1}+h_i}{2}(z_i - z_{i-1}) - g\frac{h_i+h_{i+1}}{2}(z_{i+1} - z_i) \end{pmatrix}, \quad (\text{A.9})$$

786 and

$$\bar{\mathbf{S}}_{i-1/2} = \begin{pmatrix} 0 \\ -g\frac{h_{i-1}+h_i}{2}(z_i - z_{i-1}) \end{pmatrix}. \quad (\text{A.10})$$

787 Concerning the property 3, a Taylor power series expansion of $\hat{\mathbf{F}}_i$ yields

$$\hat{\mathbf{F}}_i = \mathbf{F}_i + \left(\partial_{xx} \mathbf{F} - \tilde{\mathbf{J}} \mathbf{U} + \begin{pmatrix} 0 \\ \frac{1}{4} g \partial_x z \partial_x h \end{pmatrix} \right) \frac{\Delta x^2}{2} \quad (\text{A.11})$$

788 which holds the requirement as it converges with second order of accuracy.

789 **Acknowledgments**

790 The present work has been partially funded by the Aragón Government through the Fondo Social
791 Europeo. This research has also been supported by the Research Project CGL2015-66114-R, funded
792 by the Spanish Ministry of Economy and Competitiveness (MINECO).

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