REVISITING LIE INTEGRABILITY BY QUADRATURES FROM A GEOMETRIC PERSPECTIVE

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Abstract. After a short review of the classical Lie theorem, a finite-dimensional Lie algebra of vector fields is considered and the most general conditions under which the integral curves of one of the fields can be obtained by quadratures in a prescribed way are discussed, determining also the number of quadratures needed to integrate the system. The theory is illustrated with examples, and an extension of the theorem where the Lie algebras are replaced by some distributions is also presented.

1. Introduction: the meaning of integrability. Integrability is a topic that has been receiving quite a lot of attention because this not clearly defined notion appears in many branches of science, and in particular in physics. The exact meaning of integrability is only well defined in each specific field and each one of the many possibilities of defining in a precise way the concept of integrability has a theoretic interest. Loosely speaking integrability refers to the possibility of finding the solutions of a given differential equation

[25]

(or a system of differential equations), but one may also look for solutions of certain types, for instance, polynomial or rational ones, or expressible in terms of elementary functions. The existence of additional geometric structures allows us to introduce other concepts of integrability, and so the notion of integrability is often identified as complete integrability or Liouville integrability [A], but we can also consider generalised Liouville integrability or even non-Hamiltonian integrability [MF]. For a recent description of other related integrability approaches see e.g. [O, MCSL].

Once a definition of integrability is accepted, systems are classified into integrable and non-integrable systems. Groups of equivalence transformations allow us to do a finer classification, all systems in the same orbits having the same integrability properties. Therefore if some integrable cases have been previously selected, we will have a related family of integrable cases. So, even if the generic Riccati equation is *not* integrable by quadratures, all Riccati equations related to inhomogeneous linear differential equations are integrable by quadratures too, and this provides us integrability conditions for Riccati equations [CL1, CLR, CR].

The knowledge of particular solutions can also be useful for transforming the original system in simpler ones, and the prototypes of this situation are the so called Lie systems admitting a superposition rule for expressing their general solutions in terms of a generic set of a finite number of solutions [CGM1, CGM2, CGM3, CGR, CIMM, CL2, CR]. This is a report of a recent collaboration of Prof. Grabowski with the members of the Department of Theoretical Physics of Zaragoza University [CFGR] on a different concept of integrability, the most classical Lie concept of integrability by quadratures, i.e. all solutions can be found by algebraic operations (including inversion of functions) and computation of integrals of functions of one variable (called quadratures).

Our approach does not resort to the existence of additional compatible structures, but simply uses modern tools of algebra and geometry. In order to avoid dependence of a particular choice of coordinates we should consider the problem from a geometric perspective, replacing the systems of differential equations by vector fields, a global concept, in such a way that the integral curves of such vector fields are the solutions of a system of differential equations in a coordinate system. The two main tools to be used are finite-dimensional Lie algebras of vector fields, in particular solvable Lie algebras (see e.g. [AKN]) or nilpotent Lie algebras [MK, G], and distributions spanned by vector fields. The aim is to extend Lie classical results of integrability [AKN].

The paper is organised as follows: the fundamental notions on Lie integrabilty and their relations with the standard Arnold–Liouville integrability are recalled in Section 2 and some concepts of cohomology needed to analyse the existence of solutions for a system of first order differential equations are recalled in Section 3. The approach to integrability recently proposed in [CFGR] is sketched in Section 4 and some interesting algebraic properties are studied in Section 5. The approach is illustrated in Section 6 with the analysis, without any recourse to the symplectic structure, of a recent example of a Holt-related potential that is not separable but is superintegrable with high order first integrals, while the last sections are devoted to extending the previous results to the more general situation in which, instead of having a Lie algebra, L, of vector fields, we

have a vector space V such that its elements do not close a finite-dimensional real Lie algebra, but rather generate a general integrable distribution of vector fields.

2. Integrability by quadratures. Given an autonomous system of first order differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^N), \qquad i = 1, \ldots, N, \tag{1}$$

we can consider changes of coordinates and then the system (1) becomes a new one. This suggests that (1) can be geometrically interpreted in terms of a vector field Γ in a N-dimensional manifold M whose local expression in the given coordinates is

$$\Gamma = f^i(x^1, \ldots, x^N) \frac{\partial}{\partial x^i}.$$

The integral curves of Γ are the solutions of the given system, and then to integrate the system means to determine the general solution of the system. More specifically, integrability by quadratures means that you can determine the solutions (i.e. the flow of Γ) by means of a finite number of algebraic operations and quadratures of some functions.

There are two main techniques in the process of solving the system:

- Determination of constants of motion: Constants of motion provide us foliations such that Γ is tangent to the leaves of the foliation, and reducing in this way the problem to a family of lower-dimensional problems, one on each leaf.
- Search for symmetries of the vector field: The knowledge of infinitesimal one-parameter groups of symmetries of the vector field (i.e. of the system of differential equations), suggests us to use adapted local coordinates, the system decoupling then into lower-dimensional subsystems.

More specifically, the knowledge of r functionally independent (i.e. such that $dF_1 \wedge \ldots \wedge dF_r \neq 0$) constants of motion, $F_1, \ldots, F_r$, allows us to reduce the problem to that of a family of vector fields $\widetilde{\Gamma}_c$ defined in the $(N - r)$-dimensional submanifolds M_c given by the level sets of the vector function of rank r, $(F_1, \ldots, F_r) : M \to \mathbb{R}^r$. Of course the best situation is when $r = N - 1$: the leaves are one-dimensional, giving us the solutions to the problem, up to a reparametrisation.

There is another way of reducing the problem. Given an infinitesimal symmetry (i.e. a vector field X such that $[X, \Gamma] = 0$), then, according to the Straightening out Theorem [AM78, AM88, CP], in a neighbourhood of a point where X is different from zero we can choose adapted coordinates, $(y^1, \ldots, y^N)$, for which X is written as

$$X = \frac{\partial}{\partial y^N}.$$

Then, the symmetry condition $[X, \Gamma] = 0$ implies that Γ has the form

$$\Gamma = \bar{f}^1(y^1, \ldots, y^{N-1}) \frac{\partial}{\partial y^1} + \ldots + \bar{f}^{N-1}(y^1, \ldots, y^{N-1}) \frac{\partial}{\partial y^{N-1}} + \bar{f}^N(y^1, \ldots, y^{N-1}) \frac{\partial}{\partial y^N},$$

and its integral curves are obtained by solving the system of differential equations

$$\begin{cases} \dfrac{dy^i}{dt} = \bar{f}^i(y^1, \ldots, y^{N-1}), & i = 1, \ldots, N-1, \\[2mm] \dfrac{dy^N}{dt} = \bar{f}^N(y^1, \ldots, y^{N-1}). \end{cases}$$

We have reduced the problem to a subsystem involving only the first $N-1$ equations, and once this has been solved, the last equation is used to obtain the function $y^N(t)$ by means of one more quadrature.

Note that the new coordinates, $y^1, \ldots, y^{N-1}$, are such that $Xy^1 = \cdots = Xy^{N-1} = 0$, i.e. they are constants of the motion for X and therefore we cannot easily find such coordinates in a general case.

Moreover, the information provided by two different symmetry vector fields cannot be used simultaneously in the general case, because it is not possible to find local coordinates $(y^1, \ldots, y^N)$ such that

$$X_1 = \frac{\partial}{\partial y^{N-1}}, \qquad X_2 = \frac{\partial}{\partial y^N},$$

unless $[X_1, X_2] = 0$.

In terms of adapted coordinates for the dynamical vector field Γ, i.e. $\Gamma = \partial/\partial y^N$, the integration is immediate, the solution curves being given by

$$y^k(t) = y_0^k, \quad k = 1, \ldots, N-1, \qquad y^N(t) = y^N(0) + t.$$

This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because in these adapted coordinates the system is easily solved.

However, it will be proved that when Γ is part of a family of vector fields satisfying appropriate conditions, then it is integrable by quadratures for any choice of initial coordinates.

Both the constants of motion and infinitesimal symmetries can be used simultaneously if some compatibility conditions are satisfied. We can say that a system admitting $r < N - 1$ functionally independent constants of motion, $F_1, \ldots, F_r$, is integrable when we know furthermore s commuting infinitesimal symmetries $X_1, \ldots, X_s$, with $r + s = N$ such that

$$[X_a, X_b] = 0, \ a, b = 1, \ldots, s, \quad \text{and} \quad X_a F_\alpha = 0, \quad \forall a = 1, \ldots, s, \ \alpha = 1, \ldots r.$$

The constants of motion determine a s-dimensional foliation (with $s = N - r$) and the former condition means that the restriction of the s vector fields X_a to the leaves are tangent to such leaves.

Sometimes we have additional geometric structures that are compatible with the dynamics. For instance, a symplectic structure ω on a $2n$-dimensional manifold M. Such a 2-form relates, by contraction, in a one-to-one way, vector fields and 1-forms. The vector fields X_F associated with exact 1-forms dF are said to be Hamiltonian vector fields. We say that ω is compatible when the dynamical vector field itself is a Hamiltonian vector field X_H.

Particularly interesting is the Arnold–Liouville definition of (Abelian) complete integrability ($r = s = n$, with $N = 2n$) [A, AKN, VVK1, L]. The vector fields are $X_a = X_{F_a}$

and, for instance, $F_1 = H$.

The regular Poisson bracket defined by ω (i.e. $\{F_1, F_2\} = X_{F_2} F_1$), allows us to express the above tangency conditions as $X_{F_b} F_a = \{F_a, F_b\} = 0$—i.e. the n functions are constants of motion in involution and their corresponding Hamiltonian vector fields commute.

Our aim is to study integrability in absence of additional compatible structures, the main tool being properties of Lie algebras of vector fields containing the given vector field, very much in the approach started by Lie.

The problem of integrability by quadratures depends on the determination by quadratures of the necessary first integrals and on finding adapted coordinates, or, in other words, in finding a sufficient number of invariant tensors.

The set $\mathfrak{X}_\Gamma(M)$ of strict infinitesimal symmetries of $\Gamma \in \mathfrak{X}(M)$ is a linear space:

$$\mathfrak{X}_\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = 0\}.$$

The flows of vector fields $X \in \mathfrak{X}_\Gamma(M)$ preserve the set of integral curves of Γ.

The set of vector fields generating flows preserving the set of integral curves of Γ up to a reparametrisation is a real linear space containing $\mathfrak{X}_\Gamma(M)$ and will be denoted by

$$\mathfrak{X}^\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = f_X \, \Gamma\}, \quad f_X \in C^\infty(M).$$

The flows of vector fields in $\mathfrak{X}^\Gamma(M)$ preserve the one-dimensional distribution generated by Γ. Moreover, for any function $f \in C^\infty(M)$, $\mathfrak{X}^\Gamma(M) \subset \mathfrak{X}^{f\Gamma}(M)$, i.e. $\mathfrak{X}^\Gamma(M)$ only depends on the distribution generated by Γ and not on Γ itself.

One can check that $\mathfrak{X}^\Gamma(M)$ is a real Lie algebra and $\mathfrak{X}_\Gamma(M)$ is a Lie subalgebra of $\mathfrak{X}^\Gamma(M)$. However, $\mathfrak{X}_\Gamma(M)$ is not an ideal in $\mathfrak{X}^\Gamma(M)$.

As indicated above, finding constants of motion for Γ is not an easy task, at least in absence of a compatible symplectic structure. However, the explicit knowledge of first integrals of a given dynamical system has proved to be of great importance in the study of the qualitative properties of the system. The important point is that an appropriate set of infinitesimal symmetries of Γ can also provide constants of motion. More specifically, let $\{X_1, \ldots, X_d\}$ be a set of d vector fields taking linearly independent values in every point and which are infinitesimal symmetries of Γ. If they generate an involutive distribution, i.e. there exist functions $f_{ij}{}^k$ such that $[X_i, X_j] = f_{ij}{}^k X_k$, then, for each triple of numbers i, j, k the functions $f_{ij}{}^k$ are constants of the motion, i.e. $\Gamma(f_{ij}{}^k) = 0$. In fact, the Jacobi identity for the vector fields Γ, X_i, X_j, i.e.

$$\big[[\Gamma, X_i], X_j\big] + \big[[X_i, X_j], \Gamma\big] + \big[[X_j, \Gamma], X_i\big] = 0,$$

leads to

$$\big[[X_i, X_j], \Gamma\big] = 0 \implies [f_{ij}{}^k X_k, \Gamma] = -\Gamma(f_{ij}{}^k) \, X_k = 0.$$

Moreover, for any other index l, $X_l(f_{ij}{}^k)$ is also a constant of motion, because if X_l is a symmetry of Γ, then $\mathcal{L}_\Gamma\big(\mathcal{L}_{X_l}(f_{ij}{}^k)\big) = \mathcal{L}_{X_l}\big(\mathcal{L}_\Gamma(f_{ij}{}^k)\big) = 0$.

The constants of motion so obtained are not functionally independent but at least this proves the usefulness of finding these families of vector fields when looking for constants of motion. This points out the convenience of extending the theory from Lie algebras of symmetries to involutive distributions, as we will do in the final part of the paper.

3. Lie theorem of integrability by quadratures. The first important result is due to Lie who established the following theorem:

THEOREM 3.1. *If n vector fields, $X_1, \ldots, X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, generate a solvable Lie algebra and are such that $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then the differential equation $\dot{x} = X_1(x)$ is solvable by quadratures in U.*

We only prove the simplest case $n = 2$. The differential equation can be integrated if we are able to find a first integral F (i.e. $X_1 F = 0$), such that $dF \neq 0$ in U. The straightening out theorem [AM78, AM88, CP], says that such a function F locally exists. F implicitly defines one variable, for instance x_2, in terms of the other one by $F(x_1, \phi(x_1)) = k$.

If X_1 and X_2 are such that $[X_1, X_2] = \lambda_2 X_1$, and α_0 is a 1-form, defined up to multiplication by a function, such that $i(X_1)\alpha_0 = 0$, as X_2 is linearly independent of X_1 at each point, $i(X_2)\alpha_0 \neq 0$, and we can see that the 1-form $\alpha = (i(X_2)\alpha_0)^{-1}\alpha_0$ is such that $i(X_1)\alpha = 0$ and satisfies, by construction, the condition $i(X_2)\alpha = 1$. Such 1-form α is closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\alpha(X_1) = 0.$$

Therefore, there exists, at least locally, a function F such that $\alpha = dF$, and it is given by

$$F(x_1, x_2) = \int_\gamma \alpha,$$

where γ is any curve with end at the point (x_1, x_2). This is the function we were looking for, because $dF = \alpha$ and then

$$i(X_1)\alpha = 0 \Longleftrightarrow X_1 F = 0, \qquad i(X_2)\alpha = 1 \Longleftrightarrow X_2 F = 1.$$

We do not present here the proof for general n because it appears as a particular case of the more general situation we consider later on. The result of this theorem has been slightly generalized for adjoint-split solvable Lie algebras in [VVK2].

4. Recalling some basic concepts of cohomology. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a $\mathfrak{g}$-module, or in other words, $\mathfrak{a}$ is a linear space that is carrier space for a linear representation Ψ of $\mathfrak{g}$, i.e. $\Psi : \mathfrak{g} \to \operatorname{End} \mathfrak{a}$ satisfies

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a) = \Psi([a, b]), \quad \forall a, b \in \mathfrak{g}.$$

By a k-cochain we mean a k-linear alternating map $\alpha : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathfrak{a}$. If $C^k(\mathfrak{g}, \mathfrak{a})$ denotes the linear space of k-cochains, for each $k \in \mathbb{N}$ we define $\delta_k : C^k(\mathfrak{g}, \mathfrak{a}) \to C^{k+1}(\mathfrak{g}, \mathfrak{a})$ by (see e.g. [CE] and [CI] and references therein)

$$(\delta_k \alpha)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \Psi(a_i)\alpha(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j], a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{k+1}),$$

where $\widehat{a_i}$ denotes, as usual, that the element $a_i \in \mathfrak{g}$ is omitted.

The linear maps δ_k can be shown to satisfy $\delta_{k+1} \circ \delta_k = 0$, and consequently the linear operator δ on $C(\mathfrak{g}, \mathfrak{a}) = \bigoplus_{k=0}^{\infty} C^k(\mathfrak{g}, \mathfrak{a})$ whose restriction to each $C^k(\mathfrak{g}, \mathfrak{a})$ is δ_k,

satisfies $\delta^2 = 0$. We will then define

$$B^k(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in C^k(\mathfrak{g}, \mathfrak{a}) \mid \exists \beta \in C^{k-1}(\mathfrak{g}, \mathfrak{a}) \text{ such that } \alpha = \delta\beta\} = \text{Image } \delta_{k-1},$$
$$Z^k(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in C^k(\mathfrak{g}, \mathfrak{a}) \mid \delta\alpha = 0\} = \ker \delta_k.$$

The elements of $Z^k(\mathfrak{g}, \mathfrak{a})$ are called k-cocycles, and those of $B^k(\mathfrak{g}, \mathfrak{a})$ are called k-coboundaries. As δ is such that $\delta^2 = 0$, we see that $B^k(\mathfrak{g}, \mathfrak{a}) \subset Z^k(\mathfrak{g}, \mathfrak{a})$. The k-th cohomology group $H^k(\mathfrak{g}, \mathfrak{a})$ is

$$H^k(\mathfrak{g}, \mathfrak{a}) := \frac{Z^k(\mathfrak{g}, \mathfrak{a})}{B^k(\mathfrak{g}, \mathfrak{a})},$$

and we will define $B^0(\mathfrak{g}, \mathfrak{a}) = 0$, by convention.

We are interested in the case where $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of $\mathfrak{X}(M)$, $\mathfrak{a} = \bigwedge^p(M)$, and we consider the action of $\mathfrak{g}$ on $\mathfrak{a}$ given by $\Psi(X)\zeta = \mathcal{L}_X\zeta$. The case $p = 0$ has been used, for instance, in the study of weakly invariant differential equations as shown in [COW]. The cases $p = 1, 2$ are also interesting in mechanics [CI].

Coming back to the particular case $p = 0$, $\mathfrak{a} = \bigwedge^0(M) = C^\infty(M)$, $\mathfrak{g} = \mathfrak{X}(M)$, the elements of $Z^1(\mathfrak{g}, \bigwedge^0(M))$ are linear maps $h : \mathfrak{g} \to C^\infty(M)$ satisfying

$$(\delta_1 h)(X, Y) = \mathcal{L}_X h(Y) - \mathcal{L}_Y h(X) - h([X, Y]) = 0, \qquad X, Y \in \mathfrak{X}(M),$$

and those of $B^1(\mathfrak{g}, C^\infty(M))$ are linear maps $h : \mathfrak{g} \to C^\infty(M)$ for which $\exists g \in C^\infty(M)$ with

$$h(X) = \mathcal{L}_X g.$$

LEMMA. *Let $\{X_1, \ldots, X_n\}$ be a set of n vector fields whose values are linearly independent at each point of an n-dimensional manifold M. Then:*

1) *The necessary and sufficient condition for the system of equations for $f \in C^\infty(M)$*

$$X_i f = h_i, \qquad h_i \in C^\infty(M), \quad i = 1, \ldots, n,$$

to have a solution is that the 1-form $\alpha \in \bigwedge^1(M)$ such that $\alpha(X_i) = h_i$ be an exact 1-form.

2) *If the previous n vector fields generate an n-dimensional real Lie algebra $\mathfrak{g}$ (i.e. there exist real numbers $c_{ij}{}^k$ such that $[X_i, X_j] = c_{ij}{}^k X_k$), then the necessary condition for the system of equations to have a solution is that the $\mathbb{R}$-linear function $h : \mathfrak{g} \to C^\infty(M)$ defined by $h(X_i) = h_i$ is a 1-cochain that is a 1-cocycle.*

Proof. 1) For any pair of indices i, j, if $X_i f = h_i$ and $X_j f = h_j$, then, as $\exists f_{ij}{}^k \in C^\infty(M)$ such that $[X_i, X_j] = f_{ij}{}^k X_k$,

$$X_i(X_j f) - X_j(X_i f) = [X_i, X_j]f = f_{ij}{}^k X_k f \implies X_i(h_j) - X_j(h_i) - f_{ij}{}^k h_k = 0,$$

and as $\alpha(X_i) = h_i$, we infer that as

$$d\alpha(X_i, X_j) = X_i \alpha(X_j) - X_j \alpha(X_i) - \alpha([X_i, X_j]) = X_i(h_j) - X_j(h_i) - f_{ij}{}^k h_k,$$

the 1-form α is closed. Consequently, a necessary condition for the existence of the solution of the system is that α be closed.

2) Consider $\mathfrak{a} = C^\infty(M)$, $\mathfrak{g}$ the n-dimensional real Lie algebra generated by the vector fields X_i, and the cochain determined by the linear map $h : \mathfrak{g} \to C^\infty(M)$. Now the necessary condition for the existence of the solution is written as:

$$X_i(h_j) - X_j(h_i) - c_{ij}{}^k h_k = (\delta_1 h)(X_i, X_j) = 0.$$

This is just the 1-cocycle condition. ∎

Most properties of differential equations are of a local character: closed forms are locally exact and we can restrict ourselves to appropriate open subsets U of M, i.e. open submanifolds, where the closed 1-form is exact. Then if α is closed, it is locally exact, $\alpha = df$ in a certain open U, $f \in C^\infty(U)$, and the solution of the system can be found by one quadrature: the solution function f is given by the quadrature

$$f(x) = \int_{\gamma_x} \alpha,$$

where γ_x is any path joining some reference point $x_0 \in U$ to $x \in U$.

We also remark that α is exact, $\alpha = df$, if and only if $\alpha(X_i) = df(X_i) = X_i f = h_i$, i.e. h is a coboundary, $h = \delta f$.

In the particular case of the appearing functions h_i being constant, the condition for the existence of local solution reduces to $\alpha([X, Y]) = 0$ for each pair of elements X and Y in $\mathfrak{g}$, i.e. α vanishes on the derived Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. In particular, when $\mathfrak{g}$ is Abelian, there is not any condition.

5. A generalisation of Lie theory of integration.
Consider a family of N vector fields, $X_1, \ldots, X_N$, defined on an N-dimensional manifold M and assume that they close a Lie algebra L over the real numbers

$$[X_i, X_j] = c_{ij}{}^k X_k, \quad i, j, k = 1, \ldots, N,$$

and that, in addition, they span a basis of $T_x M$ at every point $x \in M$. We pick up an element in the family, X_1, the dynamical vector field. To emphasize its special role we will often denote it by $\Gamma \equiv X_1$.

Our goal is to obtain the integral curves $\Phi_t : M \to M$ of Γ

$$(\Gamma f)(\Phi_t(x)) = \frac{d}{dt} f(\Phi_t(x)), \quad \forall f \in C^\infty(x), \; x \in M,$$

by using quadratures (operations of integration, elimination and partial differentiation). The number of quadratures is given by the number of integrals of known functions depending on a finite number of parameters, that are performed. Γ plays a distinguished role, since it represents the dynamics to be integrated.

Our approach is concerned with the construction of a sequence of nested Lie subalgebras $L_{\Gamma,k}$ of the Lie algebra L, and it will be essential that Γ belongs to all these subalgebras. This construction, for which more details can be found in [CFGR], will be carried out in several steps.

The first one will be to reduce, by one quadrature, the original problem to a similar one but with a Lie subalgebra $L_{\Gamma,1}$ of the Lie algebra L (with $\Gamma \in L_{\Gamma,1}$) whose elements span at every point the tangent space of the leaves of a certain foliation.

If iterating the procedure we end up with an Abelian Lie algebra, we can, with one more quadrature, obtain the flow of the dynamical vector field.

We determine the foliation through a family of functions that are constant on the leaves. We first consider the ideal in L,

$$L_{\Gamma,1} = \langle \Gamma \rangle + [L, L], \quad \dim L_{\Gamma,1} = n_1,$$

that, in order to make the notation simpler, we will assume to be generated by the first n_1 vector fields of the family (i.e. $L_{\Gamma,1} = \langle \Gamma, X_2, \dots, X_{n_1} \rangle$). This can always be achieved by choosing appropriately the basis of L.

Now take ζ_1 in the annihilator of $L_{\Gamma,1}$, i.e. ζ_1 is in the set $L_{\Gamma,1}^0$ made up by the elements of L^* killing vectors of $L_{\Gamma,1}$, and define the 1-form α_{ζ_1} on M by its action on the vector fields in L:

$$\alpha_{\zeta_1}(X) = \zeta_1(X), \quad \text{for } X \in L.$$

As $\alpha_{\zeta_1}(X)$ is a constant function on M, for any vector field in L we have

$$d\alpha_{\zeta_1}(X, Y) = \alpha_{\zeta_1}([X, Y]) = \zeta_1([X, Y]) = 0, \quad \text{for } X, Y \in L, \ \zeta_1 \in L_{\Gamma,1}^0.$$

Therefore the 1-form α_{ζ_1} is closed and by application of the result of the lemma the system of partial differential equations

$$X_i Q_{\zeta_1} = \alpha_{\zeta_1}(X_i), \quad i = 1, \dots, n, \quad Q_{\zeta_1} \in C^\infty(M),$$

has a unique (up to the addition of a constant) local solution which can be obtained by one quadrature. Moreover, if we fix the same reference point x_0 for any ζ_1, then α_{ζ_1} depends linearly on ζ_1 and, if γ_x is independent of ζ_1, we see that the correspondence

$$L_{\Gamma,1}^0 \ni \zeta_1 \mapsto Q_{\zeta_1} \in C^\infty(M)$$

defines an injective linear map.

The system expresses that the vector fields in $L_{\Gamma,1}$ (including Γ) are tangent to

$$N_1^{[Y_1]} = \{x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1), \ \zeta_1 \in L_{\Gamma,1}^0\} \subset M$$

for any $[Y_1] \in L/L_{\Gamma,1}$. Locally, for an open neighbourhood U, the $N_1^{[Y_1]}$'s define a smooth foliation of n_1-dimensional leaves.

Now, we repeat the previous procedure by taking $L_{\Gamma,1}$ as the Lie algebra, and any leaf $N_1^{[Y_1]}$ as the manifold. The new subalgebra $L_{\Gamma,2} \subset L_{\Gamma,1}$ is defined by

$$L_{\Gamma,2} = \langle \Gamma \rangle + [L_{\Gamma,1}, L_{\Gamma,1}], \quad \dim L_{\Gamma,2} = n_2,$$

and taking $\zeta_2 \in L_{\Gamma,2}^0 \subset L_{\Gamma,1}^*$ (the annihilator of $L_{\Gamma,2}$), we arrive at a new system of partial differential equations

$$X_i Q_{\zeta_2}^{[Y_1]} = \zeta_2(X_i), \quad i = 1, \dots, n_1, \quad Q_{\zeta_2}^{[Y_1]} \in C^\infty(N_1^{[Y_1]}),$$

that can be solved with one quadrature and such $Q_{\zeta_2}^{[Y_1]}$ depends linearly on ζ_2.

It will be useful to extend $Q_{\zeta_2}^{[Y_1]}$ to U. We first introduce the map

$$U \ni x \mapsto [Y_1^x] \in L_{\Gamma,0}/L_{\Gamma,1},$$

where x and $[Y_1^x]$ are related by the equation $Q_{\zeta_1}(x) = \zeta_1(Y_1^x)$, that correctly determines the map. Now, we define $Q_{\zeta_2} \in C^\infty(U)$ by $Q_{\zeta_2}(x) = Q_{\zeta_2}^{[Y_1^x]}(x)$. Note that by construction

$x \in N_1^{[Y_1^x]}$ and therefore the definition makes sense. The resulting function $Q_{\zeta_2}(x)$ is smooth provided the reference point of the lemma changes smoothly from leaf to leaf.

The construction is then iterated by defining

$$N_2^{[Y_1][Y_2]} = \big\{ x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1), \ Q_{\zeta_2}(x) = \zeta_2(Y_2), \text{ with } \zeta_1 \in L_{\Gamma,1}^0, \zeta_2 \in L_{\Gamma,2}^0 \big\} \subset M,$$

for $[Y_1] \in L_{\Gamma,0}/L_{\Gamma,1}$ and $[Y_2] \in L_{\Gamma,1}/L_{\Gamma,2}$. Note that $L_{\Gamma,2}$ generates at every point the tangent space of $N_2^{[Y_1][Y_2]}$, therefore we can proceed as before.

The algorithm ends if after some steps, say k, the Lie algebra $L_{\Gamma,k} = \langle X_1, \ldots, X_{n_k} \rangle$, whose vector fields are tangent to the n_k-dimensional leaf $N_k^{[Y_1],\ldots,[Y_k]}$, is Abelian. In this moment the system of equations

$$X_i Q_{\zeta_k}^{[Y_1],\ldots,[Y_k]} = \zeta_k(X_i), \quad i = 1, \ldots, n_{k-1}, \quad Q_{\zeta_k}^{[Y_1],\ldots,[Y_k]} \in C^\infty(N_k^{[Y_1],\ldots,[Y_k]}),$$

can be solved locally by one more quadrature for any $\zeta_k \in L_{\Gamma,k}^*$.

Remark that, as the Lie algebra $L_{\Gamma,k}$ is Abelian, the integrability condition is always satisfied and we can take ζ_k in the whole of $L_{\Gamma,k}^*$ instead of $L_{\Gamma,k}^0$. Then, as before, we extend the solutions to U and call them Q_{ζ_k}.

With all these ingredients we can find the flow of Γ by performing only algebraic operations. In fact, consider the formal direct sum

$$\Xi = L_{\Gamma,1}^0 \oplus L_{\Gamma,2}^0 \oplus \cdots \oplus L_{\Gamma,k}^0 \oplus L_{\Gamma,k}^*,$$

that, as one can check, has dimension n.

The linear maps $L_{\Gamma,i}^0 \ni \zeta_i \mapsto Q_{\zeta_i} \in C^\infty(U)$ can be extended to Ξ so that to any $\xi \in \Xi$ we assign a $Q_\xi \in C^\infty(U)$. Now consider a basis

$$\{\xi_1, \ldots, \xi_n\} \subset \Xi.$$

The associated functions Q_{ξ_j}, $j = 1, \ldots, n$ are independent and satisfy

$$\Gamma Q_{\xi_j}(x) = \xi_j(\Gamma), \quad j = 1, 2, \ldots, n,$$

where it should be noticed that, as $\Gamma \in L_{\Gamma,l}$ for any $l = 0, \ldots, k$, the right hand side is well defined, and we see from here that in the coordinates given by the Q_{ξ_j}'s the vector field Γ has constant components and, then, it is trivially integrated

$$Q_{\xi_j}(\Phi_t(x)) = Q_{\xi_j}(x) + \xi_j(\Gamma)t.$$

Now, with algebraic operations, one can derive the flow $\Phi_t(x)$. Altogether we have performed $k + 1$ quadratures.

6. Algebraic properties. The previous procedure works if it reaches an end point (i.e. if there is a smallest nonnegative integer $k > 0$ such that

$$L_{\Gamma,k} = \langle \Gamma \rangle + [L_{\Gamma,k-1}, L_{\Gamma,k-1}],$$

is an Abelian algebra). In that case we say that (M, L, Γ) is Lie integrable of order $k + 1$.

The content of the previous section can, thus, be summarized in the following

PROPOSITION 6.1. *If (M, L, Γ) is Lie integrable of order r, then the integral curves of Γ can be obtained by r quadratures.*

We will discuss below some necessary and sufficient conditions for Lie integrability.

PROPOSITION 6.2. *If (M, L, Γ) is Lie integrable then L is solvable.*

Proof. Let $L_{(i)}$ be the elements of the derived series, $L_{(i+1)} = [L_{(i)}, L_{(i)}]$, $L_{(0)} = L$ (note that $L_{(i)} = L_{0,i}$). Then,

$$L_{(i)} \subset L_{\Gamma,i},$$

and if the system is Lie integrable (i.e. $L_{\Gamma,k}$ is Abelian for some k), then $L_{(k+1)} = 0$ and, therefore, L is solvable. ■

PROPOSITION 6.3. *If L is solvable and A is an Abelian ideal of L, then (M, L, Γ) is Lie integrable for any $\Gamma \in A$.*

Proof. Since A is an ideal containing Γ, we can show that

$$A + L_{\Gamma,i} = A + L_{(i)}.$$

We proceed again by induction: if the previous holds, then

$$A + L_{\Gamma,i+1} = A + [L_{\Gamma,i}, L_{\Gamma,i}] = A + [A + L_{\Gamma,i}, A + L_{\Gamma,i}]$$
$$= A + [A + L_{(i)}, A + L_{(i)}] = A + L_{(i+1)}.$$

Now L is solvable if some $L_{(k)} = 0$ and therefore $L_{\Gamma,k} \subset A$, i.e. it is Abelian and henceforth the system is Lie integrable. Note that the particular case $A = \langle \Gamma \rangle$ corresponds to the standard Lie theorem. ■

Nilpotent algebras of vector fields also play an interesting role in the integrability of vector fields.

PROPOSITION 6.4. *If L is nilpotent, (M, L, Γ) is Lie integrable for any $\Gamma \in L$.*

Proof. Let us consider the central series $L^{(i+1)} = [L, L^{(i)}]$ with $L^{(0)} = L$. Now, L nilpotent means that there is a k such that $L^{(k)} = 0$. It is easy to see, by induction, that $L_{\Gamma,i} \subset \langle \Gamma \rangle + L^{(i)}$ and therefore $L_{\Gamma,k} = \langle \Gamma \rangle$ is Abelian and the system is Lie integrable. ■

From the previous propositions, we can derive the following

COROLLARY 6.5. *Let (M, L, Γ) be Lie integrable of order r. Then*
(a) *If r_s is the minimum positive integer such that $L_{(r_s)} = 0$, then $r \geq r_s$.*
(b) *If L is nilpotent, r_n is the smallest natural number such that $L^{(r_n)} = 0$, $r \leq r_n$.*

7. An interesting example. We now analyse the particular case of a recently studied superintegrable system [CCR], where we dealt with an example of a potential that is not separable but is superintegrable with high order first integrals [PW], by studying limits of some potentials related to Holt potential [H]. Even if the system is Hamiltonian, that is, the dynamical vector field $\Gamma = X_H$ is obtained from a Hamiltonian function H by making use of a symplectic structure ω_0 defined in a cotangent bundle T^*Q, we deliberately forget this fact and analyse the situation by simply considering this system just as a dynamical system (without mentioning the existence of a symplectic structure) and focusing our attention on the Lie algebra structure of the symmetries.

Suppose that the dynamics is given by the vector field $\Gamma = X_1$ defined in $M = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, y, p_x, p_y) given by

$$\Gamma = X_1 = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - \frac{k_2}{y^{2/3}} \frac{\partial}{\partial p_x} + \frac{2}{3} \frac{k_2\, x + k_3}{y^{5/3}} \frac{\partial}{\partial p_y},$$

where k_2 and k_3 are arbitrary constants.

Consider in this case the following three vector fields:

$$X_2 = \left(6p_x^2 + 3p_y^2 + k_2 \frac{6x}{y^{2/3}} + k_3 \frac{6}{y^{2/3}} \right) \frac{\partial}{\partial x} + \left(6p_x p_y + 9k_2 y^{1/3} \right) \frac{\partial}{\partial y}$$

$$- k_2 \frac{6}{y^{2/3}} p_x \frac{\partial}{\partial p_x} + \left(4k_2 \frac{x}{y^{5/3}} - 3 \frac{1}{y^{2/3}} p_y \right) \frac{\partial}{\partial p_y},$$

$$X_3 = \left(4p_x^3 + 4p_x p_y^2 + \frac{8(k_2 x + k_3)}{y^{2/3}} p_x + 12k_2\, y^{1/3} p_y \right) \frac{\partial}{\partial x}$$

$$+ \left(4p_x^2 p_y + 12k_2\, y^{1/3} p_x \right) \frac{\partial}{\partial y} - 4k_2 \frac{1}{y^{2/3}} p_x^2 \frac{\partial}{\partial p_x}$$

$$+ \left(\frac{8}{3} \frac{k_2 x + k_3}{y^{5/3}} p_x^2 - 4k_2 \frac{1}{y^{2/3}} p_x p_y - 12k_2^2 \frac{1}{y^{1/3}} \right) \frac{\partial}{\partial p_y},$$

and

$$X_4 = \left(6p_x^5 + 12p_x^3 p_y^2 + 24 \frac{k_3 + k_2 x}{y^{2/3}} p_x^3 + 108k_2 y^{1/3} p_x^2 p_y + 324k_2^2 y^{2/3} p_x \right) \frac{\partial}{\partial x}$$

$$+ \left(6p_x^4 p_y + 36k_2 y^{1/3} p_x^3 \right) \frac{\partial}{\partial y} - 6 \left(\frac{k_2}{y^{2/3}} p_x^4 - 972k_2^3 \right) \frac{\partial}{\partial p_x}$$

$$+ \left(4 \frac{k_3 + k_2 x}{y^{5/3}} p_x^4 - 12 \frac{k_2}{y^{2/3}} - 108k_2^2 \frac{1}{y^{1/3}} p_x^2 \right) \frac{\partial}{\partial p_y}.$$

In order to apply the theory developed above, it suffices to compute the commutation relations among the fields:

$$[X_2, X_3] = 0, \qquad [X_2, X_4] = 1944k_2^3\, \Gamma, \qquad [X_3, X_4] = 432k_2^3 X_2 \qquad (2)$$

together with

$$[X_1, X_i] = 0, \quad i = 2, 3, 4. \qquad (3)$$

Therefore, Γ and the three vector fields X_2, X_3, X_4 generate a four-dimensional real Lie algebra L, whose center is generated by $\Gamma = X_1$. The derived algebra $L_{(1)} \subset L$ is two-dimensional and it is generated by X_1 and X_2, i.e. $L_{(1)}$ is Abelian. Finally, the second derived algebra $L_{(2)}$ reduces to the trivial algebra, because $L_{(1)}$ is Abelian. That is, $L_{(2)} = [L_{(1)}, L_{(1)}] = \{0\}$.

In summary, the Lie algebra L is solvable with solvability index $r_s = 2$. However, $L^{(2)} = [L, L_{(1)}]$ is not trivial but $L_{(1)}$ is the one-dimensional ideal in L generated by X_1, and this implies that the Lie algebra is nilpotent with $r_n = 3$.

According to the previous results, we can conclude that (M, L, Γ) is Lie integrable for any $\Gamma \in L$, but the order of integrability of the system depends on the choice of the dynamical field, because:

a) (M, L, Γ) is Lie integrable of order 2 (the minimum possible value) for $\Gamma = X_i$, $i = 1, 2, 3$, or any combination of them;

b) (M, L, Γ) is Lie integrable of order 3 (the maximum possible value according to the result of the corollary) for $\Gamma = X_4$ (or any combination in which the coefficient of X_4 does not vanish).

8. Distributional integrability. It is clear that the preceding construction is too rigid or restrictive, because there are simple examples which cannot be analysed in the framework considered here. For instance, the system in $\mathbb{R}^n$ with dynamical vector field

$$\Gamma = f(x)\partial_1 \Longleftrightarrow \dot{x}^1 = f(x), \ \dot{x}^2 = 0, \ \ldots, \ \dot{x}^n = 0, \tag{4}$$

can be easily solved by quadratures but the vector fields of the natural choice

$$L = \langle \Gamma, \partial_2, \ldots, \partial_n \rangle, \tag{5}$$

do not close on a real Lie algebra. Note however that if f is a never vanishing function the dynamical vector field Γ is conformally equivalent to $\partial/\partial x^1$. Moreover, we pointed out before that we can also consider non-strict symmetries of the dynamics which means that the set of solutions is preserved but with a reparametrisation of the integral curves. This suggests to extend the framework by considering $C^\infty(M)$-modules of vector fields instead of $\mathbb{R}$-linear spaces. The price to be paid is that we do not have Lie algebras of vector fields anymore. However the idea of the construction developed in our approach can be maintained as it was proved in [CFGR]. We quickly sketch the generalisation developed in [CFGR] and refer the interested reader to such paper.

First, for any subset $S \subset \mathfrak{X}(M)$, let $\mathcal{D}_S$ denote the $C^\infty(M)$-module generated by S:

$$\mathcal{D}_S = \left\{ \sum_i f^i X_i \in \mathfrak{X}(M) \mid f^i \in C^\infty(M), \ X_i \in S \right\}.$$

As $\mathcal{D}_S$ is the module of vector fields in the corresponding generalised distribution, we will also refer to $\mathcal{D}_S$ as a distribution.

We say that a real vector space, $V \subset \mathfrak{X}(M)$, is *regular* if V is isomorphic to its restriction, $V_p \subset T_pM$, at any point $p \in M$, and *completely regular* if it is regular and $V_p = T_pM$.

One basic definition is the following:

DEFINITION 8.1. Given a completely regular vector space, $V \subset \mathfrak{X}(M)$, and a subset, $S \subset \mathfrak{X}(M)$, we define the *core* of S in V, denoted by S_*, to be the *smallest* subspace of V such that $S \subset \mathcal{D}_{S_*}$.

One can prove that such a smallest subspace does exist: any subset of $\mathfrak{X}(M)$ has a core. This concept of core of a generalised distribution is essential to extend the strategy for integration by quadratures from the Lie algebra setting to that of the $C^\infty(M)$-module case.

First, in full analogy to the Lie integrability property, we introduce the concept of *distributional integrability.*

Let $V \in \mathfrak{X}(M)$ be a completely regular vector space and $\Gamma \in V$ a dynamical vector field. We introduce the following sequence: $V_{\Gamma,0} = V$ and

$$V_{\Gamma,m} = \langle \Gamma \rangle + [V_{\Gamma,m-1}, V_{\Gamma,m-1}]_*.$$

We always have $V_{\Gamma,m} \subset V_{\Gamma,m-1}$.

The sequence $V_{\Gamma,k}$ coincides with previously introduced $L_{\Gamma,k}$ when $V = L$ closes a real Lie algebra. In fact, one easily sees that in this case $[V_{\Gamma,m-1}, V_{\Gamma,m-1}]_* = [V_{\Gamma,m-1}, V_{\Gamma,m-1}]$. It will play a similar role in the more general case we are considering.

DEFINITION 8.2. We say that (M, V, Γ) is *distributionally integrable of order* $k+1$ if $V_{\Gamma,k}$ is the first Abelian (with respect to the commutator of vector fields) linear subspace in the decreasing sequence
$$V_{\Gamma,0} \supset V_{\Gamma,1} \supset V_{\Gamma,2} \supset \dots .$$

We can now state the main result of this section [CFGR].

THEOREM 8.3. *If* (M, V, Γ) *is distributionally integrable of order* r, *then the vector field* Γ *can be integrated by* r *quadratures.*

Two examples were used in [CFGR] to illustrate the theory. The first example is mentioned at the beginning of this section and explicitly given by (4), and then V is given by the right hand side of (5), i.e. $V = \langle \Gamma, \partial_2, \dots, \partial_n \rangle$. Then, we immediately see that $[\Gamma, \partial_i] \in \mathcal{D}_{\langle \Gamma \rangle}$ for any i, and therefore $V_1 = \langle \Gamma \rangle$, so the system of equations is solved with two quadratures.

As a second example (it requires n quadratures), we can consider
$$\Gamma = f(x)\big(\partial_1 + g^2(x^1)\partial_2 + \dots + g^{n-1}(x^1, \dots, x^{n-2})\partial_{n-1} + g^n(x^1, \dots, x^{n-1})\partial_n\big),$$
with $f(x) \neq 0$ everywhere and $V = \langle \Gamma, \partial_2, \dots, \partial_n \rangle$. In this case, $V_{\Gamma,1} = \langle \Gamma, \partial_3, \dots, \partial_n \rangle$, $V_{\Gamma,2} = \langle \Gamma, \partial_4, \dots, \partial_n \rangle$, and finally $V_{\Gamma,n-1} = \langle \Gamma \rangle$. This shows that the system is distributionally integrable and requires n quadratures for its solution.

Remark the appearance of a function f multiplying the dynamical vector field in the previous examples. This is, actually, the general situation as it was proved in [CFGR].

PROPOSITION 8.4. *Suppose that* (M, V, Γ), *with* $V = \langle \Gamma, X_2, \dots, X_n \rangle$, *is distributionally integrable of order* r. *Then, for any nowhere-vanishing* $f \in C^\infty(M)$, *the system* $(M, V', f\Gamma)$ *with* $V' = \langle f\Gamma, X_2, \dots, X_n \rangle$ *is distributionally integrable of order* $|r' - r| \leq 1$.

The conformally related vector fields Γ and $f\Gamma$ have the same constants of motion, and therefore the unparametrised orbits of both vector fields coincide [CIL, M]. In other words, as the integral curves of both are related by a time-reparametrisation, we can interpret the change of dynamical vector field from Γ to $f\Gamma$ as a local, position dependent, redefinition of time. Consequently, our formalism allows for such arbitrary changes of time, a property that it is true neither in the Arnold–Liouville nor in the the standard Lie theory of integration by quadratures.

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