

1 **ON WELL-POSEDNESS OF VECTOR-VALUED FRACTIONAL**
 2 **DIFFERENTIAL-DIFFERENCE EQUATIONS**

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ABSTRACT. We develop an operator-theoretical method for the analysis on well posedness of partial differential-difference equations that can be modeled in the form

$$(*) \begin{cases} \Delta^\alpha u(n) &= Au(n+2) + f(n, u(n)), & n \in \mathbb{N}_0, \quad 1 < \alpha \leq 2; \\ u(0) &= u_0; \\ u(1) &= u_1, \end{cases}$$

where A is a closed linear operator defined on a Banach space X . Our ideas are inspired on the Poisson distribution as a tool to sampling fractional differential operators into fractional differences. Using our abstract approach, we are able to show existence and uniqueness of solutions for the problem (*) on a distinguished class of weighted Lebesgue spaces of sequences, under mild conditions on sequences of strongly continuous families of bounded operators generated by A , and natural restrictions on the nonlinearity f . Finally we present some original examples to illustrate our results.

4 1. INTRODUCTION

5 Many physical signals, such as electrical voltages produced by a sound or image recording
 6 instrument or a measuring device, are essentially continuous-time signals. Computers and
 7 related devices operate on a discrete-time axis. Continuous-time signals that are to be pro-
 8 cessed by such devices therefore first need be converted to discrete-time signals. One way to
 9 do it is by sampling.

10 Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X , and
 11 let $u : \mathbb{R}_+ \rightarrow X$ be a function that represents a continuous-time signal. Motivated by
 12 previous studies of Cuesta, Lubich and Palencia [17, 18], Lizama introduced in [24] a method
 13 of sampling for the time-fractional abstract Cauchy problem $D_t^\alpha u(t) = Au(t)$, where D_t^α is
 14 the Riemann-Liouville differential operator of order $\alpha > 0$, by means of what he called the
 15 Poisson transformation, which is defined by:

$$(1.1) \quad u(n) := \mathcal{P}(u)(n) = \int_0^\infty p_n(t)u(t)dt, \quad n \in \mathbb{N}_0, \quad 0 < \alpha \leq 1,$$

16 where $p_n(t) = e^{-t} \frac{t^n}{n!}$ denotes the Poisson distribution. This transformation has remarkable
 17 properties, and some of them have been studied in [24]. A notable property related with the
 18 Riemann-Liouville differential operator of order $0 < \alpha < 1$, D_t^α , is the following:

$$(1.2) \quad \Delta^\alpha u(n) = \int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt, \quad n \in \mathbb{N}_0,$$

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where Δ^α is the operator defined for $m - 1 < \alpha < m$, $m \in \mathbb{N}$, by

$$\Delta^\alpha v(n) = \Delta^m \left(\sum_{j=0}^n k^{m-\alpha}(n-j)v(j) \right), \quad k^\beta(n) := \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)} \quad n \in \mathbb{N}_0, \quad \beta > 0,$$

1 and Δ^m is the m -th order forward difference operator. See [24] for the proof. Observe
 2 that the operator Δ^α is equivalent by translation with the fractional difference operator
 3 introduced by Atici and Eloe [9], which has been used recently by some authors in order
 4 to obtain several qualitative properties of fractional difference equations concerning stability
 5 [16, 34, 37]. However, these properties are given only in the finite dimensional case of X .
 6 Also, the kernel k^β appears in many mathematical areas of interest and has some interesting
 7 properties. Some of them have been recently discussed in [3]. It is worthwhile to mention
 8 that $k^\alpha = \mathcal{P}(g_\alpha)$ where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$.

9 Recently, some interesting applications of fractional difference equations have appeared in
 10 the literature. In [34] the authors proposed a class of linear fractional difference equations
 11 with discrete time delay and impulse effects. They obtained the exact solution via discrete
 12 Mittag-Leffler functions and they provided some results about stability and numerical exper-
 13 iments. Some recent applications to image encryption are revisited in [12]. In particular the
 14 stability of impulsive fractional difference equations and impulsive effects were introduced
 15 jointly with some numerical results which provided support to the analysis. Generalized dif-
 16 ferential optimization problems driven by the Caputo derivative were treated in [37]. Then
 17 existence of weak Carathéodory and a numerical approximation algorithm were also intro-
 18 duced, and some convergence theorems were established. Finally these results were verified
 19 by an algorithm in a nonlinear programming problem, see more details in [37].

20 In [24], it is used the procedure of sampling, by means of (1.1), in order to prove the
 21 existence of a unique solution to the initial value problem

$$(1.3) \quad \begin{cases} \Delta^\alpha u(n) &= Au(n+1) + f(n), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0 \in X, \end{cases}$$

22 where $0 < \alpha \leq 1$, and it is derived several sufficient conditions for stability in terms of A .
 23 For instance, when A is the generator of a C_0 -semigroup. However, the study of the case
 24 $1 < \alpha \leq 2$ was left as an open problem. We observe that in the scalar case, a characterization
 25 of stability has been recently proven (see [15, Theorem 3.1]). It is remarkable that such
 26 characterization coincides with the corresponding one, which appears in [24], when $A = \lambda I$
 27 where I denotes the identity operator.

28 In the paper [25], the author proved a characterization of maximal regularity for the model
 29 (1.3) on vector valued ℓ_p spaces of sequences for $0 < \alpha \leq 1$ and A being a bounded operator.
 30 The case $1 < \alpha \leq 2$ was settled in [27]. However, in both cases, a characterization for
 31 unbounded operators A is an untreated topic. This problem is important in view of recent
 32 advances in numerical methods for Partial Differential Equations [6, 22]. In the work [26],
 33 sufficient conditions for the well posedness of the corresponding semilinear problem (1.3) (i.e.
 34 with $f(n)$ replaced by $f(n, u(n))$) were proven but, again, the case $1 < \alpha \leq 2$ was left as an
 35 open problem. Note that more recently, a new method to establish the well-posedness in the
 36 time scale \mathbb{Z} was proposed in [28]. In such paper, the handling of the full range $\alpha > 0$ was
 37 done successfully.

38 The general aim of this paper is to continue in this new avenue of research. In particular,
 39 we are interested in the following specific problem:

1 (Q) Given $A : D(A) \subset X \rightarrow X$ a closed linear operator defined on a Banach space X , can
 2 we establish well-posedness for the linearized version of the abstract problem

$$(1.4) \quad \begin{cases} \Delta^\alpha u(n) &= Au(n+2) + f(n, u(n)), & n \in \mathbb{N}_0; \\ u(0) &= u_0 \in X; \\ u(1) &= u_1 \in X, \end{cases}$$

3 in case $1 < \alpha \leq 2$?. Can we obtain an explicit representation of the solution in terms of
 4 operator families when A is the generator of a C_0 -semigroup? Or a cosine/sine family?.

5 Initial studies on the model (1.4) when A is a complex or real valued matrix, have only
 6 appeared [11, 20]. From a numerical point of view, our analysis refers to schemes that are
 7 discretized only in time. The theory of time-discrete fractional equations is a promising tool
 8 for several biological and physical applications where the memory effect appears [10, 13].
 9 For example, an application to a model of growth of tumors has been analyzed by Atici and
 10 Sengül in [10]. Fractional differences do not only exhibit the advantages of memory effects,
 11 as does the continuous case, but they also involve fewer numerical computations. Recent the
 12 work of Wu, Baleanu and Xie [35] on fractional chaotic maps reveals this interest. See also
 13 the references therein. The study of the chaotic behaviour of the fractional discrete logistic
 14 map with delay was recently proposed in an interesting work by Wu and Baleanu [36]. Also,
 15 the rate of convergence in the approximation of discrete solutions to continuous solutions has
 16 a great importance (see [1, 12]).

The outline of this paper is as follows: In Section 2, we give some background on the
 definitions to be used, and provide some key properties. For example, we check that

$${}_C\Delta^\alpha u(n) = \Delta^\alpha u(n) - k^{2-\alpha}(n+1)[u(1) - 2u(0)] - k^{2-\alpha}(n+2)u(0), \quad n \in \mathbb{N}_0,$$

and

$$\Delta^\alpha(u * v)(n) = (\Delta^\alpha u * v)(n) + (u(1) - \alpha u(0))v(n+1) + u(0)v(n+2), \quad n \in \mathbb{N}_0,$$

17 for $1 < \alpha \leq 2$ and suitable sequences u and v (Theorems 2.5 and 2.6).

In Section 3, we use successfully the preceding definitions and properties to solve the
 problem (1.4), firstly, in the homogeneous linear case. In order to do that, we construct a
 distinguished sequence of bounded and linear operators $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ that solves the homo-
 geneous linear initial value problem

$$\begin{cases} \Delta^\alpha u(n) &= Au(n+2), & n \in \mathbb{N}_0, & 1 < \alpha \leq 2; \\ u(0) &= u_0 \in D(A); \\ u(1) &= u_1 \in D(A), \end{cases}$$

18 see Theorem 3.5. This sequence of bounded and linear operators, is defined axiomatically by
 19 means of the following two properties:

- 20 (i) $\mathcal{S}_\alpha(n)Ax = A\mathcal{S}_\alpha(n)x$ for $n \in \mathbb{N}_0$ and $x \in D(A)$;
 21 (ii) $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

In particular, when the operator A is bounded with $\|A\| < 1$ we derive an explicit represen-
 tation of the solution, namely

$$\mathcal{S}_\alpha(n) = \sum_{j=0}^{\infty} k^{\alpha(j+1)}(n)A^j, \quad n \in \mathbb{N}_0,$$

22 (see Proposition 3.2). From a different point of view, this representation can be considered as
 23 the discrete counterpart of the Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ (when A is a complex
 24 number) which interpolates between the exponential and hyperbolic sine function for $1 <$
 25 $\alpha < 2$. In contrast, when A is unbounded, we give an interesting characterization of the

1 sequence $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ in terms of a series of powers of the bounded operator $(I - A)^{-1}$. See
 2 Theorem 3.3.

In Section 4 we study the fully nonlinear problem (1.4). After to introduce the notion of solution, which is motivated by the representation of the solution in the non-homogeneous linear case (Corollary 3.6), we consider a distinguished class of vector-valued spaces of weighted sequences, that behaves like

$$\ell_w^\infty(\mathbb{N}_0; X) := \left\{ \xi : \mathbb{N}_0 \rightarrow X \quad / \quad \sup_{n \in \mathbb{N}_0} \frac{\|\xi(n)\|}{nn!} < \infty \right\}.$$

3 This vector-valued Banach spaces of sequences will play a central role in the development of
 4 this section. The main ingredient for the success of our analysis is the observation that the
 5 special weight $w(n) = nn!$, that represents the factorial representation of a positive integer,
 6 proves to be suitable to find existence and uniqueness of solutions for (1.4) in the above defined
 7 space $\ell_w^\infty(\mathbb{N}_0; X)$ under the hypothesis of only boundedness of the sequence of operators
 8 $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$. We give two positive results in this direction. The first, requires a Lipschitz
 9 type condition on the nonlinear term f and uses the Banach fixed point theorem as main
 10 tool. The second, replaces the Lipschitz type condition by compactness of a distinguished
 11 set of points that depends on the sequence $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ and f . In this case, we used the
 12 Leray-Schauder alternative theorem. See Theorems 4.4 and 4.6, respectively.

In Section 5, we study the Poisson transformation. In order to do this, we first study in detail the Poisson distribution from a functional-analytical point of view (Proposition 5.1). Then, we prove several relations between the continuous and discrete setting, including a generalization of the notable identity (1.2):

$$\mathcal{P}(D_t^\alpha u)(n + m) = \int_0^\infty p_{n+m}(t) D_t^\alpha u(t) dt = \Delta^\alpha \mathcal{P}(u)(n), \quad n \in \mathbb{N}_0,$$

13 where $m - 1 < \alpha \leq m$. See Theorem 5.5 below. This relations are obtained in the context
 14 of the Poisson transformation (1.1) whose main properties are established in Theorem 5.2,
 15 extending the pioneer results in [24]. Note that the idea of discretization of the fractional
 16 derivative in time was employed in the paper [19] (see also [17] and references therein).
 17 We finish this section with concrete examples of Poisson transforms for some well known
 18 functions.

Section 6 is devoted to the construction of sequences of operators $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ via subordination by the Poisson transformation of α -resolvent families $(S_\alpha(t))_{t>0}$ generated by A , i.e.,

$$\mathcal{S}_\alpha(n) := \mathcal{P}(S_\alpha)(n), \quad n \in \mathbb{N}_0,$$

19 (Theorem 6.3). Then, two interesting examples are provided. The first, corresponds to
 20 the discrete counterpart of the Mittag-Leffler function $t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha)$ and, the second, cor-
 21 responds to a sequence of operators $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ that originates from the generator of a
 22 bounded sine family (Examples 6.4 and 6.5, respectively).

23 A remarkable consequence is Theorem 6.6, which proves existence and uniqueness of so-
 24 lution for the nonlinear problem (1.4) in the space $\ell_w^\infty(\mathbb{N}_0; X)$ under the hypothesis that A
 25 is the generator of a bounded sine family such that the resolvent operator $(\lambda - A)^{-1}$ is a
 26 compact operator for some λ large enough.

Finally, Section 7 provides us with several examples and applications of our general theorems, notably concerning the cases where either A is a multiplication operator, $Af(x) = m(x)f(x)$ defined on $L^2(a, b)$ and m a continuous function, see example 7.1, or the second order partial differential operator $\frac{\partial^2}{\partial x^2}$, see example 7.2. We also pay special attention to the

case $\alpha = 2$ and to some related problems formatted in a slightly different way than (1.4). In particular, we consider difference equations in the form

$$\begin{cases} \Delta^2 u(n) &= Bu(n+1) + g(n, u(n)), & n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1, \end{cases}$$

- 1 where B is a linear operator defined on a Banach space X , see Propositions 7.3 and 7.4.
2

Notation We denote by $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, the set of non-negative integer numbers and X a complex Banach space. We denote by $s(\mathbb{N}_0; X)$ the vectorial space consisting of all vector-valued sequences $u : \mathbb{N}_0 \rightarrow X$. We recall that the Z -transform of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\tilde{u}(z) := \sum_{j=0}^{\infty} z^{-j} u(j)$$

- 3 where z is a complex number. Note that convergence of the series is given for $|z| > R$ with
4 R sufficiently large.

Recall that the finite convolution $*$ of two sequences $u \in s(\mathbb{N}_0; \mathbb{C})$ and $v \in s(\mathbb{N}_0; X)$ is defined by

$$(u * v)(n) := \sum_{j=0}^n u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

- 5 It is well known that

$$(1.5) \quad \widetilde{(u * v)}(z) = \tilde{u}(z)\tilde{v}(z), \quad |z| > \max\{R_1, R_2\},$$

where R_1 and R_2 are the radius of convergence of the Z -transforms of u and v respectively. The Banach space $\ell^1(\mathbb{N}_0; X)$ is the subset of $s(\mathbb{N}_0; X)$ such that $\|u\|_1 := \sum_{n=0}^{\infty} \|u(n)\| < \infty$; and the Lebesgue space $L^1(\mathbb{R}_+; X)$ is formed by measurable functions $f : \mathbb{R}_+ \rightarrow X$ such that

$$\|f\|_1 := \int_0^{\infty} \|f(t)\| dt < \infty.$$

The usual Laplace transform is given by

$$\hat{f}(\lambda) := \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \Re \lambda > 0, \quad f \in L^1(\mathbb{R}_+; X).$$

In the case $X = \mathbb{C}$, the Banach space $L^1(\mathbb{R}_+)$ is, in fact, a Banach algebra with the usual convolution product $*$ given by

$$f * g(t) := \int_0^t f(t-s)g(s) ds, \quad t \geq 0, \quad f, g \in L^1(\mathbb{R}_+).$$

- 6 The same holds in the case of $(\ell^1, *)$. The Banach space $C^{(m)}(\mathbb{R}_+; X)$ is formed for continuous
7 functions which have m -continuous derivatives defined on \mathbb{R}_+ with $m \in \mathbb{N}_0$.

Let $S : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ be strongly continuous, that is, for all $x \in X$ the map $t \rightarrow S(t)x$ is continuous on \mathbb{R}_+ . We say that a family of bounded and linear operators $\{S(t)\}_{t \geq 0}$ is exponentially bounded if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq M e^{\omega t}, \quad t \geq 0.$$

- 8 We say that $\{S(t)\}_{t \geq 0}$ is bounded if $\omega = 0$. Note that if $\{S(t)\}_{t \geq 0}$ is exponentially bounded
9 then the Laplace transform $\hat{S}(\lambda)x$ exists for all $\Re(\lambda) > \omega$.

1

2. FRACTIONAL DIFFERENCE OPERATORS

The forward Euler operator $\Delta : s(\mathbb{N}_0; X) \rightarrow s(\mathbb{N}_0; X)$ is defined by

$$\Delta u(n) := u(n+1) - u(n), \quad n \in \mathbb{N}_0.$$

For $m \in \mathbb{N}$, we define recursively $\Delta^m : s(\mathbb{N}_0; X) \rightarrow s(\mathbb{N}_0; X)$ by $\Delta^1 = \Delta$ and

$$\Delta^m := \Delta^{m-1} \circ \Delta.$$

2 The operator Δ^m is called the m -th order forward difference operator and

$$(2.1) \quad \Delta^m u(n) = \sum_{j=0}^m \binom{m}{j} (-1)^{m-j} u(n+j), \quad n \in \mathbb{N}_0,$$

3 for $u \in s(\mathbb{N}_0; X)$. We also denote by $\Delta^0 = I$, where I is the identity operator.

4 We define

$$(2.2) \quad k^\alpha(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0, \quad \alpha > 0.$$

5 This sequence, introduced in [24], has appeared in [2], [25] among others, in connection with
6 fractional difference operators. The semigroup property $k^\alpha * k^\beta = k^{\alpha+\beta}$ and the generating
7 formula

$$(2.3) \quad \sum_{n=0}^{\infty} k^\alpha(n) z^n = \frac{1}{(1-z)^\alpha}, \quad |z| < 1,$$

8 hold for $\alpha, \beta > 0$, see for example [38, Vol. I, p.77].9 The following definition of fractional sum (also called Cesàro sum in [38]) has appeared
10 recently in some papers, see for example [2, 24, 25]. It has proven to be useful in the
11 treatment of fractional difference equations. Note that this definition is implicitly included
12 in e.g. [5, 9, 29].13 **Definition 2.1.** [25, Definition 2.5] Let $\alpha > 0$. The α -th fractional sum of a sequence
14 $u : \mathbb{N}_0 \rightarrow X$ is defined as follows

$$(2.4) \quad \Delta^{-\alpha} u(n) := \sum_{j=0}^n k^\alpha(n-j) u(j) = (k^\alpha * u)(n), \quad n \in \mathbb{N}_0.$$

15 One of the reasons to choose this operator in this paper is because their flexibility to be
16 handled by means of Z -transform methods. Moreover, it has a better behavior for mathe-
17 matical analysis when we ask, for example, for definitions of fractional sums and differences
18 on subspaces of $s(\mathbb{N}_0; X)$ like e.g. ℓ_p spaces. We notice that, recently, this approach by means
19 of the Z -transform has been followed by other authors, see [15, 16].20 The next concept is analogous to the definition of a fractional derivative in the sense
21 of Riemann-Liouville, see [8, 29]. In other words, to a given vector-valued sequence, first
22 fractional summation and then integer difference are applied.

Definition 2.2. [25, Definition 2.7] Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The fractional difference operator of
order α in the sense of Riemann-Liouville, $\Delta^\alpha : s(\mathbb{N}_0; X) \rightarrow s(\mathbb{N}_0; X)$, is defined by

$$\Delta^\alpha u(n) := \Delta^m (\Delta^{-(m-\alpha)} u)(n), \quad n \in \mathbb{N}_0,$$

23 where $m-1 < \alpha < m$.

Example 2.3. Let $1 < \beta$. Then

$$\Delta k^\beta(n) = \frac{\Gamma(\beta + n + 1)}{\Gamma(\beta)\Gamma(n + 2)} - \frac{\Gamma(\beta + n)}{\Gamma(\beta)\Gamma(n + 1)} = \frac{(\beta - 1)\Gamma(\beta + n)}{\Gamma(\beta)\Gamma(n + 2)} = k^{\beta-1}(n + 1), \quad n \in \mathbb{N}_0.$$

We iterate m -times with $m \in \mathbb{N}$ to get for $\beta > m$ that

$$\Delta^m k^\beta(n) = k^{\beta-m}(n + m), \quad n \in \mathbb{N}_0.$$

Let $0 < \alpha < \beta$ and $m - 1 < \alpha < m$ for $m \in \mathbb{N}$. By Definition 2.2 and (2.1), we get that

$$\Delta^\alpha k^\beta(n) = \Delta^m(\Delta^{-(m-\alpha)}k^\beta)(n) = \Delta^m(k^{m-\alpha} * k^\beta)(n) = \Delta^m(k^{m-\alpha+\beta}) = k^{\beta-\alpha}(n + m),$$

1 for $n \in \mathbb{N}_0$. This equality extends [24, Corollary 3.6] given for $0 < \alpha < 1$.

2 Interchanging the order of the operators in the definition of fractional difference in the
3 sense of Riemann-Liouville, and in analogous way as above, we can introduce the notion of
4 fractional difference in the sense of Caputo as follows.

Definition 2.4. [25, Definition 2.8] Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The fractional difference operator of order α in the sense of Caputo, ${}_C\Delta^\alpha : s(\mathbb{N}_0; X) \rightarrow s(\mathbb{N}_0; X)$, is defined by

$${}_C\Delta^\alpha u(n) := \Delta^{-(m-\alpha)}(\Delta^m u)(n), \quad n \in \mathbb{N}_0,$$

5 where $m - 1 < \alpha < m$.

6 For further use, we note the following relation between the Caputo and Riemann-Liouville
7 fractional differences of order $1 < \alpha < 2$. The connection between the Caputo and Riemann-
8 Liouville fractional differences of order $0 < \alpha < 1$ is given in [24, Theorem 2.4].

Theorem 2.5. For each $1 < \alpha < 2$ and $u \in s(\mathbb{N}_0; X)$, we have

$${}_C\Delta^\alpha u(n) = \Delta^\alpha u(n) - k^{2-\alpha}(n + 1)[u(1) - 2u(0)] - k^{2-\alpha}(n + 2)u(0), \quad n \in \mathbb{N}_0.$$

Proof. By definition and (2.4) we have

$$\begin{aligned} \Delta^{-(2-\alpha)}(\Delta^2 u)(n) &= \sum_{j=0}^n k^{2-\alpha}(n-j)\Delta^2 u(j) = \sum_{j=0}^n k^{2-\alpha}(n-j)u(j+2) \\ &\quad - 2 \sum_{j=0}^n k^{2-\alpha}(n-j)u(j+1) + \sum_{j=0}^n k^{2-\alpha}(n-j)u(j) \\ &= \sum_{j=2}^{n+2} k^{2-\alpha}(n+2-j)u(j) - 2 \sum_{j=1}^{n+1} k^{2-\alpha}(n+1-j)u(j) \\ &\quad + \sum_{j=0}^n k^{2-\alpha}(n-j)u(j) \\ &= \sum_{j=0}^{n+2} k^{2-\alpha}(n+2-j)u(j) \\ &\quad - 2 \sum_{j=0}^{n+1} k^{2-\alpha}(n+1-j)u(j) + \sum_{j=0}^n k^{2-\alpha}(n-j)u(j) \\ &\quad - k^{2-\alpha}(n+2)u(0) - k^{2-\alpha}(n+1)u(1) + 2k^{2-\alpha}(n+1)u(0) \\ &= \Delta^2(\Delta^{-(2-\alpha)}u)(n) - k^{2-\alpha}(n+1)(u(1) - 2u(0)) - k^{2-\alpha}(n+2)u(0), \end{aligned}$$

9 and so we obtain the desired result. \square

1 We also have the following property for the Riemann-Liouville fractional difference of the
2 convolution.

Theorem 2.6. *Let $1 < \alpha \leq 2$, $u \in s(\mathbb{N}_0; \mathbb{C})$ and $v \in s(\mathbb{N}_0; X)$. Then, for each $n \in \mathbb{N}_0$ the following identity holds*

$$\Delta^\alpha(u * v)(n) = (\Delta^\alpha u * v)(n) + (u(1) - \alpha u(0))v(n+1) + u(0)v(n+2).$$

3 *Proof.* For each $n \in \mathbb{N}_0$,

$$\begin{aligned} \Delta^\alpha(u * v)(n) &= \Delta^{-(2-\alpha)}(u * v)(n+2) - 2\Delta^{-(2-\alpha)}(u * v)(n+1) + \Delta^{-(2-\alpha)}(u * v)(n) \\ &= \sum_{j=0}^{n+2} (k^{2-\alpha} * u)(n+2-j)v(j) - 2 \sum_{j=0}^{n+1} (k^{2-\alpha} * u)(n+1-j)v(j) \\ &\quad + \sum_{j=0}^n (k^{2-\alpha} * u)(n-j)v(j) \\ &= \sum_{j=0}^n (k^{2-\alpha} * u)(n+2-j)v(j) - 2 \sum_{j=0}^n (k^{2-\alpha} * u)(n+1-j)v(j) \\ &\quad + \sum_{j=0}^n (k^{2-\alpha} * u)(n-j)v(j) + (k^{2-\alpha} * u)(1)v(n+1) \\ &\quad + (k^{2-\alpha} * u)(0)v(n+2) - 2(k^{2-\alpha} * u)(0)v(n+1) \\ &= \sum_{j=0}^n \Delta^2(k^{2-\alpha} * u)(n-j)v(j) + (k^{2-\alpha}(0)u(1) + k^{2-\alpha}(1)u(0))v(n+1) \\ &\quad + (k^{2-\alpha}(0)u(0))v(n+2) - 2(k^{2-\alpha}(0)u(0))v(n+1) \\ &= \sum_{j=0}^n \Delta^\alpha u(n-j)v(j) + (u(1) + (2-\alpha)u(0))v(n+1) + u(0)v(n+2) \\ &\quad - 2u(0)v(n+1) \\ &= (\Delta^\alpha u * v)(n) + (u(1) - \alpha u(0))v(n+1) + u(0)v(n+2), \end{aligned}$$

4 proving the claim. □

We notice that for $0 < \alpha \leq 1$ the above property reads

$$\Delta^\alpha(u * v)(n) = (\Delta^\alpha u * v)(n) + u(0)v(n+1), \quad n \in \mathbb{N}_0.$$

5 It has been proved only recently in [25, Lemma 3.6].

6 3. LINEAR FRACTIONAL DIFFERENCE EQUATIONS ON BANACH SPACES

7 Let A be a closed linear operator defined on a Banach space X . In this section we study
8 the problem

$$(3.1) \quad \begin{cases} \Delta^\alpha u(n) = Au(n+2) + f(n), & n \in \mathbb{N}_0, \quad 1 < \alpha \leq 2; \\ u(0) = u_0; \\ u(1) = u_1. \end{cases}$$

9 Following [24], we say that a vector valued sequence $u \in s(\mathbb{N}_0; X)$ is a solution of (3.1) if
10 $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$ and u satisfies (3.1).

11 We will use the notion of discrete α -resolvent family introduced in [2, Definition 3.1] to
12 obtain the solution of the problem (3.1). Note that the knowledge of the abstract properties of

1 this family of bounded operators provide insights on the qualitative behavior of the solutions
2 of fractional difference equations.

3 **Definition 3.1.** Let $\alpha > 0$ and A be a closed linear operator with domain $D(A)$ defined
4 on a Banach space X . An operator-valued sequence $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete
5 α -resolvent family generated by A if it satisfies the following conditions

- 6 (i) $\mathcal{S}_\alpha(n)Ax = A\mathcal{S}_\alpha(n)x$ for $n \in \mathbb{N}_0$ and $x \in D(A)$;
7 (ii) $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.

8 The family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ is said bounded if $\|\mathcal{S}\|_\infty := \sup_{n \in \mathbb{N}_0} \|\mathcal{S}_\alpha(n)\| < \infty$.

9 An explicit representation of a discrete α -resolvent family generated by a bounded operator
10 A with $\|A\| < 1$ is given in the following proposition.

Proposition 3.2. Let $\alpha > 0$ and $A \in \mathcal{B}(X)$, with $\|A\| < 1$. Then the operator A generates
a discrete α -resolvent family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ given by

$$\mathcal{S}_\alpha(n) = \sum_{j=0}^{\infty} k^{\alpha(j+1)}(n)A^j, \quad n \in \mathbb{N}_0.$$

11 *Proof.* Since $k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(\frac{1}{n}))$, for $n \in \mathbb{N}$ (see for example [38, Vol. I, (1.18)]), then
12 the series is convergent for $\|A\| < 1$. Take $x \in X$ and $n \in \mathbb{N}_0$. Then we get that

$$\begin{aligned} A(k^\alpha * \mathcal{S}_\alpha)(n)x &= A \sum_{j=0}^n k^\alpha(n-j)\mathcal{S}_\alpha(j)x = A \sum_{j=0}^n k^\alpha(n-j) \sum_{i=0}^{\infty} k^{\alpha(i+1)}(j)A^i x \\ &= \sum_{i=0}^{\infty} A^{i+1}x \sum_{j=0}^n k^\alpha(n-j)k^{\alpha(i+1)}(j) = \sum_{i=0}^{\infty} k^{\alpha(i+2)}(n)A^{i+1}x, \end{aligned}$$

where we have applied the semigroup property of the kernel k^α . Then we obtain

$$k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x = \sum_{i=0}^{\infty} k^{\alpha(i+1)}(n)A^i x = \mathcal{S}_\alpha(n)x, \quad n \in \mathbb{N}_0,$$

13 and we conclude the proof. □

14 For $\alpha > 0$ fixed and each $n \in \mathbb{N}$ the sequence $\{\beta_{\alpha,n}(j)\}_{j=1,\dots,n}$ was introduced in [2, Section
15 3.1] as follows:

For $n = 1$,

$$\beta_{\alpha,1}(1) = k^\alpha(1) = \alpha.$$

16 For $n = 2$,

$$\begin{aligned} \beta_{\alpha,2}(1) &= k^\alpha(2) - k^\alpha(1)\beta_{\alpha,1}(1) = k^\alpha(2) - (k^\alpha(1))^2, \\ \beta_{\alpha,2}(2) &= k^\alpha(1)\beta_{\alpha,1}(1) = (k^\alpha(1))^2 = \alpha^2. \end{aligned}$$

17 For $n = 3$,

$$\begin{aligned} \beta_{\alpha,3}(1) &= k^\alpha(3) - k^\alpha(2)\beta_{\alpha,1}(1) - k^\alpha(1)\beta_{\alpha,2}(1) = k^\alpha(3) - 2k^\alpha(2)k^\alpha(1) + (k^\alpha(1))^3, \\ \beta_{\alpha,3}(2) &= k^\alpha(2)\beta_{\alpha,1}(1) + k^\alpha(1)\beta_{\alpha,2}(1) - k^\alpha(1)\beta_{\alpha,2}(2) = 2k^\alpha(2)k^\alpha(1) - 2(k^\alpha(1))^3, \\ \beta_{\alpha,3}(3) &= k^\alpha(1)\beta_{\alpha,2}(2) = (k^\alpha(1))^3 = \alpha^3. \end{aligned}$$

18 For $n \geq 4$,

$$\beta_{\alpha,n}(1) = k^\alpha(n) - \sum_{j=1}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(1),$$

$$\begin{aligned}\beta_{\alpha,n}(l) &= \sum_{j=l-1}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(l-1) - \sum_{j=l}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(l) \quad \text{for } 2 \leq l \leq n-1, \\ \beta_{\alpha,n}(n) &= k^\alpha(1)\beta_{\alpha,n-1}(n-1) = (k^\alpha(1))^n = \alpha^n\end{aligned}$$

In case that A is closed, but not necessarily bounded, the authors in [2, Theorem 3.2] proven that given $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ a discrete α -resolvent family generated by A , then $1 \in \rho(A)$ and $\mathcal{S}_\alpha(0) = (I - A)^{-1}$; $\mathcal{S}_\alpha(0)x \in D(A)$ and $\mathcal{S}_\alpha(n)x \in D(A^2)$ for all $n \in \mathbb{N}$, and $x \in X$; and

$$\mathcal{S}_\alpha(n)x = \sum_{j=1}^n \beta_{\alpha,n}(j)(I - A)^{-(j+1)}x, \quad n \in \mathbb{N}, \quad x \in X.$$

1 The last equality provides an explicit representation of discrete α -resolvent families in terms
2 of a bounded linear operators which is, in fact, a characterization of this family of operators
3 as the next theorem shows.

4 **Theorem 3.3.** *Let $\lambda, \alpha > 0$, $(A, D(A))$ be a closed operator on the Banach space X and*
5 *$\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ be a sequence of bounded operators. Then the following conditions are*
6 *equivalent.*

- 7 (i) *The family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ is a discrete α -resolvent family generated by A .*
8 (ii) *$1 \in \rho(A)$, the operator $\mathcal{S}_\alpha(0) = (I - A)^{-1}$ and*

$$\mathcal{S}_\alpha(n)x = \sum_{j=1}^n \beta_{\alpha,n}(j)(I - A)^{-(j+1)}x, \quad n \in \mathbb{N}, \quad x \in X.$$

8 *If there exists $\lambda_0 > 0$ such that $\sup_{n \in \mathbb{N}_0} \lambda_0^{-n} \|\mathcal{S}_\alpha(n)\| < \infty$, both equations are equivalent to*

- 9 (iii) *$\left(\frac{\lambda - 1}{\lambda}\right)^\alpha \in \rho(A)$ and*

$$(3.2) \quad \left(\left(\frac{\lambda - 1}{\lambda} \right)^\alpha - A \right)^{-1} x = \sum_{n=0}^{\infty} \lambda^{-n} \mathcal{S}_\alpha(n)x, \quad x \in X, |\lambda| > \max\{\lambda_0, 1\}.$$

10 *Proof.* The condition (i) implies the condition (ii) is given in [2, Theorem 3.2]. Now we
11 suppose that the condition (ii) holds. Then $\mathcal{S}_\alpha(n)x \in D(A)$ for any $x \in X$ and $n \in \mathbb{N}_0$. For
12 $n \in \mathbb{N}$ and $x \in X$ we have that

$$\begin{aligned}(I - A)\mathcal{S}_\alpha(n)x &= \sum_{j=1}^n \beta_{\alpha,n}(j)(I - A)^{-j}x = (k^\alpha(n) - \sum_{i=1}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(1))(I - A)^{-1}x \\ &\quad + \sum_{j=2}^{n-1} \left(\sum_{i=j-1}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j-1) - \sum_{i=j}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j) \right) (I - A)^{-j}x \\ &\quad + k^\alpha(1)\beta_{\alpha,n-1}(n-1)(I - A)^{-n}x \\ &= k^\alpha(n)(I - A)^{-1}x + \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} k^\alpha(n-i)\beta_{\alpha,i}(j)((I - A)^{-(j+1)} - (I - A)^{-j})x.\end{aligned}$$

13 Applying that $(I - A)^{-1} - I = A(I - A)^{-1}$ and $\mathcal{S}_\alpha(0) = (I - A)^{-1}$ we get that

$$(I - A)\mathcal{S}_\alpha(n)x = k^\alpha(n)(I + A\mathcal{S}_\alpha(0))x + A \sum_{i=1}^{n-1} k^\alpha(n-i) \sum_{j=1}^i \beta_{\alpha,i}(j)(I - A)^{-(j+1)}x$$

$$= k^\alpha(n)(I + A\mathcal{S}_\alpha(0))x + A \sum_{i=1}^{n-1} k^\alpha(n-i)\mathcal{S}_\alpha(i)x,$$

1 and clearly it follows that $\mathcal{S}_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * \mathcal{S}_\alpha)(n)x$ for $n \in \mathbb{N}$. The case $n = 0$ is a
2 simple check.

3 Finally we prove that if there exists $\lambda_0 > 0$ such that $\sup_{n \in \mathbb{N}_0} \lambda_0^{-n} \|\mathcal{S}_\alpha(n)\| < \infty$, (iii) is
4 equivalent to (i). Assume that $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ is a discrete α -resolvent family generated
5 by A , then applying Z -transform we get for $|\lambda| > \max\{\lambda_0, 1\}$ that

$$\begin{aligned} \widetilde{\mathcal{S}}_\alpha(\lambda)x &= \sum_{j=0}^{\infty} \lambda^{-j} \mathcal{S}_\alpha(j)x = \widetilde{k}^\alpha(\lambda)x + A\widetilde{k}^\alpha(\lambda)\widetilde{\mathcal{S}}_\alpha(\lambda)x \\ &= \left(\frac{\lambda}{\lambda-1}\right)^\alpha x + \left(\frac{\lambda}{\lambda-1}\right)^\alpha A\widetilde{\mathcal{S}}_\alpha(\lambda)x, \quad x \in X, \end{aligned}$$

and

$$\widetilde{\mathcal{S}}_\alpha(\lambda)x = \left(\frac{\lambda}{\lambda-1}\right)^\alpha x + \left(\frac{\lambda}{\lambda-1}\right)^\alpha \widetilde{\mathcal{S}}_\alpha(\lambda)Ax, \quad x \in D(A),$$

6 where we have used that (1.5) and (2.3). Thus the operator $\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A$ is invertible, and
7 we get (3.2). Conversely, let $|\lambda|, |\mu| > \max\{\lambda_0, 1\}$ and $x \in D(A)$, then there exists $y \in X$
8 such that $x = \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} y$. Using that $\left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right)^{-1}$ and $\left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1}$ are
9 bounded operators and commute, and A is closed we have that

$$\begin{aligned} \widetilde{\mathcal{S}}_\alpha(\lambda)x &= \widetilde{\mathcal{S}}_\alpha(\lambda) \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} y \\ &= \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} \left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right)^{-1} y \\ &= \sum_{n=0}^{\infty} \lambda^{-n} \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} \mathcal{S}_\alpha(n) \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right) x. \end{aligned}$$

The uniqueness of Z -transform proves that

$$\mathcal{S}_\alpha(n)x = \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right)^{-1} \mathcal{S}_\alpha(n) \left(\left(\frac{\mu-1}{\mu}\right)^\alpha - A\right) x.$$

10 Then we have $\mathcal{S}_\alpha(n)x \in D(A)$, and therefore $A\mathcal{S}_\alpha(n)x = \mathcal{S}_\alpha(n)Ax$ for all $x \in X$. Finally note
11 that for $|\lambda| > \max\{\lambda_0, 1\}$ and $x \in D(A)$ we have using (2.3) that

$$\begin{aligned} \widetilde{k}^\alpha(\lambda)x &= \widetilde{k}^\alpha(\lambda)\widetilde{\mathcal{S}}_\alpha(\lambda) \left(\left(\frac{\lambda-1}{\lambda}\right)^\alpha - A\right) x \\ &= \widetilde{\mathcal{S}}_\alpha(\lambda)x - \widetilde{(k^\alpha * \mathcal{S}_\alpha)}(\lambda)Ax, \end{aligned}$$

12 and by the uniqueness of Z -transform we get the result. \square

13 A nice consequence of Theorem 3.3 is the following result about sums of combinatorial
14 numbers which seems to be new.

15 **Corollary 3.4.** *Take $\alpha > 0$, $n \in \mathbb{N}$ and $\{\beta_{\alpha,n}(j)\}_{j=1,\dots,n}$ defined as above. Then*

16 (i)
$$\sum_{j=1}^n \frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty} \lambda^l \frac{\Gamma(\alpha(l+1) + n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}, \quad \text{for } |\lambda| < 1.$$

$$1 \quad \text{(ii)} \quad \sum_{j=1}^n \beta_{\alpha,n}(j) \frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)} = \frac{\Gamma(\alpha(l+1)+n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}, \quad \text{for } l \in \mathbb{N}.$$

Proof. (i) We take $|\lambda| < 1$, then using Proposition 3.2 and Theorem 3.3 in the scalar case we have that

$$\sum_{j=1}^n \frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty} \lambda^l k^{\alpha(l+1)}(n) = \sum_{l=0}^{\infty} \lambda^l \frac{\Gamma(\alpha(l+1)+n)}{\Gamma(\alpha(l+1))\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

(ii) Let $|\lambda| < 1$, then

$$\sum_{j=1}^n \frac{\beta_{\alpha,n}(j)}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty} \lambda^l \sum_{j=1}^n \beta_{\alpha,n}(j) \frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)}, \quad n \in \mathbb{N},$$

2 where we have applied that $\frac{1}{(1-\lambda)^{j+1}} = \sum_{l=0}^{\infty} \lambda^l \frac{\Gamma(l+1+j)}{\Gamma(l+1)\Gamma(j+1)}$. Then we apply (i) to get
3 the result. \square

4 Our main result in this section is the following theorem.

5 **Theorem 3.5.** *Let $1 < \alpha \leq 2$. Suppose that A is the generator of a discrete α -resolvent*
6 *family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ on a Banach space X . Then the fractional difference equation*

$$(3.3) \quad \Delta^\alpha u(n) = Au(n+2), \quad n \in \mathbb{N}_0,$$

with initial conditions $u(0) = u_0 \in D(A)$ and $u(1) = u_1 \in D(A)$ admits the unique solution

$$u(n) = \mathcal{S}_\alpha(n)(I-A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I-A)u(1), \quad n \in \mathbb{N}_0.$$

Proof. Convolving the identity given in Definition 3.1(ii) by $k^{2-\alpha}$, we obtain

$$(k^{2-\alpha} * \mathcal{S}_\alpha)(n)x = (k^{2-\alpha} * k^\alpha)(n)x + A(k^{2-\alpha} * k^\alpha * \mathcal{S}_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

Using the semigroup property for the kernels k^α we have

$$(k^{2-\alpha} * \mathcal{S}_\alpha)(n)x = k^2(n)x + A(k^2 * \mathcal{S}_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

This is equivalent, by definition of fractional sum and convolution, to the following identity

$$\Delta^{-(2-\alpha)}\mathcal{S}_\alpha(n)x = k^2(n)x + A \sum_{j=0}^n k^2(n-j)\mathcal{S}_\alpha(j)x, \quad n \in \mathbb{N}_0.$$

Therefore, we get using $\Delta^2 k^2(j) = 0$ for $j \in \mathbb{N}_0$ that

$$\begin{aligned} \Delta^2 \circ \Delta^{-(2-\alpha)}\mathcal{S}_\alpha(n)x &= \Delta^2 k^2(n)x + A\Delta^2 \sum_{j=0}^n k^2(n-j)\mathcal{S}_\alpha(j)x \\ &= A \left[\sum_{j=0}^{n+2} k^2(j)\mathcal{S}_\alpha(n+2-j)x - 2 \sum_{j=0}^{n+1} k^2(j)\mathcal{S}_\alpha(n+1-j)x \right. \\ &\quad \left. + \sum_{j=0}^n k^2(j)\mathcal{S}_\alpha(n-j)x \right] \\ &= A \left[\sum_{j=2}^{n+2} k^2(j)\mathcal{S}_\alpha(n+2-j)x - 2 \sum_{j=1}^{n+1} k^2(j)\mathcal{S}_\alpha(n+1-j)x \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^n k^2(j) \mathcal{S}_\alpha(n-j)x + \mathcal{S}_\alpha(n+2)k^2(0)x \\
& + \mathcal{S}_\alpha(n+1)k^2(1)x - 2\mathcal{S}_\alpha(n+1)k^2(0)x \Big] \\
= & A \left[\sum_{j=0}^n k^2(j+2) \mathcal{S}_\alpha(n-j)x - 2 \sum_{j=0}^n k^2(j+1) \mathcal{S}_\alpha(n-j)x \right. \\
& \left. + \sum_{j=0}^n k^2(j) \mathcal{S}_\alpha(n-j)x + \mathcal{S}_\alpha(n+2)x \right]
\end{aligned}$$

1 for all $n \in \mathbb{N}_0$. We note that the left hand side in the above identity corresponds to the
2 fractional difference of order $\alpha \in (0, 2)$ in the sense of Riemann-Liouville. Therefore, we
3 obtain

$$(3.4) \quad \Delta^\alpha \mathcal{S}_\alpha(n)x = A\mathcal{S}_\alpha(n+2)x,$$

for all $n \in \mathbb{N}_0$ and all $x \in X$. Define $u(n)$ as

$$u(n) := \mathcal{S}_\alpha(n)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I - A)u(1), \quad n \in \mathbb{N}_0.$$

It then follows from (3.4) that u solves (3.3). Finally, from the identities

$$\mathcal{S}_\alpha(0)x = k^\alpha(0)x + A(k^\alpha * \mathcal{S}_\alpha)(0)x = x + Ak^\alpha(0)\mathcal{S}_\alpha(0)x = x + A\mathcal{S}_\alpha(0)x,$$

and

$$\mathcal{S}_\alpha(1)x = k^\alpha(1)x + A(k^\alpha * \mathcal{S}_\alpha)(1)x = \alpha\mathcal{S}_\alpha(0)x + A\mathcal{S}_\alpha(1)x,$$

4 which follow from Definition 3.1 (ii), we obtain $u(0) = \mathcal{S}_\alpha(0)(I - A)u_0 = u_0$ and $u(1) =$
5 $\mathcal{S}_\alpha(1)(I - A)u_0 - \alpha\mathcal{S}_\alpha(0)u_0 + \mathcal{S}_\alpha(0)(I - A)u_1 = u_1$, and we conclude the proof. \square

6 In the non homogeneous case, we derive the following result.

7 **Corollary 3.6.** *Let $1 < \alpha \leq 2$. Suppose that A is the generator of a discrete α -resolvent*
8 *family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ on a Banach space X and f be a vector-valued sequence. The fractional*
9 *difference equation*

$$(3.5) \quad \Delta^\alpha u(n) = Au(n+2) + f(n), \quad n \in \mathbb{N}_0, \quad 1 < \alpha \leq 2,$$

with initial conditions $u(0) = u_0 \in D(A)$ and $u(1) = u_1 \in D(A)$, admits the unique solution

$$u(n) = \mathcal{S}_\alpha(n)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I - A)u(1) + (\mathcal{S}_\alpha * f)(n-2),$$

10 for all $n \geq 2$.

11 *Proof.* Indeed, by Theorem 3.5 and Theorem 2.6 we have $u(n) \in D(A)$ for all $n \geq 2$ and

$$\begin{aligned}
\Delta^\alpha u(n) & = \Delta^\alpha (\mathcal{S}_\alpha(n)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n-1)u(0) + \mathcal{S}_\alpha(n-1)(I - A)u(1)) \\
& \quad + \Delta^\alpha (\mathcal{S}_\alpha * f)(n-2) \\
& = A(\mathcal{S}_\alpha(n+2)(I - A)u(0) - \alpha\mathcal{S}_\alpha(n+1)u(0) + \mathcal{S}_\alpha(n)(I - A)u(1)) \\
& \quad + (\Delta^\alpha \mathcal{S}_\alpha * f)(n-2) + (\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1) + \mathcal{S}_\alpha(0)f(n) \\
& = Au(n+2) - (A\mathcal{S}_\alpha * f)(n) + (\Delta^\alpha \mathcal{S}_\alpha * f)(n-2) \\
& \quad + (\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1) + \mathcal{S}_\alpha(0)f(n).
\end{aligned}$$

12 From (3.4) it follows $(\Delta^\alpha \mathcal{S}_\alpha * f)(n-2) = (A\mathcal{S}_\alpha * f)(n) - A\mathcal{S}_\alpha(1)(n-1) - A\mathcal{S}_\alpha(0)(n)$, and
13 hence we obtain

$$\Delta^\alpha u(n) = Au(n+2) + (I - A)\mathcal{S}_\alpha(0)f(n) + ((I - A)\mathcal{S}_\alpha(1) - \alpha\mathcal{S}_\alpha(0))f(n-1)$$

$$= Au(n+2) + f(n),$$

1 where we have used that $\mathcal{S}_\alpha(0) = (I - A)^{-1}$ and $(I + A)\mathcal{S}_\alpha(1) = \alpha\mathcal{S}_\alpha(0)$. For $n = 0$ and
2 $n = 1$ it is a simple check, using the same above arguments, that u is solution of (3.5). \square

3 4. NON-LINEAR FRACTIONAL DIFFERENCE EQUATIONS ON BANACH SPACES

4 Let A be a closed linear operator defined on a Banach space X . In this section we study
5 the non linear problem

$$(4.1) \quad \begin{cases} \Delta^\alpha u(n) = Au(n+2) + f(n, u(n)), & n \in \mathbb{N}_0, 1 < \alpha \leq 2; \\ u(0) = 0; \\ u(1) = 0. \end{cases}$$

6 For the case $0 < \alpha \leq 1$ see the reference [26]. The following definition is motivated by
7 Corollary 3.6. In particular, it shows consistence with the problem (4.1).

Definition 4.1. Under the assumption that the operator A is the generator of a discrete
 α -resolvent family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ on a Banach space X , we say that $u : \mathbb{N}_0 \rightarrow X$ is a solution
of the non-linear problem (4.1) if u satisfies

$$u(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k)f(k, u(k)), \quad n = 2, 3, 4, \dots$$

8 The next concept of admissibility is one of the keys ingredients for the estimates that we
9 will use in the proofs of our main results on existence of solutions to (4.1).

Definition 4.2. We say that a sequence $h : \mathbb{N}_0 \rightarrow (0, \infty)$ is an admissible weight if

$$\lim_{n \rightarrow \infty} h(n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) = 0.$$

10 **Example 4.3.** The sequence $h(n) = nn!$, that represents the factorial number system, is an
11 admissible weight function, since by [30, formula 33 p.598], we have $\sum_{k=1}^n kk! = (n+1)! - 1$.

For each admissible weight sequence h , we consider the vector-valued weighted space

$$\ell_h^\infty(\mathbb{N}_0; X) = \{\xi : \mathbb{N}_0 \rightarrow X \mid \|\xi\|_h < \infty\},$$

12 where the norm $\|\cdot\|_h$ is defined by $\|\xi\|_h := \sup_{n \in \mathbb{N}_0} \frac{\|\xi(n)\|}{h(n)}$.

13 The following is our first positive result on existence of solutions for the problem (4.1). It
14 uses a Lipschitz type condition.

Theorem 4.4. *Let h be an admissible weight and define*

$$H := \sup_{n \in \{2, 3, \dots\}} \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k).$$

15 *Let A be the generator of a bounded discrete α -resolvent family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ on a Banach
16 space X for some $1 < \alpha \leq 2$, and let $f : \mathbb{N}_0 \times X \rightarrow X$ be given and verifying the following
17 hypothesis:*

18 (F) *$f(k, 0) \neq 0$ for all $k \in \mathbb{N}_0$ and there exists a positive sequence $a \in \ell^1(\mathbb{N}_0)$ and constants
19 $c \geq 0, b > 0$ such that $\|f(k, x)\| \leq a(k)(c\|x\| + b)$ for all $k \in \mathbb{N}_0$ and $x \in X$.*

- 1 (L) The function f satisfies a Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, that
 2 is, there exists a constant $L > 0$ such that $\|f(k, x) - f(k, y)\| \leq L\|x - y\|$, for all
 3 $x, y \in X$, $k \in \mathbb{N}_0$, with $L < (\|\mathcal{S}_\alpha\|_\infty H)^{-1}$.
 4 Then the problem (4.1) has an unique solution in $\ell_h^\infty(\mathbb{N}_0; X)$.

Proof. Let us define the operator $G : \ell_h^\infty(\mathbb{N}_0; X) \rightarrow \ell_h^\infty(\mathbb{N}_0; X)$ given by

$$Gu(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k) f(k, u(k)), \quad n \geq 2.$$

First, we show that G is well defined: Let $u \in \ell_h^\infty(\mathbb{N}_0; X)$ be given. By using the assumption (F) and the boundedness of $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ we get that,

$$\begin{aligned} \|Gu(n)\| &\leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k))\| \leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| a(k) [c\|u(k)\| + b] \\ &\leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \sum_{k=0}^{n-2} h(k) + b\|\mathcal{S}_\alpha\|_\infty \|a\|_1 \end{aligned}$$

for each $n \in \mathbb{N}_0$. Hence,

$$\frac{\|Gu(n)\|}{h(n)} \leq c\|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{1}{h(n)} b\|\mathcal{S}_\alpha\|_\infty \|a\|_1.$$

- 5 and since h is an admissible weight, the above inequality proves that $Gu \in \ell_h^\infty(\mathbb{N}_0; X)$. We
 6 next prove that G is a contraction on $\ell_h^\infty(\mathbb{N}_0; X)$. Indeed, let $u, v \in \ell_h^\infty(\mathbb{N}_0; X)$ be given.
 7 Then, for each $n \in \mathbb{N}_0$,

$$\begin{aligned} \|Gu(n) - Gv(n)\| &\leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k)) - f(k, v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \sum_{k=0}^{n-2} \|f(k, u(k)) - f(k, v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \sum_{k=0}^{n-2} L\|u(k) - v(k)\| \leq \|\mathcal{S}_\alpha\|_\infty L\|u - v\|_h \sum_{k=0}^{n-2} h(k), \end{aligned}$$

where we have used the assumption (L). Therefore

$$\frac{\|Gu(n) - Gv(n)\|}{h(n)} \leq \|\mathcal{S}_\alpha\|_\infty L\|u - v\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k),$$

and consequently

$$\|Gu - Gv\|_h \leq \|\mathcal{S}_\alpha\|_\infty HL\|u - v\|_h,$$

- 8 with $\|\mathcal{S}_\alpha\|_\infty HL < 1$. Then, G has a unique fixed point in $\ell_h^\infty(\mathbb{N}_0; X)$, by the Banach fixed
 9 point theorem. \square

10 The next Lemma provide a necessary tool for the use of the Schauder's fixed point theorem,
 11 needed in the second main result on existence and uniqueness of solutions to (4.1).

12 **Lemma 4.5.** Let h be an admissible weight and $U \subset \ell_h^\infty(\mathbb{N}_0; X)$ such that:

- 13 (a) The set $H_n(U) = \left\{ \frac{u(n)}{h(n)} : u \in U \right\}$ is relatively compact in X , for all $n \in \mathbb{N}_0$.

- 1 (b) $\lim_{n \rightarrow \infty} \frac{1}{h(n)} \sup_{u \in U} \|u(n)\| = 0$, that is, for each $\varepsilon > 0$, there are $N > 0$ such that
 2 $\frac{\|u(n)\|}{h(n)} < \varepsilon$, for each $n \geq N$ and for all $u \in U$.
 3 Then U is relatively compact in $\ell_h^\infty(\mathbb{N}_0; X)$.

Proof. Let $\{u_m\}_m$ be a sequence in U , then by (a) for $n \in \mathbb{N}_0$ there is a convergent subsequence $\{u_{m_j}\}_j \subset \{u_m\}_m$ such that $\lim_{j \rightarrow \infty} \frac{u_{m_j}(n)}{h(n)} = a(n)$, that is, for each $\varepsilon > 0$ there exists $N(n, \varepsilon) > 0$ such that $\|\frac{u_{m_j}(n)}{h(n)} - a(n)\| < \varepsilon$ for all $j \geq N(n, \varepsilon)$. Let $\varepsilon > 0$ and N the value of the assumption (b). If we consider $N^* := \min_{0 \leq n < N} N(n, \varepsilon)$, then for $j, k \geq N^*$ we have

$$\sup_{0 \leq n < N} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} \leq \sup_{0 \leq n < N} \|\frac{u_{m_j}(n)}{h(n)} - a(n)\| + \sup_{0 \leq n < N} \|\frac{u_{m_k}(n)}{h(n)} - a(n)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and also

$$\sup_{n \geq N} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} \leq \sup_{n \geq N} \frac{\|u_{m_j}(n)\|}{h(n)} + \sup_{n \geq N} \frac{\|u_{m_k}(n)\|}{h(n)} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consequently,

$$\|u_{m_j} - u_{m_k}\|_h = \sup_{n \in \mathbb{N}_0} \frac{\|u_{m_j}(n) - u_{m_k}(n)\|}{h(n)} < \varepsilon,$$

- 4 therefore $\{u_{m_j}\}_j$ is a Cauchy subsequence in $\ell_h^\infty(\mathbb{N}_0; X)$ which finishes the proof. \square

For $f : \mathbb{N}_0 \times X \rightarrow X$ we recall that the Nemytskii operator $\mathcal{N}_f : \ell_h^\infty(\mathbb{N}_0; X) \rightarrow \ell_h^\infty(\mathbb{N}_0; X)$ is defined by

$$\mathcal{N}_f(u)(n) := f(n, u(n)), \quad n \in \mathbb{N}_0.$$

- 5 The next theorem is the second main result for this section. It gives one useful criteria for
 6 the existence of solutions without use of Lipchitz type conditions.

7 **Theorem 4.6.** *Let h be an admissible weight function. Let A be the generator of a bounded
 8 discrete α -resolvent family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ on a Banach space X for some $1 < \alpha \leq 2$, and
 9 $f : \mathbb{N}_0 \times X \rightarrow X$. Suppose that the condition (F) and following conditions are satisfied:*

- 10 (i) *The Nemytskii operator is continuous in $\ell_h^\infty(\mathbb{N}_0; X)$, that is, for each $\varepsilon > 0$, there is
 11 $\delta > 0$ such that for all $u, v \in \ell_h^\infty(\mathbb{N}_0; X)$, $\|u - v\|_h < \delta$ implies that $\|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h < \varepsilon$.
 12 ε .*
 13 (ii) *For all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set $\{\mathcal{S}_\alpha(n)f(k, x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is relatively
 14 compact in X for all $n \in \mathbb{N}_0$.*

15 Then the problem (4.1) has an unique solution in $\ell_h^\infty(\mathbb{N}; X)$.

Proof. Let us define the operator $G : \ell_h^\infty(\mathbb{N}_0; X) \rightarrow \ell_h^\infty(\mathbb{N}_0; X)$ given by

$$Gu(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k)f(k, u(k)), \quad n \geq 2.$$

- 16 To prove that G has a fixed point in $\ell_h^\infty(\mathbb{N}_0)$, we will use Leray-Schauder alternative theorem.
 17 We verify that the conditions of the theorem are satisfied:

18

- 19 • G is well defined: It follows from condition (F) and was proved in the first part of
 20 the proof of Theorem 4.4.

- 1 • G is continuous: Let $\varepsilon > 0$ and $u, v \in \ell_h^\infty(\mathbb{N}_0)$. Then, for each $n \in \mathbb{N}_0$,

$$\begin{aligned} \|Gu(n) - Gv(n)\| &\leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k)) - f(k, v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \sum_{k=0}^{n-2} \|f(k, u(k)) - f(k, v(k))\| \\ &\leq \|\mathcal{S}_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h \sum_{k=0}^{n-2} h(k). \end{aligned}$$

Therefore

$$\frac{\|Gu(n) - Gv(n)\|}{h(n)} \leq \|\mathcal{S}_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k).$$

2 Hence, by the assumption (ii) and admissibility of h we obtain $\|Gu - Gv\|_h < \varepsilon$.

- 3 • G is compact: For $R > 0$ given, let $B_R(\ell_h^\infty(\mathbb{N}_0; X)) := \{w \in \ell_h^\infty(\mathbb{N}_0; X) : \|w\|_h < R\}$.
4 To prove that $V := G(B_R(\ell_h^\infty(\mathbb{N}_0; X)))$ is relatively compact, we will use Lemma 4.5.
5 We check that the conditions in such Lemma are satisfied:

(a) Let $u \in B_R(\ell_h^\infty(\mathbb{N}_0; X))$ and $v = Gu$. We have

$$v(n) = Gu(n) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(n-2-k) f(k, u(k)) = \sum_{k=0}^{n-2} \mathcal{S}_\alpha(k) f(n-2-k, u(n-2-k)),$$

and then,

$$\frac{v(n)}{h(n)} = \frac{n-1}{h(n)} \left(\frac{1}{n-1} \sum_{k=0}^{n-2} \mathcal{S}_\alpha(k) f(n-2-k, u(n-2-k)) \right).$$

Therefore $\frac{v(n)}{h(n)} \in \frac{n-1}{h(n)} \text{co}(K_n)$, where $\text{co}(K_n)$ denotes the convex hull of K_n for the set

$$K_n = \bigcup_{k=0}^{n-2} \{\mathcal{S}_\alpha(k) f(\xi, x) : \xi \in \{0, 1, 2, \dots, n-2\}, \|x\| \leq R\}, \quad n \in \mathbb{N}_0.$$

6 Note that each set K_n is relatively compact by the assumption (ii). From the
7 inclusions $H_n(V) = \left\{ \frac{v(n)}{h(n)} : v \in V \right\} \subseteq \frac{n-1}{h(n)} \text{co}(K_n) \subseteq \frac{n-1}{h(n)} \text{co}(\overline{K_n})$, we conclude
8 that the set $H_n(V)$ is relatively compact in X , for all $n \in \mathbb{N}_0$.
9

- 10 (b) Let $u \in B_R(\ell_h^\infty(\mathbb{N}_0; X))$ and $v = Gu$. Using condition (F), for each $n \in \{2, 3, \dots\}$
11 we have

12 For each $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \frac{\|v(n)\|}{h(n)} &\leq \frac{1}{h(n)} \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k))\| \\ &\leq c \|\mathcal{S}_\alpha\|_\infty \|a\|_\infty \|u\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{1}{h(n)} b \|\mathcal{S}_\alpha\|_\infty \|a\|_1 \end{aligned}$$

$$\leq c\|\mathcal{S}_\alpha\|_\infty\|a\|_\infty R \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{1}{h(n)} b\|\mathcal{S}_\alpha\|_\infty\|a\|_1,$$

1 then the admissibility of h implies $\lim_{n \rightarrow \infty} \frac{\|v(n)\|}{h(n)} = 0$ independently of $u \in B_R(\ell_h^\infty(\mathbb{N}_0; X))$.

2 Therefore, $V = G(B_R(\ell_h^\infty(\mathbb{N}_0; X)))$ is relatively compact in $\ell_h^\infty(\mathbb{N}_0; X)$ by Lemma 4.5
3 and we conclude that G is a compact operator.

- The set $U := \{u \in \ell_h^\infty(\mathbb{N}_0; X) : u = \gamma Gu, \gamma \in (0, 1)\}$ is bounded: In fact, let us consider $u \in \ell_h^\infty(\mathbb{N}_0; X)$ such that $u = \gamma Gu$, $\gamma \in (0, 1)$. Again by condition (F),

$$\begin{aligned} \|u(n)\| &= \|\gamma Gu(n)\| \leq \sum_{k=0}^{n-2} \|\mathcal{S}_\alpha(n-2-k)\| \|f(k, u(k))\| \\ &\leq c\|\mathcal{S}_\alpha\|_\infty\|a\|_\infty\|u\|_h \sum_{k=0}^{n-2} h(k) + b\|\mathcal{S}_\infty\| \|a\|_1 \end{aligned}$$

Then for each $n \in \{2, 3, \dots\}$ we have

$$\frac{\|u(n)\|}{h(n)} \leq c\|\mathcal{S}_\alpha\|_\infty\|a\|_\infty\|u\|_h \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) + \frac{b}{h(n)} \|\mathcal{S}_\infty\| \|a\|_1.$$

4 Since h is admissible, we deduce that U is a bounded set in $\ell_h^\infty(\mathbb{N}_0; X)$.

5 Finally, by using the Leray-Schauder alternative theorem, we conclude that G has a fixed
6 point $u \in \ell_h^\infty(\mathbb{N}_0; X)$. \square

7 5. THE POISSON TRANSFORMATION OF FRACTIONAL DIFFERENCE OPERATORS

8 For each $n \in \mathbb{N}_0$, the Poisson distribution is defined by

$$(5.1) \quad p_n(t) := e^{-t} \frac{t^n}{n!}, \quad t \geq 0.$$

9 The Poisson distribution arises in connection with classical Poisson processes and semigroups
10 of functions; note that it is also called fractional integral semigroup in [31, Theorem 2.6].
11 In this section we study in detail this sequence of functions (Proposition 5.1), the Poisson
12 transformation (considered deeply in Theorem 5.2) and give their connection with fractional
13 difference and differential operators in Theorem 5.5.

14 **Proposition 5.1.** *Let $n \in \mathbb{N}_0$ and $(p_n)_{n \geq 0}$ given by (5.1). Then*

- (i) *For $t \geq 0$, the inequality $p_n(t) \geq 0$ holds, $\int_0^\infty p_n(t) dt = 1$, and*

$$\int_0^\infty p_n(t) p_m(t) dt = \frac{1}{2^{n+m+1}} \frac{(n+m)!}{n!m!}, \quad n, m \in \mathbb{N}_0.$$

- 15 (ii) *The semigroup property $p_n * p_m = p_{n+m}$ holds for $n, m \in \mathbb{N}_0$.*
16 (iii) *Given $t \geq 0$, then*

$$(p_{(\cdot)}(t) * p_{(\cdot)}(t))(n) = 2^n e^{-t} p_n(t), \quad n \in \mathbb{N}_0.$$

- 16 (iv) *For $m, n \in \mathbb{N}_0$, we have $\Delta^m p_n = (-1)^m p_{n+m}^{(m)}$.*

- 17 (v) *The Z-transform and the Laplace transform are given by*

$$\begin{aligned} \widetilde{p_{(\cdot)}(t)}(z) &= e^{-t(1-\frac{1}{z})}, \quad z \neq 0, \quad t > 0; \\ \widehat{p_n}(\lambda) &= \frac{1}{(\lambda+1)^{n+1}}, \quad \Re \lambda > -1, \quad n \in \mathbb{N}_0. \end{aligned}$$

Proof. The proof of (i) and (ii) is straightforward, and also may be found in [31, Theorem 2.6]. To show (iii), note that

$$(p_{(\cdot)}(t) * p_{(\cdot)}(t))(n) = e^{-2t} \frac{t^n}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} = 2^n e^{-t} p_n(t),$$

for $n \in \mathbb{N}_0$ and $t \geq 0$. Now we get that

$$(p_{n+1})'(t) = -e^{-t} \frac{t^{n+1}}{(n+1)!} + e^{-t} \frac{t^n}{n!} = -\Delta p_n(t),$$

1 and we iterate to obtain the equality $\Delta^m p_n = (-1)^m p_{n+m}^{(m)}$ for $m, n \in \mathbb{N}_0$. Finally the Z -
2 transform and the Laplace transform of $(p_n)_{n \geq 0}$ are easily obtained. \square

3 Now we introduce an integral transform using the Poisson distribution as integral kernel.
4 Some of their properties are inspired in results included in [24, Section 3] in particular a
5 remarkable connection between the vector-valued Z -transform and the vector-valued Laplace
6 transform, Theorem 5.2 (ii).

7 **Theorem 5.2.** Let $\psi \in L^1(\mathbb{R}_+; X)$ and we define $(\mathcal{P}\psi) \in s(\mathbb{N}_0; X)$ by

$$(5.2) \quad (\mathcal{P}\psi)(n) := \int_0^\infty p_n(t) \psi(t) dt, \quad n \in \mathbb{N}_0.$$

8 Then the following properties hold.

- 9 (i) The map \mathcal{P} defines a bounded linear operator from $L^1(\mathbb{R}_+; X)$ to $\ell^1(\mathbb{N}_0; X)$ and $\|\mathcal{P}\| =$
10 1.
11 (ii) For $\psi \in L^1(\mathbb{R}_+; X)$, we have that

$$\mathcal{P}(\psi)(n) = \frac{(-1)^n}{n!} \left[\widehat{\psi}(\lambda) \right]^{(n)} \Big|_{\lambda=1}, \quad n \in \mathbb{N}_0.$$

12 In particular the map \mathcal{P} is injective.

- 13 (iii) We have that $(\widetilde{\mathcal{P}\psi})(z) = \widehat{\psi}(1 - 1/z)$, for $|z| > 1$.
14 (iv) For $a \in L^1(\mathbb{R}_+)$ and $\psi \in L^1(\mathbb{R}_+; X)$ then $\mathcal{P}(a * \psi) = \mathcal{P}(a) * \mathcal{P}(\psi)$.
15 (v) If there are constants $M > 0$ and $\omega \geq 0$ such that $\|\psi(t)\| \leq M e^{-\omega t}$ for a.e. $t \geq 0$ then
16 $\|\mathcal{P}(\psi)(n)\| \leq \frac{M}{(1 + \omega)^{n+1}}$ for all $n \in \mathbb{N}_0$. In particular if ψ is bounded then $\{\mathcal{P}(\psi)(n)$
17 for $n \in \mathbb{N}_0$ is well-defined by (5.2) and $\{\mathcal{P}(\psi)(n)\}_{n \in \mathbb{N}_0}$ is bounded.
18 (vi) Let X be a Banach lattice and $\psi(t) \geq 0$ for a.e. $t \geq 0$, then $\mathcal{P}(\psi)(n) \geq 0$ for $n \in \mathbb{N}_0$.
19 (vii) Suppose that $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is a uniformly bounded family of operators. If
20 $\{S(t)\}_{t \geq 0}$ is compact then $\{\mathcal{P}(S)(n)\}_{n \in \mathbb{N}_0}$ is compact.

Proof. To prove (i) is enough to observe that

$$\|\mathcal{P}\psi\|_1 \leq \sum_{n=0}^{\infty} \int_0^\infty p_n(t) \|\psi(t)\| dt = \int_0^\infty \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-t} \|\psi(t)\| dt = \int_0^\infty \|\psi(t)\| dt = \|\psi\|_1,$$

for $\psi \in L^1(\mathbb{R}_+; X)$. Take $0 \neq x \in X$ and define $(e_\lambda \otimes x)(t) := e^{-\lambda t} x$ for $t, \lambda > 0$. Note that $e_\lambda \otimes x \in L^1(\mathbb{R}_+; X)$ and $\|e_\lambda \otimes x\|_1 = \frac{1}{\lambda} \|x\|$ for $\lambda > 0$. It is straightforward to check that

$$\mathcal{P}(e_\lambda \otimes x)(n) = \frac{1}{(1 + \lambda)^{n+1}} x, \quad \lambda > 0, \quad n \in \mathbb{N}_0,$$

21 and $\|\mathcal{P}(e_\lambda \otimes x)\|_1 = \frac{1}{\lambda} \|x\|$ for $\lambda > 0$. We conclude that $\|\mathcal{P}\| = 1$.

By properties of Laplace transform, see for example [7, Theorem 1.5.1], we have that

$$\mathcal{P}(\psi)(n) = \frac{(-1)^n}{n!} \left[\widehat{\psi}(\lambda) \right]^{(n)} \Big|_{\lambda=1}, \quad \psi \in L^1(\mathbb{R}_+; X).$$

1 Now take $\psi \in L^1(\mathbb{R}_+; X)$ such that $\mathcal{P}(\psi)(n) = 0$ for all $n \in \mathbb{N}_0$. Then we also get that
 2 $\left[\widehat{\psi}(\lambda) \right]^{(n)} \Big|_{\lambda=1} = 0$ for $n \in \mathbb{N}_0$. Since $\widehat{\psi}$ is an holomorphic function, we conclude that $\widehat{\psi} = 0$
 3 and then $\psi = 0$ where we apply that the Laplace transform is injective, see for example [7,
 4 Theorem 1.7.3].

5 Part (iii) is proved following similar ideas than in [24, Theorem 3.1]. For (iv) note that
 6 because $a \in L^1(\mathbb{R}_+)$ and $\psi \in L^1(\mathbb{R}_+; X)$ we have $a * \psi \in L^1(\mathbb{R}_+; X)$ and

$$\begin{aligned} \mathcal{P}(a * \psi)(n) &= \int_0^\infty \frac{t^n}{n!} e^{-t} \int_0^t a(s) \psi(t-s) ds dt = \int_0^\infty a(s) e^{-s} \int_0^\infty \frac{(s+u)^n}{n!} e^{-u} \psi(u) du ds \\ &= \sum_{j=0}^\infty \int_0^\infty a(s) e^{-s} \frac{s^j}{j!} ds \int_0^\infty \frac{u^{n-j}}{(n-j)!} e^{-u} \psi(u) du = (\mathcal{P}(a) * \mathcal{P}(\psi))(n), \end{aligned}$$

7 for $n \in \mathbb{N}_0$. Assertion (v) and (vi) are easily checked and assertion (vii) is obtained from [33,
 8 Corollary 2.3]. \square

We check Poisson transforms of some known functions in the next example. Note that, in fact, the Poisson transform can be extended to other sets than $L^1(\mathbb{R}_+; X)$, for example, $\mathcal{P}(f)(n)$ is well-defined for measurable functions f such that

$$\text{ess sup}_{t \geq 0} \|e^{\omega t} f(t)\| < \infty.$$

9 Also the identity given in Theorem 5.2 (iii) holds for the Dirac distribution δ_t for $t > 0$, see
 10 Proposition 5.1 (v).

11 **Definition 5.3.** The map $\mathcal{P} : L^1(\mathbb{R}_+; X) \rightarrow \ell^1(\mathbb{N}_0; X)$ defined by (5.2) is called the Poisson
 12 transformation.

13 **Example 5.4.** (i) Note that $\mathcal{P}(e_\lambda)(n) = \frac{1}{(\lambda+1)^{n+1}}$ for $n \in \mathbb{N}_0$, where $e_\lambda(t) := e^{-\lambda t}$ for
 14 $t > 0$.

(ii) By Proposition 5.1(i),

$$\mathcal{P}(p_m)(n) = \frac{1}{2^{n+m+1}} \frac{(n+m)!}{n!m!} = \frac{1}{2^{n+m+1}} k^{m+1}(n), \quad n, m \in \mathbb{N}_0,$$

15 where the kernel k^α is defined in (2.2).

(iii) For $\alpha > 0$, define

$$g_\alpha(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0; \\ 0, & t = 0. \end{cases}$$

16 Then the identity $\mathcal{P}(g_\alpha) = k^\alpha$ holds, see more details in [24, Example 3.3].

(iv) The Mittag-Leffler function is an entire function defined by

$$E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

see for example [14, Section 1.3]. Now take $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ and $s_{\alpha,\beta}(t) := t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)$, for $t > 0$. Then

$$\mathcal{P}(s_{\alpha,\beta})(n) = \sum_{k=0}^{\infty} \frac{\lambda^k}{n!\Gamma(\alpha k + \beta)} \int_0^{\infty} t^{n+\alpha k+\beta-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{\lambda^k \Gamma(n + \alpha k + \beta)}{n!\Gamma(\alpha k + \beta)}, \quad n \in \mathbb{N}_0,$$

which extends the result [24, Theorem 4.7] proved for $\beta = 1$. In the particular case $\beta = \alpha$, we get that

$$\mathcal{P}(s_{\alpha,\alpha})(n) = \sum_{j=0}^{\infty} \frac{\lambda^j \Gamma(n + \alpha(j+1))}{n!\Gamma(\alpha(j+1))} = \sum_{j=0}^{\infty} \lambda^j k^{\alpha(j+1)}(n), \quad n \in \mathbb{N}_0.$$

Now we are interested to establish a notable relation between the discrete and continuous fractional concepts in the sense of Riemann-Liouville. In order to give our next result, we recall that the Riemann-Liouville fractional integral of order $\alpha > 0$, of a locally integrable function $u : [0, \infty) \rightarrow X$ is given by:

$$I_t^\alpha u(t) := (g_\alpha * u)(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t > 0.$$

- 1 The Riemann-Liouville fractional derivative of order α , for $m-1 < \alpha < m$, $m \in \mathbb{N}$, is defined
2 by

$$(5.3) \quad D_t^\alpha u(t) := \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s)u(s)ds = \frac{d^m}{dt^m} (g_{m-\alpha} * u)(t), \quad t > 0,$$

- 3 for $u \in C^{(m)}(\mathbb{R}_+; X)$, see for example [14, Section 1.2] and [13, Section 1.3]. Compare these
4 definitions with Definitions 2.1 and 2.2.

Theorem 5.5. *Let $m \in \mathbb{N}$ and $m-1 < \alpha \leq m$. Take $u \in C^{(m)}(\mathbb{R}_+; X)$ such that $e_{-\omega}u^{(m)}$ is integrable for some $0 < \omega < 1$. Then we have*

$$\mathcal{P}(D_t^\alpha u)(n+m) = \int_0^{\infty} p_{n+m}(t)D_t^\alpha u(t)dt = \Delta^\alpha \mathcal{P}(u)(n), \quad n \in \mathbb{N}_0.$$

- 5 *Proof.* Set $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $u \in C^{(m)}(\mathbb{R}_+; X)$ such that $e_{-\omega}u^{(m)}$ is integrable for some
6 $0 < \omega < 1$. We integrate by parts m -times to get

$$\begin{aligned} \mathcal{P}(D_t^m u)(n+m) &= \int_0^{\infty} p_{n+m}(t)D_t^m u(t)dt = \dots = (-1)^m \int_0^{\infty} p_{n+m}^{(m)}(t)u(t)dt \\ &= \int_0^{\infty} \Delta^m p_n(t)u(t)dt = \Delta^m \mathcal{P}(u)(n) \end{aligned}$$

- 7 where we have applied Proposition 5.1 (iv).

- 8 Now consider $m-1 < \alpha < m$. By the definition of Riemann-Liouville fractional derivative,
9 see formula (5.3), we have that

$$\begin{aligned} \mathcal{P}(D_t^\alpha u)(n+m) &= \int_0^{\infty} p_{n+m}(t)D_t^\alpha u(t)dt = \int_0^{\infty} p_{n+m}(t) \frac{d^m}{dt^m} (g_{m-\alpha} * u)(t)dt \\ &= \Delta^m \mathcal{P}(g_{m-\alpha} * u)(n) = \Delta^m (k^{m-\alpha} * \mathcal{P}(u))(n) \\ &= \Delta^m \left(\Delta^{-(m-\alpha)} \mathcal{P}(u) \right) (n) = \Delta^\alpha \mathcal{P}(u)(n), \end{aligned}$$

- 10 where we have applied Theorem 5.2 (iv), Example 5.4 (iii) and Definition 2.1. □

6. DISCRETE α -RESOLVENT FAMILIES VIA POISSON SUBORDINATION

We recall the following concept (see [4, 23] and references therein).

Definition 6.1. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an α -resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0, \infty) \rightarrow \mathcal{B}(X)$ (respectively $S_\alpha : (0, \infty) \rightarrow \mathcal{B}(X)$ in case $0 < \alpha < 1$) such that $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$, the resolvent set of A , and

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad \operatorname{Re}(\lambda) > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the α -resolvent family generated by A .

By the uniqueness theorem for the Laplace transform, a 1-resolvent family is the same as a C_0 -semigroup, while a 2-resolvent family corresponds to a strongly continuous sine family. See for example [7] and the references therein for an overview on these concepts. Some properties of $(S_\alpha(t))_{t>0}$ are included in the following Lemma. For a proof, see for example [23].

Lemma 6.2. Let $\alpha > 0$. The following properties hold:

(i) $\lim_{t \rightarrow 0^+} \frac{S_\alpha(t)x}{g_\alpha(t)} = x$ for all $x \in X$.

(ii) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t > 0$.

(iii) For all $x \in D(A)$: $S_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)AS_\alpha(s)x ds$, $t > 0$.

(iv) For all $x \in X$: $(g_\alpha * S_\alpha)(t)x \in D(A)$ and

$$S_\alpha(t)x = g_\alpha(t)x + A \int_0^t g_\alpha(t-s)S_\alpha(s)x ds, \quad t > 0.$$

The next theorem is the main result of this section.

Theorem 6.3. Suppose that A is the generator of an α -resolvent family $(S_\alpha(t))_{t>0}$ on a Banach space X , of exponential bound less than 1. Then A is the generator of a discrete α -resolvent family $(\mathcal{S}_\alpha(n))_{n \in \mathbb{N}_0}$ defined by

$$\mathcal{S}_\alpha(n) := \mathcal{P}(S_\alpha)(n), \quad n \in \mathbb{N}_0.$$

Proof. Take $x \in D(A)$. Since $(A, D(A))$ is a closed operator and the condition in Lemma 6.2(ii) we have that

$$\mathcal{S}_\alpha(n)Ax = \int_0^\infty p_n(t)S(t)Ax dt = \int_0^\infty p_n(t)AS(t)x dt = A\mathcal{S}_\alpha(n)x.$$

From the identity

$$S_\alpha(t)x = g_\alpha(t)x + A \int_0^t g_\alpha(t-s)S_\alpha(s)x ds, \quad t > 0,$$

valid for all $x \in X$, we obtain

$$\mathcal{S}_\alpha(n)x = \mathcal{P}(S_\alpha)(n)x = \mathcal{P}(g_\alpha)(n) + A\mathcal{P}(g_\alpha * S_\alpha)(n)x = k^\alpha(n) + A(k^\alpha * \mathcal{S}_\alpha)(n)x,$$

where we have applied Example 5.4(iii) and Theorem 5.2 (iv) and the second condition in Definition 3.1. The theorem is proved. \square

Example 6.4. Consider the Mittag-Leffler function $E_{\alpha,\beta}$ studied in Example 5.4 (iv). Suppose that A is a bounded operator on the Banach space X . It follows from Definition 6.1 that

$$S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha), \quad t > 0, \alpha > 0$$

is the α -resolvent family generated by A . If $\|A\| < 1$, then

$$\mathcal{S}_\alpha(n)x := \int_0^\infty e^{-t} \frac{t^n}{n!} t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) x dt = \sum_{k=0}^\infty \frac{\Gamma(\alpha(k+1) + n)}{\Gamma(\alpha(k+1))\Gamma(n+1)} A^k x, \quad n \in \mathbb{N}_0,$$

1 for $x \in X$. Compare with Proposition 3.2.

Example 6.5. Suppose that A is the generator of a bounded sine family $(S(t))_{t>0}$ on X . Then A is the generator of a bounded α -resolvent family $(S_\alpha(t))_{t>0}$ on X for $1 < \alpha < 2$ given by

$$S_\alpha(t)x = \int_0^\infty \psi_{\alpha/2,0}(t,s)S(s)x ds, \quad t > 0, \quad x \in X,$$

where $\psi_{\alpha/2,0}(t,s)$ is the stable Lévy process, see [4, Corollary 14]. Then, by [4, Theorem 3 (vi)]

$$\|S_\alpha(t)\| \leq M \int_0^\infty \psi_{\alpha/2,0}(t,s) ds = Mg_{\alpha/2}(t), \quad t > 0,$$

2 and since $S_\alpha(0) = 0$, $\frac{1}{2} < \frac{\alpha}{2} < 1$ and $(S_\alpha(t))_{t>0}$ is strongly continuous we conclude that
3 $(S_\alpha(t))_{t>0}$ is bounded. Hence, by Theorem 6.3, we obtain a bounded discrete α -resolvent
4 family $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$.

5 Our next result imposes a natural and useful condition of compactness on a given family
6 of operators in order to obtain existence and uniqueness of solutions.

7 **Theorem 6.6.** *Suppose that A is the generator of a bounded sine family $(S(t))_{t>0}$ on X such
8 that $(\lambda - A)^{-1}$ is a compact operator for some λ large enough. Let $f : \mathbb{N}_0 \times X \rightarrow X$ be given
9 and suppose that the condition (F) and the following conditions are satisfied:*

10 (N) *The Nemytskii operator \mathcal{N}_f is continuous in $\ell_h^\infty(\mathbb{N}_0; X)$.*

11 *Then, for each $1 < \alpha \leq 2$, the problem (4.1) has an unique solution in $\ell_h^\infty(\mathbb{N}_0; X)$.*

12 *Proof.* To prove this result we only have to check that the assumption (ii) in Theorem 4.6 is
13 satisfied. Indeed, by hypothesis we have that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda^\alpha \in \rho(A)$ and
14 all $1 < \alpha \leq 2$. By Example 6.5 we obtain that A is the generator of a bounded α -resolvent
15 family $(S_\alpha(t))_{t>0}$, which is moreover compact by [33, Corollary 2.3]. From Theorem 5.2
16 (vii) it follows that $\{\mathcal{S}_\alpha(n)\}_{n \in \mathbb{N}_0}$ is compact. Also, for all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set
17 $\{f(k, x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is bounded because $\|f(k, x)\| \leq \|a\|_\infty(c\|x\| + b) \leq \|a\|_\infty(c\sigma + b)$
18 for all $0 \leq k \leq a$ and $\|x\| \leq \sigma$. Consequently, the set $\{\mathcal{S}_\alpha(n)f(k, x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is
19 relatively compact in X for all $n \in \mathbb{N}_0$. \square

20 7. EXAMPLES, APPLICATIONS AND FINAL COMMENTS

21 In this section, we provide several concrete examples and applications of the abstract
22 results developed in the previous sections. Finally we present some related problems with
23 problem (1.4) for $\alpha = 2$.

Example 7.1. Let $m : [a, b] \rightarrow (0, 1)$ be a continuous function. Let A be the multiplication operator given by $Af(x) = m(x)f(x)$ defined on $L^2(a, b)$. We know that A is a bounded

operator [21, Proposition 4.10, Chapter I]. Since $(\lambda^2 - A)^{-1} = \frac{1}{\lambda^2 - m(x)}$ for λ sufficiently large, we have by Definition 6.1 that A generates a sine family $(S(t))_{t>0}$ on $L^2(a, b)$, given by

$$S(t)f(x) = \frac{1}{2\sqrt{m(x)}} \left(e^{\sqrt{m(x)}t} - e^{-\sqrt{m(x)}t} \right) f(x), \quad t > 0.$$

- 1 Since $0 < m(x) < 1$ we obtain by subordination

$$\begin{aligned} \mathcal{S}(n)f(x) &= \int_0^\infty p_n(t)S(t)f(x)dt \\ &= \int_0^\infty e^{-t} \frac{t^n}{n!} \frac{1}{2\sqrt{m(x)}} \left(e^{\sqrt{m(x)}t} - e^{-\sqrt{m(x)}t} \right) f(x)dt \\ &= \frac{1}{n!2\sqrt{m(x)}} \left(\int_0^\infty t^n e^{-(1-\sqrt{m(x)})t} f(x)dt - \int_0^\infty t^n e^{-(1+\sqrt{m(x)})t} f(x)dt \right) \\ &= \frac{1}{2\sqrt{m(x)}} \left(\frac{1}{(1-\sqrt{m(x)})^{n+1}} - \frac{1}{(1+\sqrt{m(x)})^{n+1}} \right) f(x), \end{aligned}$$

for $n \in \mathbb{N}_0$. By Theorem 3.5 and Theorem 6.3, we conclude that the fractional difference equation

$$\Delta^2 u(n) = Au(n+2), \quad n \in \mathbb{N}_0,$$

- 2 with initial conditions $u(0) = u_0$ and $u(1) = u_1$, admits the explicit solution

$$\begin{aligned} u(n) &= (\mathcal{S}(n)(I - A) - 2\mathcal{S}(n-1))u_0 + \mathcal{S}(n-1)(I - A)u_1 \\ &= \sqrt{A^{-1}} \left(\frac{I - A}{2}(1 - \sqrt{A})^{-(n+1)} - \frac{I - A}{2}(1 + \sqrt{A})^{-(n+1)} \right) u_0 \\ &\quad - \sqrt{A^{-1}} \left((1 - \sqrt{A})^{-n} - (1 + \sqrt{A})^{-n} \right) u_0 \\ &\quad + \frac{1}{2}\sqrt{A^{-1}} \left((1 - \sqrt{A})^{-n} - (1 + \sqrt{A})^{-n} \right) (I - A)u_1, \quad n \in \mathbb{N}_0. \end{aligned}$$

- 3 **Example 7.2.** We study the existence of solutions for the problem

$$(7.1) \quad \begin{cases} \Delta^\alpha u(n, x) = u_{xx}(n+2, x) + \frac{1}{1+n^3} \frac{\cos(u(n, x))}{1 + \left(\int_0^\pi |u(n, s)|^2 ds \right)^{1/2}}, & n \in \mathbb{N}_0, \quad 0 < x < \pi; \\ u(0, x) = 0; \quad u(1, x) = 0; \\ u(n, 0) = 0; \quad u(n, \pi) = 0; \end{cases}$$

- 4 for $1 < \alpha < 2$. We will use Corollary 6.6.

5

Let $X = L^2[0, \pi]$ and let us define the operator $A = \frac{\partial^2}{\partial x^2}$, on the domain

$$D(A) = \{v \in L^2[0, \pi] / v, v' \text{ absolutely continuous, } v'' \in L^2[0, \pi], v(0) = v(\pi) = 0\}.$$

Observe that the operator A can be written as

$$Av = - \sum_{n=1}^{\infty} n^2 (v, z_n) z_n, \quad v \in D(A),$$

where $z_n(s) := \sqrt{2/\pi} \sin ns$, $n = 1, 2, \dots$, is an orthonormal set of eigenvectors of A . Note that A is the infinitesimal generator of a sine family $(S(t))_{t \in \mathbb{R}}$ in $L^2[0, \pi]$, given by

$$S(t)v = \sum_{n=1}^{\infty} \frac{\sin nt}{n} (v, z_n) z_n, \quad v \in L^2[0, \pi].$$

The resolvent of A is given by

$$R(\lambda; A)v = \sum_{n=1}^{\infty} \frac{1}{\lambda + n^2} (v, z_n) z_n, \quad v \in L^2[0, \pi], \quad -\lambda \neq k^2, k \in \mathbb{N}.$$

1 The compactness of $R(\lambda; A)$ follows from the fact that eigenvalues of $R(\lambda; A)$ are $\lambda_n = \frac{1}{\lambda + n^2}$,
 2 $n = 1, 2, \dots$, and thus $\lim_{n \rightarrow \infty} \lambda_n = 0$, see for example [32].
 3

Let us consider the weighted space

$$\ell_h^\infty(\mathbb{N}_0; L^2[0, \pi]) = \left\{ \xi : \mathbb{N}_0 \rightarrow L^2[0, \pi] / \sup_{n \in \mathbb{N}} \frac{\|\xi(n)\|_{L^2}}{nn!} < \infty \right\},$$

4 where the function $h(n) = nn!$ is an admissible weight function (see Example 4.3).

5 For the function $f : \mathbb{N}_0 \times L^2[0, \pi] \rightarrow L^2[0, \pi]$, defined by $f(n, v) := \frac{1}{1+n^3} \frac{\cos(v)}{1+\|v\|}$, we
 6 consider the Nemystkii operator \mathcal{N}_f associated to f . That is, $N_f(u) : \mathbb{N}_0 \rightarrow L^2[0, \pi]$ is such
 7 that $N_f(u)(n) := f(n, u(n))$ for $u : \mathbb{N}_0 \rightarrow L^2[0, \pi]$. Then:

8 (i) $f(n, 0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists $a(n) = \frac{1}{1+n^3}$ in $l^1(\mathbb{N}_0)$ and $c = 1, b = 0$
 9 such that $\|f(n, v)\| \leq a(n)(c\|v\| + b)$, for all $n \in \mathbb{N}_0$ and $v \in L^2[0, \pi]$.

10
 11 (ii) Is clear from the definition.

Consequently, by Theorem 6.6, we conclude that the problem (7.1) has an unique solution $u \in \ell_h^\infty(\mathbb{N}_0)$, that is, u satisfies

$$\sup_{n \in \mathbb{N}_0} \frac{\|u(n)\|_{L^2}}{nn!} = \sup_{n \in \mathbb{N}_0} \frac{1}{nn!} \left(\int_0^\pi |u(n, x)|^2 dx \right)^{1/2} < \infty,$$

Therefore, there exist a constant $K > 0$ such that

$$\int_0^\pi |u(n, x)|^2 dx < K(nn!)^2, \quad n \in \mathbb{N}.$$

12 **Final comments.** In some circumstances, the equation (1.4) for $\alpha = 2$ may have a different
 13 format on the right hand side. For instance, the problem

$$(7.2) \quad \begin{cases} \Delta^2 u(n) &= Bu(n+1) + g(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases}$$

14 where B is a linear operator defined on a Banach space X . In such cases, and under mild
 15 conditions, we can still handle this problem with our theory. That is the content of the
 16 following two results.

17 **Proposition 7.3.** *Let B be a linear operator defined on a Banach space X , such that $-2 \in$
 18 $\rho(B)$. Then, (7.2) is equivalent to the problem*

$$(7.3) \quad \begin{cases} \Delta^2 u(n) &= Tu(n+2) + Tu(n) + (I - T)g(n, u(n)), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases}$$

1 where $T = I - 2(2 + B)^{-1}$.

2 *Proof.* From the definition

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n),$$

we obtain

$$u(n+1) = \frac{1}{2}(u(n+2) - \Delta^2 u(n) + u(n)).$$

On the other hand, by (7.2) we have

$$u(n+2) - 2u(n+1) + u(n) = Bu(n+1) + g(n, u(n))$$

that is, for $-2 \in \rho(B)$ we have

$$u(n+1) = (2+B)^{-1}u(n+2) + (2+B)^{-1}u(n) - (2+B)^{-1}g(n, u(n)).$$

By identifying both expressions for $u(n+1)$, we achieve

$$(2+B)^{-1}u(n+2) + (2+B)^{-1}u(n) - (2+B)^{-1}g(n, u(n)) = \frac{1}{2}(u(n+2) - \Delta^2 u(n) + u(n)),$$

and therefore

$$\Delta^2 u(n) = (I - 2(2+B)^{-1})u(n+2) + (I - 2(2+B)^{-1})u(n) + 2(2+B)^{-1}g(n, u(n)).$$

3 So, assuming $-2 \in \rho(B)$, the original problem (7.2) is equivalent to the problem (7.3), with
4 $T = I - 2(2+B)^{-1}$. \square

5 Observe that the operator T in the above proposition is bounded whenever B is a closed
6 linear operator and $-2 \in \rho(B)$. A second case of interest is the following.

7 **Proposition 7.4.** *Let B be a linear operator defined on a Banach space X , such that $1 \in$
8 $\rho(B)$. Then, the problem*

$$(7.4) \quad \begin{cases} \Delta^2 u(n) &= Bu(n) + g(n+1, u(n+1)), & n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases}$$

9 *is equivalent to the problem*

$$(7.5) \quad \begin{cases} \Delta^2 u(n) &= Tu(n+2) - 2Tu(n+1) + (I-T)g(n+1, u(n+1)), & n \in \mathbb{N}_0; \\ u(0) &= u_0; \\ u(1) &= u_1. \end{cases}$$

10 where $T = I - (I - B)^{-1}$.

11 *Proof.* From the definition

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n),$$

we obtain

$$u(n) = \Delta^2 u(n) - u(n+2) + 2u(n+1).$$

On the other hand, by (7.4) we have

$$u(n+2) - 2u(n+1) + u(n) = Bu(n) + g(n+1, u(n+1))$$

that is, for $1 \in \rho(B)$ we have

$$u(n) = -(I - B)^{-1}u(n+2) + 2(I - B)^{-1}u(n+1) + (I - B)^{-1}g(n+1, u(n+1)).$$

By identifying both expressions for $u(n)$, we achieve

$$(I - B)^{-1}(-u(n+2) + 2u(n+1) + g(n+1, u(n+1))) = \Delta^2 u(n) - u(n+2) + 2u(n+1),$$

and therefore

$$\Delta^2 u(n) = (I - (I - B)^{-1})u(n+2) - 2(I - (I - B)^{-1})u(n+1) + (I - B)^{-1}g(n+1, u(n+1)).$$

1 So, assuming $1 \in \rho(B)$, the problem (7.4) is equivalent to the problem (7.5), with $T =$
2 $I - (I - B)^{-1}$. \square

3 For instance, let B be a linear operator defined on a Banach space X , and γ a positive
4 constant. We study the existence of solutions of the problem

$$(7.6) \quad \begin{cases} \Delta^2 u(n, x) = (B + 2\gamma)u(n+1, x), & n \in \mathbb{N}_0, x \in [a, b]; \\ u(0, x) = 0; & u(1, x) = 0, \\ u(n, a) = 0; & u(n, b) = 0. \end{cases}$$

5 By Proposition 7.3 the solution of (7.6) coincides with the solution of the problem

$$(7.7) \quad \begin{cases} \Delta^2 u(n, x) = Tu(n+2, x) + Tu(n, x), & n \in \mathbb{N}_0, x \in [a, b]; \\ u(0, x) = 0; & u(1, x) = 0, \\ u(n, a) = 0; & u(n, b) = 0. \end{cases}$$

6 where $T = I - 2(2(1 + \gamma) + B)^{-1}$, provided that $2 + 2\gamma \in \rho(-B)$.

As an example of application to Theorem 4.4 with $\alpha = 2$, let us consider $X = L^2(\pi, 2\pi)$ and define

$$Bf(x) = 2\left(\frac{1}{1+x} - (1+\gamma)\right)f(x), \quad x \in [\pi, 2\pi].$$

Note that B is bounded. A computation shows that $Tf(x) = -xf(x)$ and therefore generates the sine family

$$S(t)f(x) = \frac{\sin(\sqrt{x}t)}{\sqrt{x}}f(x), \quad x \in [\pi, 2\pi]$$

7 It follows that $\|T\| \leq 2\pi$ and $\|S\|_\infty \leq \sqrt{\pi}$.

Let h the admissible weight function defined by $h(n) = nn!$, for which we have

$$\sup_{n \in \mathbb{N}_0} \frac{1}{h(n)} \sum_{k=0}^{n-2} h(k) = \frac{1}{18}$$

8 since $\frac{1}{h(n)} \sum_{k=0}^{n-2} h(k)$ is a decreasing sequence for $n \geq 3$. Let us consider the function $f :$

9 $\mathbb{N}_0 \times X \rightarrow X$ defined by $f(n, \xi) = \frac{1}{n^2}T\xi + \frac{1}{1+n^2}$. Then the function f verifies:

10 (F) $f(n, 0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists $a(n) = \frac{1}{n^2}$ in $\ell^1(\mathbb{N}_0)$ and $c = \|T\|, b = 1$
11 such that $\|f(n, \xi)\| \leq a(n)(c\|\xi\| + b)$ for all $n \in \mathbb{N}_0$ and $\xi \in X$.

(L) There exists $L := \|T\|$ such that

$$\|f(n, x) - f(n, y)\| \leq \|T\|\|x - y\|,$$

for all $x, y \in X$. Moreover,

$$\|T\|\|S\|_\infty \frac{1}{18} < \frac{2\pi\sqrt{\pi}}{18} < 1.$$

12 Therefore, by Theorem 4.4 we conclude that the problem (7.7) has an unique solution or,
13 equivalently, the problem (7.6) has an unique solution $u \in \ell_h^\infty(\mathbb{N}_0; X)$.

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