

On Hamiltonian alternating cycles and paths*

Mercè Claverol¹ Alfredo García² Delia Garijo³ Carlos Seara¹
Javier Tejel²

Abstract

We undertake a study on computing Hamiltonian alternating cycles and paths on bicolored point sets. This has been an intensively studied problem, not always with a solution, when the paths and cycles are also required to be plane. In this paper, we relax the constraint on the cycles and paths from being plane to being 1-plane, and deal with the same type of questions as those for the plane case, obtaining a remarkable variety of results. For point sets in general position, our main result is that it is always possible to obtain a 1-plane Hamiltonian alternating cycle. When the point set is in convex position, we prove that every Hamiltonian alternating cycle with minimum number of crossings is 1-plane, and provide $O(n)$ and $O(n^2)$ time algorithms for computing, respectively, Hamiltonian alternating cycles and paths with minimum number of crossings.

1 Introduction

A *geometric graph* is a graph drawn in the plane whose vertex set is a set of points and whose edges are straight-line segments connecting pairs of points. Two edges of a geometric graph *cross* if they have an intersection point lying in the relative interior of both edges. When there are no crossings, the geometric graph is said to be *plane*, and it is *alternating* if its vertex set is a bicolored point set and all its edges are *bichromatic*, i.e., they connect points of different colors; when all edges connect points of the same color, the geometric graph is *monochromatic*. In this paper, we deal with alternating (geometric) cycles and paths.

The study of alternating paths was initiated, according to Pach [15], by Erdős [9] around 1989, who proposed the following problem:

Determine or estimate the largest number $\ell(n)$ such that, for every set of n red and n blue points on a circle, there exists a non-crossing alternating path consisting of $\ell(n)$ vertices.

This problem comes from the fact that it is not always possible to obtain a plane Hamiltonian alternating path on that point configuration. Erdős conjectured that $\ell(n) = \frac{3}{2}n + 2 + o(n)$, but one can find point configurations for which $\ell(n) < \frac{4}{3}n + o(n)$, as described in [1, 15, 16]. The best bounds up to date for $\ell(n)$ are due to Kynčl, Pach and Tóth [15], and valid for bicolored point sets in general position (i.e, no three points lie on the same line). However, the conjecture that $|\ell(n) - \frac{4}{3}n| = o(n)$ remains open, even for points in convex position. For more information on this topic see [2, 5].

In this context, Akiyama and Urrutia [3] gave an $O(n^2)$ time algorithm for computing a plane alternating path visiting the maximum possible number of points, provided that the point set is in

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¹Departament de Matemàtiques, Universitat Politècnica de Catalunya, Spain. merce.claverol@upc.edu and carlos.seara@upc.edu. Partially supported by projects Gen. Cat. DGR 2014SGR46 and MTM2015-63791-R MINECO/FEDER.

²Departamento de Métodos Estadísticos, IUMA, Universidad de Zaragoza, Spain. olaverri@unizar.es and jtejel@unizar.es. Partially supported by projects E58 (ESF) (Gobierno de Aragón) and MTM2015-63791-R MINECO/FEDER.

³Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain. dgarijo@us.es. Partially supported by projects MTM2014-60127-P and FQM-164.

convex position and the endpoints of the path are previously fixed. Other references on bicolored point sets are [13] (which is the book of reference) and [7, 8, 20] (where monochromatic paths and matchings are studied). See also [5] for results on alternating paths on double chains, and [2] for results on plane alternating spanning trees. For other problems on geometric graphs see [4, 12, 18].

Since the plane character for Hamiltonian alternating cycles and paths is not always possible to achieve, it is natural to relax the constraint in order to obtain better results. Thus, different approaches arise based on allowing crossings. Kaneko, Kano and Yoshimoto [14] focused in getting few crossings in global, and proved that there always exists a Hamiltonian alternating cycle on a bicolored point set in general position with at most $n - 1$ crossings, where n is the number of points of each color. Claverol et al. [6] studied the *1-plane* character, i.e., every edge is allowed to have at most one crossing. They proved that one can always obtain a 1-plane Hamiltonian alternating cycle on a point set in convex position and on a double chain. They also conjectured the existence of a 1-plane Hamiltonian alternating path for point sets in general position. Observe that the terms plane graph and 1-plane graph refer to a geometric object, while to be planar or 1-planar are properties of the underlying abstract graph; see [19] where the concept of 1-planar graph was introduced (in relation to a coloring problem for planar graphs).

Our paper can be considered as a continuation of the study developed by Claverol et al. [6] since we explore the relaxation from being plane to being 1-plane for paths and cycles on bicolored point sets. To be more precise on the content of this paper, we next provide some notation and additional definitions.

Let R and B be two disjoint sets of red and blue points in the plane, respectively, such that $S = R \cup B$ is in general position⁴. Unless otherwise stated, $|B| = n$ and $n \leq |R| \leq n + 1$. For short, we shall use 1-PHAC and 1-PHAP to refer to a 1-plane Hamiltonian alternating cycle and a 1-plane Hamiltonian alternating path, respectively. As usual, $\text{CH}(X)$ denotes the convex hull of a point set $X \subseteq S$. We say that $\text{CH}(X)$ is *bichromatic* if its boundary contains points of both colors, and *monochromatic* otherwise. A *run* of $X \subseteq S$ is a maximal set of consecutive points of the same color on the boundary of $\text{CH}(X)$; it can be *red* or *blue* depending on the color of the points, and its *cardinality* is the number of its points. Let $\mathbf{r}(X)$ and $\mathbf{b}(X)$ be the number of red and blue runs of X , respectively. Note that $\mathbf{r}(X) = \mathbf{b}(X)$ whenever $\text{CH}(X)$ is bichromatic; otherwise one of the values equals 1 and the other zero.

Our main result in Section 2 is that a 1-PHAC can always be obtained on all instances of set S in general position, also upper-bounding the number of crossings by $n - \max\{\mathbf{r}(S), \mathbf{b}(S)\}$. The key tool to prove this result is that, except for some special configurations, one can always draw a 1-PHAP on S , fixing previously its endpoints on the boundary of $\text{CH}(S)$. We conclude the section by showing that a 1-PHAC on S can be computed in $O(n^2)$ time and space.

Section 3 concerns Hamiltonian alternating cycles and paths on point sets S in convex position. Subsection 3.1 deals with cycles: $n - \mathbf{r}(S)$ is shown to be a lower bound for the number of crossings, and only some cycles that are 1-plane attain the bound. We also prove that a 1-PHAC on S with minimum number of crossings can be computed in $O(n)$ time and space, and as a consequence of the process for cycles, the same complexity is obtained for computing a 1-PHAP with minimum number of crossings, provided that the endpoints of the path are consecutive points of S .

Subsection 3.2 contains our results for paths on sets in convex position. We first prove that the above-mentioned special configurations are the unique point configurations that do not admit a 1-PHAP with given endpoints. Then, we show that, except for those special configurations, every Hamiltonian alternating path with minimum number of crossings is 1-plane. This result allows us to design an $O(n^2)$ time and space algorithm to compute a 1-PHAP with fixed endpoints and minimum number of crossings.

We conclude the paper in Section 4 with some open problems.

⁴In all the figures in this paper, red points are illustrated as solid red points, and blue points are depicted as hollow blue points.

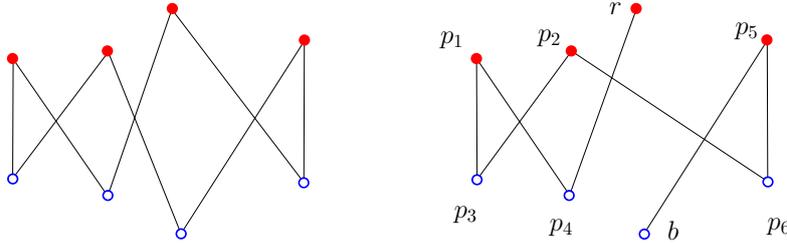


Figure 1: Left: A 1-PHAC. Right: A point configuration that admits no 1-PHAP with endpoints r and b .

2 1-PHAC for point sets in general position

Our aim in this section is to prove that there always exists a 1-PHAC on a set $S = R \cup B$ in general position with at most $n - \max\{\mathbf{r}(S), \mathbf{b}(S)\}$ crossings, where $n = |R| = |B| \geq 2$ (Figure 1 left shows an example); see Theorem 1 below.

The proof of Theorem 1 follows that given by Kaneko, Kano and Yoshimoto [14] to construct a Hamiltonian alternating cycle with at most $n - 1$ crossings. Although both proofs are similar, there are two fundamental differences. On the one hand, the construction of the cycle in [14] is based on the fact that one can always draw a Hamiltonian alternating path with at most $n - 1$ crossings connecting any red point to any blue point, both located on the boundary of $\text{CH}(S)$. Nevertheless, this is not true for 1-plane geometric graphs (see Figure 1 right) and so our proof is adapted to this new scenario. On the other hand, our proof makes possible upper bounding the number of crossings, not only by $n - 1$ as in [14], but using the number of runs of S .

The following technical lemma, which can be viewed as an adaptation of Lemma 1 from Kaneko, Kano and Yoshimoto [14], is crucial to reach some of the main results in this paper.

Lemma 1 *Let $S = R \cup B$ be in general position, and let r be an arbitrary red point of S . Then, the following statements hold.*

- (i) *Suppose that $|R| = |B| = n \geq 1$, $\{p_1, \dots, p_{2n-1}\}$ is the counterclockwise radial order of the points of $S \setminus \{r\}$ around r , and p_{i+1} is the first point in that sequence such that*

$$|\{r, p_1, \dots, p_{i+1}\} \cap R| = |\{r, p_1, \dots, p_{i+1}\} \cap B|.$$

Then, (a) $p_{i+1} = p_1$ if p_1 is blue, (b) p_i and p_{i+1} both exist and are blue whenever p_1 is red. In this last case, we say that $S_1 = \{p_1, \dots, p_i\}$ and $S_2 = \{p_{i+1}, \dots, p_{2n-1}\}$ form a counterclockwise partition of $S \setminus \{r\}$ around r ; see Figure 2 left.

- (ii) *Suppose that $|R| = |B| + 1 = n + 1 \geq 2$, $\{p_1, \dots, p_{2n}\}$ is the counterclockwise radial order of the points of $S \setminus \{r\}$ around r , point p_{2n} is red, and p_{i+1} is the first point in that sequence such that*

$$|\{r, p_1, \dots, p_{i+1}\} \cap R| = |\{r, p_1, \dots, p_{i+1}\} \cap B|.$$

Then, (a) $p_{i+1} = p_1$ if p_1 is blue, (b) p_i and p_{i+1} both exist and are blue whenever p_1 is red. In this last case, we say that $S_1 = \{p_1, \dots, p_i\}$ and $S_2 = \{p_{i+1}, \dots, p_{2n}\}$ form a counterclockwise partition of $S \setminus \{r\}$ around r .

Proof. We prove statement (i). An analogous argument, by removing point p_{2n} , proves statement (ii). If p_1 is blue, the result is obvious. If p_1 is red, let $\Delta_i = |\{r, p_1, \dots, p_i\} \cap R| - |\{r, p_1, \dots, p_i\} \cap B|$ for $1 \leq i \leq 2n - 1$. The sequence of values of Δ_i begins with value 2, ends with value 0, increases by 1 if the visited point is red, and decreases by 1 if the visited point is blue. Therefore, when the value 0 appears the first time in that sequence, it is produced by a subsequence $\dots, 2, 1, 0$, which implies that the last two visited points must be blue. See Figure 2 left. \square

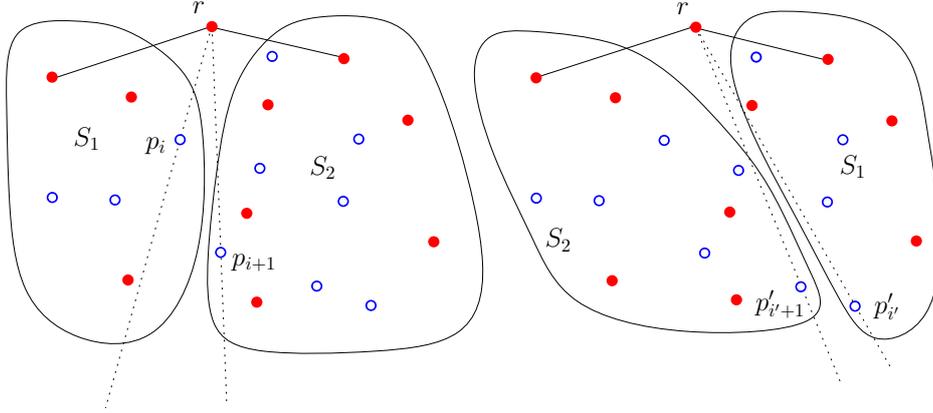


Figure 2: A counterclockwise partition (left) and a clockwise partition (right) of S around r .

A totally symmetric result holds if we explore the points of $S \setminus \{r\}$ in clockwise order: if $\{p'_1, \dots, p'_{2n-1}\}$ ($\{p'_1, \dots, p'_{2n}\}$) is the clockwise radial order of the points of $S \setminus \{r\}$ around r and point p'_1 is red, then there is a first clockwise point $p'_{i'+1}$ such that

$$|\{r, p'_1, \dots, p'_{i'+1}\} \cap R| = |\{r, p'_1, \dots, p'_{i'+1}\} \cap B|$$

and points $p'_{i'}$, $p'_{i'+1}$ are blue. We say that $S_1 = \{p'_1, \dots, p'_{i'}\}$ and $S_2 = \{p'_{i'+1}, \dots, p'_{2n-1}\}$ ($S_2 = \{p'_{i'+1}, \dots, p'_{2n}\}$) form a *clockwise partition of $S \setminus \{r\}$ around r* . See Figure 2 right.

Obviously, clockwise and counterclockwise partitions of $S \setminus \{b\}$ around a blue point b can also be defined when $|R| = |B| = n \geq 1$ or when $|B| = |R| + 1 = n + 1 \geq 2$ and p_{2n} is blue.

Notation according to Lemma 1. Hereafter, given a partition $S_1 \cup S_2$ of $S \setminus \{r\}$ around r , we shall use $n_1 = |R \cap S_1| = |B \cap S_1|$ and $n_2 = |R \cap S_2| = |B \cap S_2|$ when $|R| = |B| + 1$. For $|R| = |B|$ it follows that $|B \cap S_2| = n_2 + 1$ and $|R \cap S_2| = n_2$. Further, we shall write $P_X(p, q)$ to indicate a 1-PHAP on a set $X \subseteq S$ with endpoints $p, q \in X$ (here, it is understood that p and q have adequate colors, and that X contains a proper number of red and blue points in order to construct such a path).

Lemma 1 and the notion of *visibility* provided below are the key tools to prove Lemma 2, which states that a 1-PHAP connecting any two points on the boundary of $\text{CH}(S)$ can always be drawn except for the so-called *special configurations* (see Definition 1), also upper-bounding the number of crossings in terms of the number of runs of S . This is the main result to reach Theorem 1.

For a point set X (not necessarily consisting of red and blue points) and a point x outside $\text{CH}(X)$, a point $p \in X$ is *visible* from x or point x *sees* point p if the segment xp does not cross $\text{CH}(X)$. Note that if p and p' are both visible from x , and p is placed counterclockwise before p' on the boundary of $\text{CH}(X)$, then all points on the boundary of $\text{CH}(X)$ placed clockwise between p' and p are also visible from x .

Definition 1 Let $S = R \cup B$, and let $r \in R$ and $b \in B$ be two arbitrary points on the boundary of $\text{CH}(S)$. The triple (S, r, b) is a special configuration if the three following conditions are satisfied:

- (i) The two neighbors of r on the boundary of $\text{CH}(S)$ are red, and those of b are blue.
- (ii) $p_{i+1} = p'_{i'+1} = b$ in the clockwise and counterclockwise partitions of S around r .
- (iii) $p_{i+1} = p'_{i'+1} = r$ in the clockwise and counterclockwise partitions of S around b .

Otherwise, the configuration (S, r, b) is non-special.

Figure 1 (right) illustrates a special configuration (S, r, b) . In the counterclockwise partition of S around r , $S_1 = \{p_1, p_2, p_3, p_4\}$ and so $p_{i+1} = b$; for the clockwise partition we have $S_1 = \{p_5, p_6\}$, and again $p'_{i'+1} = b$. The same happens for the clockwise and counterclockwise partitions of S around b : points p_{i+1} and $p'_{i'+1}$ coincide with r .

Lemma 2 *Let $S = R \cup B$, and let r be an arbitrary red point on the boundary of $CH(S)$. Then, the following statements hold.*

- (i) *If $|R| = |B| = n \geq 1$ and there is a blue point b on the boundary of $CH(S)$ such that the configuration (S, r, b) is non-special, then there exists a 1-PHAP on S with endpoints r and b and at most $n - \mathbf{r}(S) = n - \mathbf{b}(S)$ crossings.*
- (ii) *If $|R| = |B| + 1 = n + 1 \geq 2$ and there is a red point $r' \in R \setminus \{r\}$ on the boundary of $CH(S)$, then there exists a 1-PHAP on S with endpoints r and r' and at most $n - \mathbf{b}(S)$ crossings.*

Proof. We prove simultaneously statements (i) and (ii) by induction on $|S|$. Note that one of the differences between the two statements is the color of the endpoints of the corresponding paths, they have either distinct color (red-blue) or the same color (red-red); obviously one can also consider the blue version of statement (ii) for $|B| = |R| + 1$ and two distinct blue points on the boundary of $CH(S)$. Thus, the three possible combinations of color will be used along the proof to construct, by induction, adequate 1-plane paths on subsets of S and connect them in order to obtain a 1-plane path on S . Assume that $|S| > 3$ (the result is straightforward for $|S| = 2$ or $|S| = 3$).

Proof of statement (i)

Suppose first that there exists a blue point $b' \neq b$ on the boundary of $CH(S \setminus \{r\})$ that is visible from r . Then, by induction, there is a 1-plane path $P_{S \setminus \{r\}}(b', b)$ with at most $n - 1 - \mathbf{r}(S \setminus \{r\})$ crossings. Clearly, $\mathbf{r}(S \setminus \{r\}) \geq \mathbf{r}(S)$ except if $\{r\}$ is a red run that $\mathbf{r}(S \setminus \{r\}) = \mathbf{r}(S) - 1$. Thus, the path $P_S(r, b) = \{rb'\} \cup P_{S \setminus \{r\}}(b', b)$ connects r and b and has at most $n - 1 - (\mathbf{r}(S) - 1) = n - \mathbf{r}(S)$ crossings. Since edge rb' does not intersect $CH(S \setminus \{r\})$, $P_S(r, b)$ is the desired 1-plane path.

The preceding argument also applies if there is a red point $r' \neq r$ on the boundary of $CH(S \setminus \{b\})$ that is visible from b . Thus, we assume the following *visibility property*: all points on the boundary of $CH(S \setminus \{r\})$ visible from r are red and all points on the boundary of $CH(S \setminus \{b\})$ visible from b are blue except for, respectively, b and r if they are consecutive on the boundary of $CH(S)$. We now distinguish two cases according to the position of r and b .

Case 1: points r and b are consecutive on the boundary of $CH(S)$.

From the fact that r and b are consecutive and the visibility property, it follows that there exist a red point r' and a blue point b' consecutive on $CH(S \setminus \{r, b\})$ and visible from both r and b (see Figure 3 left). Since condition (i) of Definition 1 is not satisfied, the configuration $(S \setminus \{r, b\}, b', r')$ is non-special and so, by induction, there is a 1-plane path $P_{S \setminus \{r, b\}}(b', r')$ with at most $n - 1 - \mathbf{r}(S \setminus \{r, b\})$ crossings. Edges rb' and $r'b$ do not intersect that path as their endpoints are visible, which implies that $P_S(r, b) = \{rb'\} \cup P_{S \setminus \{r, b\}}(b', r') \cup \{r'b\}$ is a 1-PHAP on S with endpoints r and b .

The red run of S containing point r has more than one point (note that r only sees b and red points) and so it remains in $S \setminus \{r, b\}$; an analogous argument follows for b . Therefore, $\mathbf{r}(S \setminus \{r, b\}) \geq \mathbf{r}(S)$ which upper bounds the number of crossings in $P_S(r, b)$ by $n - 1 - \mathbf{r}(S) + 1 = n - \mathbf{r}(S)$ (edges rb' and $r'b$ might intersect).

Case 2: points r and b are non-consecutive on the boundary of $CH(S)$.

By the visibility property, the two neighbors of r on the boundary of $CH(S)$ are red points (blue points for b), and so we can apply Lemma 1 around r or b to partition S . Since the configuration (S, r, b) is non-special, condition (ii) or (iii) of Definition 1 is not satisfied for some partition; assume, without loss of generality, that $p_{i+1} \neq b$ in the counterclockwise partition of S around r .

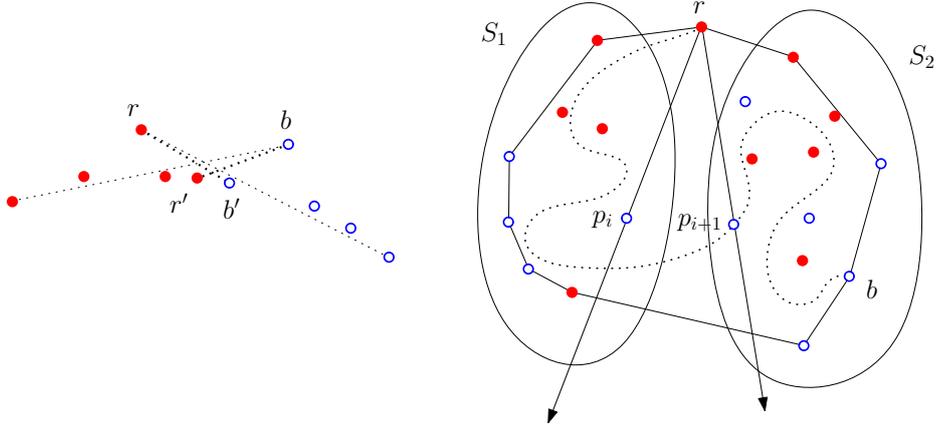


Figure 3: Left: r and b are consecutive points on the boundary of $\text{CH}(S)$. Right: r and b are not consecutive points.

Recall that p_i and p_{i+1} are consecutive blue points in the counterclockwise radial order around r , as specified in Lemma 1.

Suppose that $b \in S_2$ (a symmetric construction can be done for $b \in S_1$); see Figure 3 right. By induction, there are 1-plane paths $P_{S_1 \cup \{r, p_{i+1}\}}(r, p_{i+1})$ and $P_{S_2}(p_{i+1}, b)$ with at most $n_1 + 1 - \mathbf{r}(S_1 \cup \{r, p_{i+1}\})$ and $n_2 - \mathbf{r}(S_2)$ crossings, respectively. Connecting both 1-plane paths, we obtain a path $P_S(r, b)$ that is also 1-plane since $\text{CH}(S_1 \cup \{r, p_{i+1}\})$ and $\text{CH}(S_2)$ do not intersect. Further, $n = n_1 + n_2 + 1$ and so it has at most $n - \mathbf{r}(S_1 \cup \{r, p_{i+1}\}) - \mathbf{r}(S_2)$ crossings.

Each red run of S is also a red run of $S_1 \cup \{r, p_{i+1}\}$ or a red run of S_2 , and the red run of S containing point r is counted twice, one in $S_1 \cup \{r, p_{i+1}\}$ and the other in S_2 . Thus $\mathbf{r}(S_1 \cup \{r, p_{i+1}\}) + \mathbf{r}(S_2) \geq \mathbf{r}(S) + 1$, which upper bounds the number of crossings in $P_S(r, b)$ by $n - \mathbf{r}(S) - 1$.

Proof of statement (ii)

Let C_r be the set of points of $\text{CH}(S \setminus \{r\})$ that are visible from r , and suppose first that it contains red and blue points. Then, there are two consecutive points $b \in B$ and $r'' \in R$ on the boundary of $\text{CH}(S \setminus \{r\})$, and so the configuration $(S \setminus \{r\}, b, r')$ is non-special (point b has a red neighbor). Hence, by induction, one can construct a 1-plane path $P_{S \setminus \{r\}}(b, r')$ with at most $n - \mathbf{r}(S \setminus \{r\}) = n - \mathbf{b}(S \setminus \{r\})$ crossings.

Since points r and b are visible, we can conclude that the path $P_S(r, r') = \{rb\} \cup P_{S \setminus \{r\}}(b, r')$ is a 1-PHAP on S with endpoints r, r' and at most $n - \mathbf{b}(S \setminus \{r\}) \leq n - \mathbf{b}(S)$ crossings (this inequality comes from the fact that C_r contains points of both colors).

The preceding argument also applies if the set $C_{r'}$ of points of $\text{CH}(S \setminus \{r'\})$ visible from r' contains red and blue points. Therefore, we may assume that C_r and $C_{r'}$ consist of points of the same color, and distinguish two cases according to it.

Case 3: C_r or $C_{r'}$ or both consist of only red points.

Suppose, without loss of generality, that C_r consists of red points. Let r_1 and r_2 be the neighbors of r on the boundary of $\text{CH}(S)$, clockwise. Since r_1 is a red point, there is a counterclockwise partition $S_1 \cup S_2$ of $S \setminus \{r\}$ around r as described in Lemma 1. Moreover, points p_i and p_{i+1} (also specified in Lemma 1) are consecutive in the radial order around r , and so rays rp_i and rp_{i+1} cross the same edge uv of the boundary of $\text{CH}(S)$, with $u \in S_1$ and $v \in S_2$. Assume that $r' \in S_2$ (analogous for $r' \in S_1$), and consider two cases depending on whether (S_2, p_{i+1}, r') is a special configuration or not.

Case 3.1: the configuration (S_2, p_{i+1}, r') is non-special.

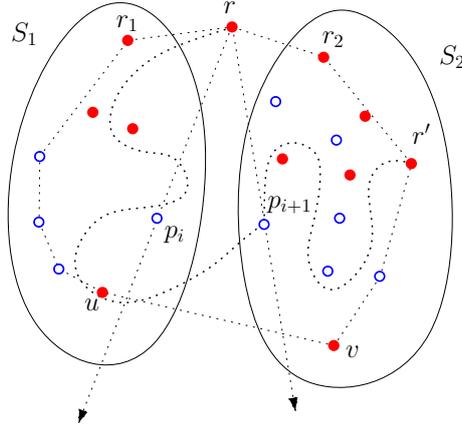


Figure 4: The configuration (S_2, p_{i+1}, r') is non-special.

By induction, there are 1-plane paths $P_{S_2}(p_{i+1}, r')$ and $P_{S_1 \cup \{r, p_{i+1}\}}(r, p_{i+1})$ with at most $n_2 - \mathbf{r}(S_2)$ and $n_1 + 1 - \mathbf{r}(S_1 \cup \{r, p_{i+1}\})$ crossings, respectively. See Figure 4. The connection of both paths gives rise to a path on S with endpoints r and r' that, in addition, is 1-plane since $\text{CH}(S_2)$ and $\text{CH}(S_1 \cup \{r, p_{i+1}\})$ do not intersect. Moreover, the number of crossings in that path is at most

$$n + 1 - \mathbf{b}(S_2) - \mathbf{b}(S_1 \cup \{r, p_{i+1}\})$$

as $\mathbf{r}(S_2) = \mathbf{b}(S_2)$, $\mathbf{r}(S_1 \cup \{r, p_{i+1}\}) = \mathbf{b}(S_1 \cup \{r, p_{i+1}\})$ and $n_1 + n_2 = n$. Therefore, it remains to prove that $\mathbf{b}(S_2) + \mathbf{b}(S_1 \cup \{r, p_{i+1}\}) \geq \mathbf{b}(S) + 1$.

If point u is red and $p_{i+1} \neq v$, then p_{i+1} is a blue run of $S_1 \cup \{r, p_{i+1}\}$ but not of S . Further, if $p_{i+1} = v$ then it is counted twice as a blue run, one in $S_1 \cup \{r, p_{i+1}\}$ and the other in S_2 . The situation is analogous if v is red. If u and v are both blue, then the blue run of S containing them is counted twice. Hence, $\mathbf{b}(S_2) + \mathbf{b}(S_1 \cup \{r, p_{i+1}\}) \geq \mathbf{b}(S) + 1$.

Case 3.2: the configuration (S_2, p_{i+1}, r') is special.

Since all points of C_r are red, there exist two consecutive red points $r'_1, r'_2 \in C_r$ such that $r'_1 \in S_1$ and $r'_2 \in S_2$. Let $C_1 = \{p_i = q_1, q_2, \dots, q_k = r'_1\}$ be the sequence of consecutive points on the boundary of $\text{CH}(S_1)$ ordered counterclockwise from $p_i = q_1$ to $r'_1 = q_k$ (see Figure 5). By construction, all these points are visible from r . Let q_{j+1} be the first red point of C_1 , which exists since $p_i = q_1$ is blue and $r'_1 = q_k$ is red. Analogously, let $C_2 = \{p_{i+1} = q'_1, q'_2, \dots, q'_{k'} = r'_2\}$ be the sequence of consecutive points on the boundary of $\text{CH}(S_2)$ ordered clockwise from $p_{i+1} = q'_1$ to $r'_2 = q'_{k'}$, and let $q'_{j'+1}$ be the first red point of C_2 .

By construction, one of the tangent points from q'_1 to S_1 , say q , belongs to C_1 and, for any other point of C_2 , its corresponding tangent point is also q or a point of C_1 placed after q counterclockwise. Let p be the tangent point of C_1 from $q'_{j'}$ to S_1 .

Suppose first that point p is red (see Figure 5 left). By induction, there are 1-plane paths $P_{S_1}(q_j, p)$ and $P_{S_2}(q'_{j'}, r')$ with at most $n_1 - \mathbf{b}(S_1)$ and $n_2 - \mathbf{b}(S_2)$ crossings, respectively; note that the corresponding configurations are non-special since points q_j and $q'_{j'}$ are blue with red neighbors q_{j+1} and $q'_{j'+1}$. Therefore,

$$P_S(r, r') = \{rq_j\} \cup P_{S_1}(q_j, p) \cup \{pq'_{j'}\} \cup P_{S_2}(q'_{j'}, r')$$

is a 1-PHAP on S with endpoints r, r' and at most $n_1 - \mathbf{b}(S_1) + n_2 - \mathbf{b}(S_2) + 1$ crossings. Observe that $\text{CH}(S_1)$ and $\text{CH}(S_2)$ do not intersect, and by construction, edges $pq'_{j'}$ and rq_j only produce one crossing that is their intersection point located outside of those convex hulls, and so one can guarantee the 1-plane character of $P_S(r, r')$. That $n - \mathbf{b}(S)$ is an upper bound for the number of crossings comes from the fact that $\mathbf{b}(S_2) + \mathbf{b}(S_1) \geq \mathbf{b}(S) + 1$ (analogous to case 3.1) and $n = n_1 + n_2$.

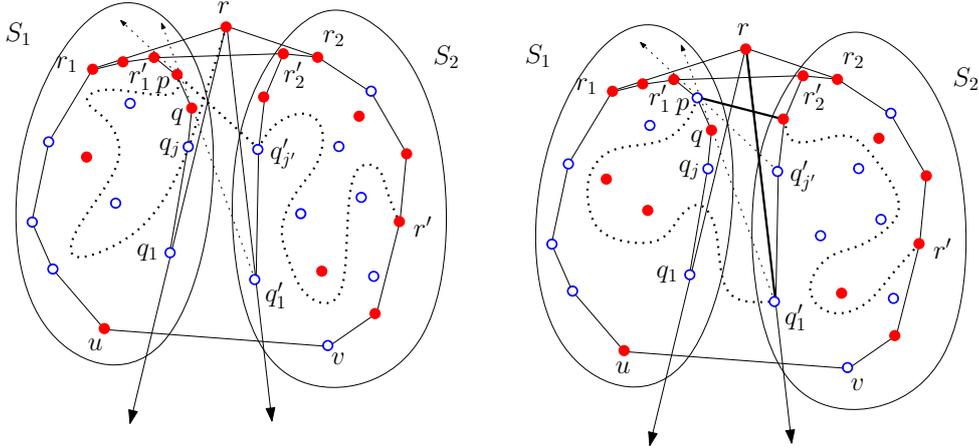


Figure 5: (S_2, p_{i+1}, r') is a special configuration. Left: point p is red. Right: point p is blue.

Assume now that point p is blue (see Figure 5 right). Point $q'_{j'+1} \in \text{CH}(S_2)$ is red and has a blue neighbor $q'_{j'}$, then $q'_{j'+1} \neq r'$ since the configuration (S_2, p_{i+1}, r') is special. Hence, by induction, there are 1-plane paths $P_{S_2 \setminus \{q'_1\}}(q'_{j'+1}, r')$ and $P_{S_1 \cup \{q'_1\}}(q'_1, p)$ with at most $n_2 - 1 - \mathbf{b}(S_2 \setminus \{q'_1\})$ and $n_1 - \mathbf{r}(S_1 \cup \{q'_1\})$ crossings, respectively. A 1-plane path $P_S(r, r')$ is obtained by connecting the two paths via the edges rq'_1 and $pq'_{j'+1}$, that is

$$P_S(r, r') = \{rq'_1\} \cup P_{S_1 \cup \{q'_1\}}(q'_1, p) \cup \{pq'_{j'+1}\} \cup P_{S_2 \setminus \{q'_1\}}(q'_{j'+1}, r').$$

Again, the 1-plane character is guaranteed since $\text{CH}(S_1 \cup \{q'_1\})$ and $\text{CH}(S_2 \setminus \{q'_1\})$ do not intersect, and edges rq'_1 and $pq'_{j'+1}$ only produce one crossing (their intersection point) that is located outside the convex hulls as the endpoints of the edges are visible.

The number of crossings in $P_S(r, r')$ is at most

$$n_2 - 1 - \mathbf{b}(S_2 \setminus \{q'_1\}) + n_1 - \mathbf{r}(S_1 \cup \{q'_1\}) + 1$$

which is $n - \mathbf{b}(S_2 \setminus \{q'_1\}) - \mathbf{b}(S_1 \cup \{q'_1\})$; note that $\text{CH}(S_1 \cup \{q'_1\})$ is bichromatic, and so $\mathbf{r}(S_1 \cup \{q'_1\}) = \mathbf{b}(S_1 \cup \{q'_1\})$. Thus, it remains to prove that $\mathbf{b}(S_2 \setminus \{q'_1\}) + \mathbf{b}(S_1 \cup \{q'_1\}) \geq \mathbf{b}(S)$.

Since (S_2, r', p_{i+1}) is special, the two neighbors of $p_{i+1} = q'_1$ on the boundary of $\text{CH}(S_2)$ are blue. Hence, either q'_1 produces a blue run in $S_1 \cup \{q'_1\}$ (if u is red), or the neighbors of q'_1 produce a blue run in $S_2 \setminus \{q'_1\}$ (if v is red), or the blue run of S containing points u and v is counted twice (if u and v both are blue). In any case, $\mathbf{b}(S_2 \setminus \{q'_1\}) + \mathbf{b}(S_1 \cup \{q'_1\}) > \mathbf{b}(S)$.

Case 4: Both C_r and $C_{r'}$ consist of blue points.

Let $b \in C_r$. The configuration $(S \setminus \{r\}, b, r')$ is non-special ($C_{r'}$ consists of blue points and so r' has a blue neighbor on the boundary of $\text{CH}(S \setminus \{r\})$) then, by induction, there is a 1-plane path $P_{S \setminus \{r\}}(b, r')$ with at most $n - \mathbf{b}(S \setminus \{r\})$ crossings. One might extend this path by the edge rb but this would give $n - \mathbf{b}(S) + 1$ as upper bound for the number of crossings. Indeed, the two neighbors of r on the boundary of $\text{CH}(S)$ are blue points, say b_1, b_2 (clockwise), and so the two blue runs of S containing, respectively, points b_1 and b_2 produce a unique blue run in $S \setminus \{r\}$. Thus $\mathbf{b}(S \setminus \{r\}) = \mathbf{b}(S) - 1$. To reach the upper bound $n - \mathbf{b}(S)$, we provide a more elaborated construction by distinguishing two cases.

Case 4.1: A red point $r'' \in \text{CH}(S \setminus \{r, b_2\})$ is visible from b_2 .

By induction, there is a 1-plane path $P_{S \setminus \{r, b_2\}}(r'', r')$ with at most $n - 1 - \mathbf{b}(S \setminus \{r, b_2\})$ crossings. Thus, $P_S(r, r') = \{rb_2\} \cup \{b_2r''\} \cup P_{S \setminus \{r, b_2\}}(r'', r')$ is clearly a 1-PHAP on S that also has at most $n - 1 - \mathbf{b}(S \setminus \{r, b_2\})$ crossings. See Figure 6.

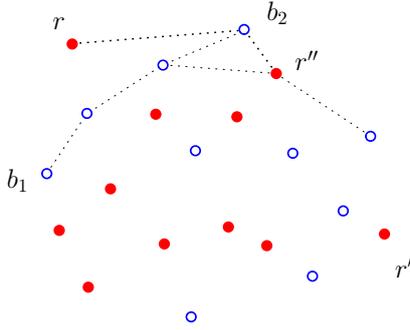


Figure 6: A red point $r'' \in \text{CH}(S \setminus \{r, b_2\})$ is visible from b_2 .

Since r'' is on the boundary of $\text{CH}(S \setminus \{r, b_2\})$ then $\mathbf{b}(S \setminus \{r, b_2\}) \geq \mathbf{b}(S \setminus \{r\}) = \mathbf{b}(S) - 1$. Therefore, the number of crossings in $P_S(r, r')$ is at most $n - 1 - (\mathbf{b}(S) - 1) = n - \mathbf{b}(S)$.

Case 4.2: Every point of $\text{CH}(S \setminus \{r, b_2\})$ visible from b_2 is blue.

By Lemma 1(i) on $S \setminus \{r\}$, there is a counterclockwise partition $S_1 \cup S_2$ of $S \setminus \{r, b_2\}$ around b_2 , where p_i and p_{i+1} are red. Suppose first that $r' \neq p_{i+1}$, and assume that $r' \in S_1$ (a symmetric construction can be done for $r' \in S_2 \setminus \{p_{i+1}\}$); see Figure 7 left.

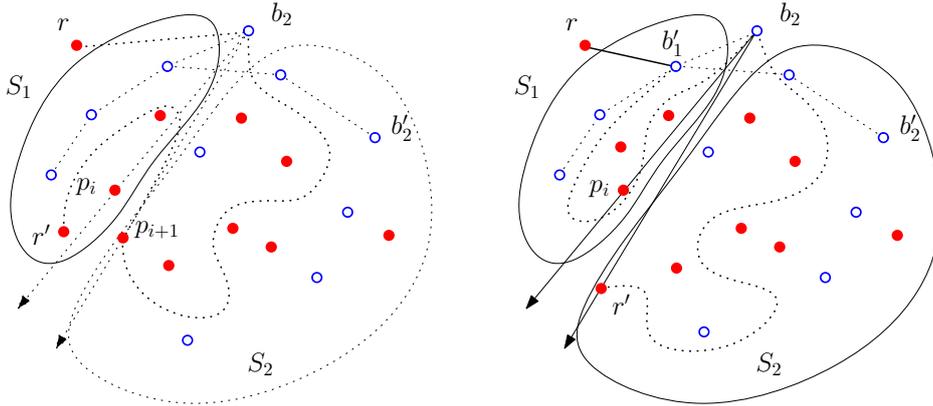


Figure 7: Every point of $\text{CH}(S \setminus \{r, b_2\})$ visible from b_2 is blue.

Points b_2 and p_{i+1} are neighbors on $\text{CH}(S_2 \cup \{b_2\})$ and have distinct color, then the configuration $(S_2 \cup \{b_2\}, p_{i+1}, b_2)$ is non-special and, by induction, there is a 1-plane path $P_{S_2 \cup \{b_2\}}(b_2, p_{i+1})$ with at most $n_2 + 1 - \mathbf{b}(S_2 \cup \{b_2\})$ crossings, where $n_2 = |(B \setminus \{b_2\}) \cap S_2|$. Again by induction, there is a 1-plane path $P_{S_1 \cup \{p_{i+1}\}}(p_{i+1}, r')$ with at most $n_1 - \mathbf{b}(S_1 \cup \{p_{i+1}\})$ crossings, where $n_1 = |(B \setminus \{b_2\}) \cap S_1|$. Thus,

$$P_S(r, r') = \{rb_2\} \cup P_{S_2 \cup \{b_2\}}(b_2, p_{i+1}) \cup P_{S_1 \cup \{p_{i+1}\}}(p_{i+1}, r')$$

is a 1-PHAP with endpoints r and r' and at most $n - \mathbf{b}(S_2 \cup \{b_2\}) - \mathbf{b}(S_1 \cup \{p_{i+1}\})$ crossings (note that $n = n_1 + n_2 + 1$). The same arguments as those of case 3 prove that $P_S(r, r')$ is 1-plane. With respect to the number of crossings, the bound of $n - \mathbf{b}(S)$ is obtained from the fact that

$$\mathbf{b}(S_2 \cup \{b_2\}) + \mathbf{b}(S_1 \cup \{p_{i+1}\}) \geq \mathbf{b}((S \setminus \{r\}) + 1 = \mathbf{b}(S),$$

note that the blue run of $S \setminus \{r\}$ containing point b_2 is counted twice, one in $S_2 \cup \{b_2\}$ and the other in $S_1 \cup \{p_{i+1}\}$.

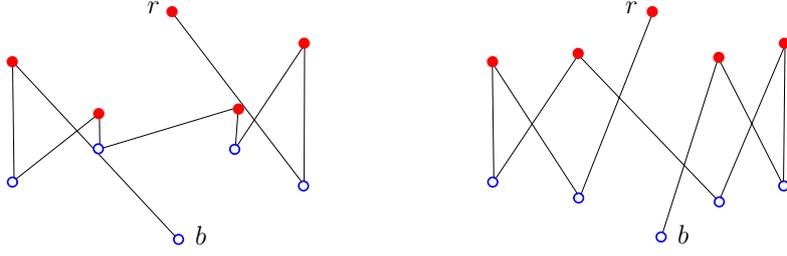


Figure 8: A special configuration (S, r, b) that: (left) admits a 1-PHAP, (right) does not admit a 1-PHAP.

Suppose now that $r' = p_{i+1}$ (see Figure 7 right). Let b'_1 be the previous point (clockwise) to b_2 on the boundary of $\text{CH}(S \setminus \{r\})$ which, by construction, is blue and visible from r . The desired 1-PHAP on S is

$$P_S(r, r') = \{rb'_1\} \cup P_{S_1 \cup b_2}(b'_1, b_2) \cup P_{S_2 \cup b_2}(b_2, r')$$

where $P_{S_1 \cup b_2}(b'_1, b_2)$ and $P_{S_2 \cup \{b_2\}}(b_2, r')$ are 1-PHAP obtained by induction on $S_1 \cup \{b_2\}$ and $S_2 \cup \{b_2\}$, respectively; note that r' and b_2 are neighbors of distinct color on $\text{CH}(S_2 \cup \{b_2\})$ which implies that the configuration $(S_2 \cup \{b_2\}, b_2, r')$ is non-special. The number of crossings in $P_S(r, r')$ is at most $n - \mathbf{b}(S)$ since again the blue run of $S \setminus \{r\}$ containing b_2 is counted twice. \square

Remark 1 One cannot guarantee the existence of a 1-PHAP for a special configuration. For instance, Figure 8 shows two similar special configurations: the one on the left admits a 1-PHAP and the other (right) does not. In this last configuration (which is also shown in Figure 1 as an example of special configuration), at least one of the edges must be crossed twice in any Hamiltonian alternating path because the points are essentially in convex position, and for point sets in convex position, the only configurations that do not admit a 1-PHAP are precisely the special configurations, as we prove in the next section (see Theorem 4).

We are now ready to prove our main result in this section.

Theorem 1 *For given $S = R \cup B$ with $|B| = |R| = n \geq 2$, there exists a 1-plane Hamiltonian alternating cycle on S with at most $n - \max\{\mathbf{r}(S), \mathbf{b}(S)\}$ crossings.*

Proof. Suppose first that $\text{CH}(S)$ is bichromatic (which implies $\mathbf{r}(S) = \mathbf{b}(S)$) and consider two consecutive points of distinct color, say a red point r and a blue point b . Thus, the configuration (S, r, b) is non-special and, by Lemma 2, there is a 1-PHAP on S with endpoints r and b and at most $n - \mathbf{r}(S)$ crossings. By adding the edge rb , the desired cycle is obtained.

Assume now that the boundary of $\text{CH}(S)$ consists of points of only one color, say red ($\mathbf{r}(S) = 1$ and $\mathbf{b}(S) = 0$), and let r be one of those points. Partition counterclockwise $S \setminus \{r\}$ around r as described in Lemma 1(i) obtaining the sets S_1, S_2 and the point p_{i+1} .

By Lemma 2, there are 1-plane paths $P_{S_1 \cup \{r, p_{i+1}\}}(r, p_{i+1})$ and $P_{S_2 \cup \{r\}}(p_{i+1}, r)$ with at most $n_1 + 1 - \mathbf{r}(S_1 \cup \{r, p_{i+1}\})$ and $n_2 + 1 - \mathbf{r}(S_2 \cup \{r\})$ crossings, respectively. Connecting both paths, we construct a 1-PHAC on S with at most

$$n_1 + n_2 + 2 - \mathbf{r}(S_1 \cup \{r, p_{i+1}\}) - \mathbf{r}(S_2 \cup \{r\})$$

crossings. Since $n_1 + n_2 + 1 = n$ and, in the worst case $\mathbf{r}(S_1 \cup \{r, p_{i+1}\}) = \mathbf{r}(S_2 \cup \{r\}) = 1$, the cycle has at most $n - 1$ crossings. \square

Remark 2 Theorem 1 can also be proved by using the Ham-sandwich theorem and Lemma 2. Further, the bound of $n - \max\{\mathbf{r}(S), \mathbf{b}(S)\}$ for the number of crossings is tight. Indeed, Kaneko et al. [14] proved that, for a point set S consisting of $2n$ points in convex position, n consecutive

red points and n consecutive blue points, any Hamiltonian alternating cycle has at least $n - 1$ crossings. However, there are configurations of points for which this bound is far to be tight. For instance, if the n red points are on the boundary of $\text{CH}(S)$ and the n blue points are inside $\text{CH}(S)$, then it is not difficult to prove that there always exists a Hamiltonian alternating cycle on S with no crossings (see Lemma 3.4 of [10]), and the bound of Theorem 1 would be $n - 1$.

To conclude this section, we show that Lemma 2 and Theorem 1 let us compute a 1-PHAC on $S = R \cup B$ (not necessarily minimizing the number of crossings) in $O(n^2)$ time and space where, as usual, $n = |R| = |B|$.

An $O(n^2 \log n)$ time and $O(n)$ space algorithm for this problem is clear, as Kaneko et al. pointed out in [14]: each time that one (or two) edges are assigned to the cycle, we must compute a convex hull of one (or two) subset, and the radial order of the points of a given subset around a given point r . In principle, computing a convex hull and a radial order can be done in $O(n \log n)$ time and, since this is done $O(n)$ times, the overall complexity is $O(n^2 \log n)$ time and $O(n)$ space. However, these two operations can be done more efficiently as follows.

All the radial orders around the $2n$ points of S can be obtained in $O(n^2)$ time and space (see for instance [17]). Thus, in a generic step of the above algorithm, computing the radial order of the points of a subset S' around a point r requires only linear time.

Once $\text{CH}(S')$ has been computed, its updates after point deletions can be done in $O(\log n)$ amortized time per deletion, using the semi-dynamic data structure of Hershberger and Suri [11]. Therefore, computing the different convex hulls in the overall process only requires $O(n \log n)$. Moreover, the same semi-dynamic data structure allows to compute a tangent from a point outside $\text{CH}(S')$ with the same complexity, $O(\log n)$ amortized time. Thus, we obtain the following corollary.

Corollary 1 *For given $S = R \cup B$ with $|B| = |R| = n \geq 2$, a 1-plane Hamiltonian alternating cycle on S can be computed in $O(n^2)$ time and space.*

3 1-PHAC and 1-PHAP for point sets in convex position

In this section, we study 1-plane hamiltonian alternating cycles and paths for a well-known restricted position of the point set: the convex position. Thus, hereafter the set $S = R \cup B$ is assumed to be in convex position, unless otherwise stated.

3.1 1-plane Hamiltonian alternating cycles

We first prove that every Hamiltonian alternating cycle on S has at least $n - \max\{\mathbf{r}(S), \mathbf{b}(S)\} = n - \mathbf{r}(S)$ crossings (note that $\mathbf{r}(S) = \mathbf{b}(S)$), and moreover, only those cycles that are 1-plane attain the bound. We shall write *optimum* for a path or cycle with minimum number of crossings. The two endpoints of a run of S are called *limits*, and a *bridge* of S is a bichromatic edge that connects consecutive runs of S , i.e., its endpoints are the two consecutive points of S that are limits of those runs.

Theorem 2 *For given $S = R \cup B$ in convex position with $|R| = |B| = n \geq 2$, the following statements hold.*

- (i) *Every Hamiltonian alternating cycle on S has at least $n - \mathbf{r}(S)$ crossings.*
- (ii) *A Hamiltonian alternating cycle on S with $n - \mathbf{r}(S)$ crossings is 1-plane.*
- (iii) *Every bridge of S belongs to any 1-PHAC on S with $n - \mathbf{r}(S)$ crossings.*

Proof. Let C be a Hamiltonian alternating cycle on S . Given a red run of S of cardinality k , there are at most two edges of C that are bridges of S and have an endpoint in the run. Thus, the remaining edges of C with an endpoint in that red run, at least $2k - 2$, are necessarily chords.

Extending this argument to every red run of S , we obtain that at least $2n - 2\mathbf{r}(S)$ edges of C are chords. Further, every chord must be crossed at least once to connect a point to the left to a point to the right of the chord. Since a crossing requires two chords and a chord must be crossed at least once, then the number of crossings in C is at least $n - \mathbf{r}(S)$; this proves statement (i). Moreover, if the number of crossings is precisely $n - \mathbf{r}(S)$, then every chord is crossed exactly once and every bridge of S is an edge of C . Thus, statements (ii) and (iii) follow. \square

Remark 3 The converse of statement (ii) in the preceding theorem is not true: it is possible to construct a 1-PHAC on a set S with more than $n - \mathbf{r}(S)$ crossings. For instance, if S consists of points alternating in color then $\mathbf{r}(S) = n$, and obviously one can draw a 1-PHAC without crossings, but also a 1-PHAC with $\lceil \frac{n}{2} \rceil$ or $\lceil \frac{n}{2} \rceil + 1$ crossings (depending on the parity of n). Moreover, the number of different 1-plane Hamiltonian alternating cycles on a point set can be exponential. Indeed, if S consists of points alternating in color, for every subset $P = \{p_j, p_{j+1}, p_{j+2}, p_{j+3}\}$ of four consecutive points we can connect the points either in the same order $p_j, p_{j+1}, p_{j+2}, p_{j+3}$ or in the order $p_j, p_{j+3}, p_{j+2}, p_{j+1}$. Now, divide S into $\frac{n}{2}$ groups of four consecutive points each, and connect each group to the next one, clockwise, to form a cycle: either using edge $p_{j+3}p_{j+4}$ or edge $p_{j+1}p_{j+4}$. Clearly, the resulting cycle is 1-plane; the only possible crossings correspond to edges of type p_jp_{j+3} and $p_{j+1}p_{j+4}$. Since for every group there are two options to connect its points, then there are $\Omega(2^{\frac{n}{2}})$ different 1-plane Hamiltonian alternating cycles on S .

Observe that, by removing a bridge of an optimum 1-PHAC, we obtain a 1-PHAP whose endpoints are two consecutive points of S of different colors. Hence, an optimum 1-PHAP connecting two consecutive points of different colors inherits the properties of an optimum 1-PHAC. When $|R| = |B| + 1 = n + 1$, a similar situation happens for an optimum 1-PHAP on S with two consecutive red points r and r' as endpoints; one can add a dummy blue point between r and r' so that the path can be completed to a cycle. Thus, we have the following corollary.

Corollary 2 *For given $S = R \cup B$ in convex position, the following statements hold.*

- (i) *Let $r \in R$ and $b \in B$ be consecutive points on S . If $|R| = |B| = n \geq 2$, then every optimum Hamiltonian alternating path on S with endpoints r and b has $n - \mathbf{r}(S)$ crossings, is 1-plane and uses all bridges of S except for rb .*
- (ii) *Let $|R| = |B| + 1 = n + 1 \geq 2$ and let $r, r' \in R$ be consecutive red points on S . Then every optimum Hamiltonian alternating path on S with endpoints r and r' has $n - \mathbf{b}(S)$ crossings, is 1-plane and uses all bridges of S .*

We now prove that an optimum 1-PHAC on a set S in convex position can be computed in $O(n)$ time and space. Recall that, by Corollary 1, to find a 1-PHAC for a set in general position takes $O(n^2)$ time and space, and the cycle in that case is not necessarily optimum.

We begin by describing a linear procedure to compute, for every red point, the first clockwise blue point such that, between them, there are the same number of red and blue points. For simplicity in the arguments, in the explanation below, we modify the notation used in the proof of Lemma 2 for the endpoints of the paths: they were called r, b, r' , etc, and now they will be denoted as p_i, p_j , etc. We shall go back to that original notation to state Theorem 3.

Consider the clockwise circular order $\langle p_1, \dots, p_{2n} \rangle$ of the points of S , where $|R| = |B|$. Let $S[p_i, p_j] = \{p_i, p_{i+1}, \dots, p_j\}$ (clockwise) and $S(p_i, p_j) = S[p_i, p_j] \setminus \{p_i, p_j\}$. For each red point p_i , let $p_{J(i)}$ be the first point of S such that $|S[p_i, p_{J(i)}] \cap R| = |S[p_i, p_{J(i)}] \cap B|$.

By Lemma 1 (in its clockwise version), if p_{i+1} is blue then $p_{J(i)} = p_{i+1}$, otherwise $p_{J(i)-1}$ and $p_{J(i)}$ are both blue. Note that $J(i) \neq J(i')$ whenever $i \neq i'$, and so the mapping that assigns to each value of i the value $J(i)$ is bijective.

The following procedure computes point $p_{J(i)}$ for each red point p_i .

PROCEDURE: J -PAIRS

Input: $S = B \cup R$ in convex position, $|B| = |R| = n$

Output: All pairs $(p_i, p_{J(i)})$ for $p_i \in R$

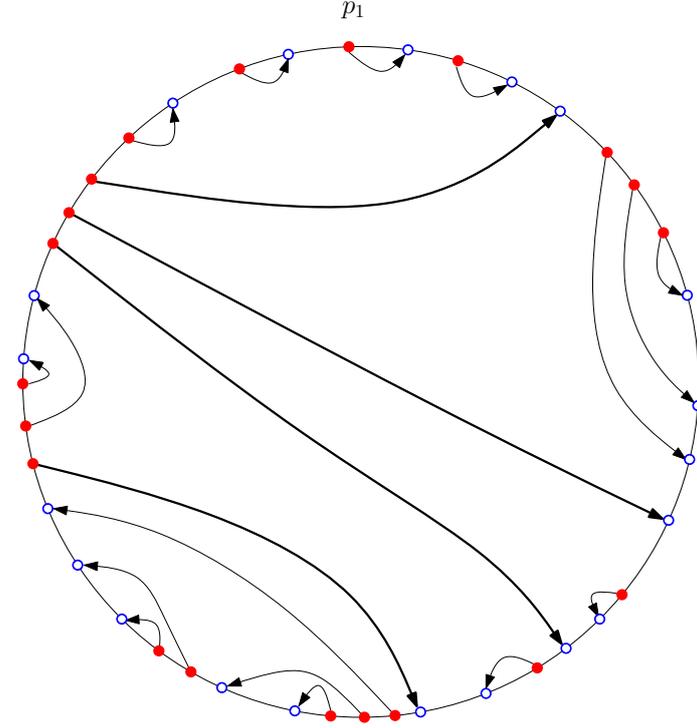


Figure 9: Computing pairs $(p_i, p_{J(i)})$.

1. Consider two sorted lists of red and blue points: L_R is the red list and L_B is the blue list. Initially both lists are empty.
2. Explore the point set $S = \langle p_1, \dots, p_{2n} \rangle$ given in clockwise circular order:
 - (a) If the explored point p_j is red, add it at the end of list L_R .
 - (b) If the explored point p_j is blue and $L_R \neq \emptyset$, assign p_j to the last red point of list L_R and remove that red point from L_R .
 - (c) If the explored point p_j is blue and $L_R = \emptyset$, add p_j at the end of list L_B .
3. When step 2 has finished, L_R and L_B have the same number of points. Assign the remaining points of L_B to the remaining points of L_R in reverse order, that is, the first point of L_B is assigned to the last point of L_R . Remove both points from the lists, and repeat the assignation last-first until both lists are empty.

END PROCEDURE

Figure 9 illustrates the assignation of points described in the above procedure. The assignments done in step 3 (with the points that remain in lists L_R and L_B) are represented by a thicker line than those of step 2.

Lemma 3 PROCEDURE *J-PAIRS* computes all pairs $(p_i, p_{J(i)})$ with $p_i \in R$ in $\Theta(n)$ time.

Proof. Observe first that, in step 2, a point in list L_B is never removed from that list, and it can be included in L_B only if $L_R = \emptyset$; a point in list L_R can be removed from the list, but in that case it is never inserted again.

We next prove that if p_j is assigned to p_i in PROCEDURE *J-PAIRS* then $p_j = p_{J(i)}$, that is, $|S[p_i, p_j] \cap B| = |S[p_i, p_j] \cap R|$ and p_j is the first point after p_i , clockwise, satisfying that equality.

Suppose that such result is true for all the assignments done in step 2 before exploring point p_j , and then p_j must be assigned to p_i . This means that p_j is blue and p_i is the last red point in list L_R . Further, all the blue points in $S(p_i, p_j)$ have been assigned to red points in the same set $S(p_i, p_j)$ (otherwise p_i cannot be in list L_R), and all the red points in $S(p_i, p_j)$ have received an assignment of a blue point in $S(p_i, p_j)$ (otherwise they should be in L_R after p_i). Therefore, $|S[p_i, p_j] \cap B| = |S[p_i, p_j] \cap R|$.

Now, if j is not the first index such that $J(i) = j$, but there is an index $j' < j$ so that $J(i) = j'$, then j' would have been previously assigned, i.e., $j' = J(i')$ for some other point $p_{i'}$. This contradicts that J is bijective. Thus, if p_j is assigned to p_i in step 2 then $p_j = p_{J(i)}$.

In step 3, before the last point $p_k \in L_R$ and the first point $p_\ell \in L_B$ (clockwise), all the red and blue points have been assigned. Reasoning as above, $|S[p_k, p_\ell] \cap B| = |S[p_k, p_\ell] \cap R|$ and p_ℓ is the first point after p_k , clockwise, satisfying that equality. Thus, $p_\ell = p_{J(k)}$. If we now remove these two points from the corresponding lists, the new last point of L_R and the new first point of L_B are again in the same conditions, and so the same argument can be applied. \square

If PROCEDURE J -PAIRS is performed counterclockwise, we obtain another blue point $p_{J'(i)}$ for each red point p_i . When $|R| = |B| + 1$, the blue point $p_{J(i)}$ does not exist for one of the red points, but the same procedure computes $p_{J(i)}$ for the remaining red points. Note that, if p_k is the red point for which $p_{J(k)}$ is not defined, then $p_{J'(k)}$ necessarily exists.

The special configuration of Definition 1 (for S in convex position) occurs when $p_{J(i)} = p_{J'(i)}$; points r and b in that definition would be $p_i \in R$ and $p_{J(i)} \in B$. Moreover, if the assignation of PROCEDURE J -PAIRS is done counterclockwise and blue points play the role of red points, we obtain the same assignation as above: the red point p_i is assigned to the blue point $p_{J(i)}$.

With PROCEDURE J -PAIRS in hand, we can compute optimum Hamiltonian alternating cycles or paths on S in $O(n)$ time and space.

Theorem 3 *For given $S = R \cup B$ in convex position, the following statements hold.*

- (i) *An optimum Hamiltonian alternating cycle on S can be computed in $O(n)$ time and space provided that $|R| = |B| = n \geq 2$.*
- (ii) *Let $r \in R$ and $b \in B$ consecutive points on S . If $|R| = |B| = n \geq 2$, then an optimum Hamiltonian alternating path on S with endpoints r and b can be computed in $O(n)$ time and space.*
- (iii) *Let $|R| = |B| + 1 = n + 1 \geq 2$, and let $r, r' \in R$ be consecutive red points on S . Then, an optimum Hamiltonian alternating path on S with endpoints r and r' can be computed in $O(n)$ time and space.*

Moreover, in all cases, the optimum is 1-plane.

Proof. By Theorem 2 and Corollary 2 the optimum is 1-plane in all cases, and proving statements (i) and (ii) is equivalent. Hence, it suffices to show that the process described in Lemma 2 to compute 1-plane Hamiltonian alternating paths can be done in linear time.

Suppose that $|R| = |B| = n$, and assume that points $p_{J(i)}$ have been pre-computed in linear time (regardless of the color of the points p_i) by applying PROCEDURE J -PAIRS twice, one for the red points and the other for the blue points.

For the special case of points in convex position, in a generic step of the process described in the proof of Lemma 2, we only have to compute optimum 1-plane Hamiltonian alternating paths with endpoints p_i and p_j on some subsets of consecutive points $S[p_i, p_j]$.

If p_i and p_j have different colors, in constant time, we can compute one or two edges of that optimum path, and study a smaller subproblem. The same happens when p_i and p_j have the same color, and one of them has a neighbor of different color. Now, if p_i, p_j and their neighbors have the same color, in constant time we can study two subproblems by using point $p_{J(i)}$ that, by Lemma 1, has a different color. Thus, in all cases, the cost of computing each new edge of the

desired optimum path can be done in constant time. This implies that the procedure described in Lemma 2 is $O(n)$.

A similar argument applies when $|R| = |B| + 1$ and the desired optimum 1-PHAP on S has two red consecutive points, $r, r' \in R$, as endpoints. In the pre-processing step, if r' is after r (clockwise), then it suffices to compute $p_{J(i)}$ beginning at r' and $p_{J'(i)}$ beginning at r . \square

3.2 1-plane Hamiltonian alternating paths

We first show that if $S = R \cup B$ is in convex position and $|R| = |B|$, the special configurations (S, r, b) of Definition 1 are the unique configurations of points that do not admit a 1-PHAP with fixed endpoints r and b . Recall that by Lemma 2, in any other case, such a path does exist, and also a 1-PHAP with fixed red endpoints provided that $|R| = |B| + 1$.

In this section, again, to simplify statements and explanations, the notation of Section 2 for the endpoints of the paths will be modified when necessary: r, r', b, b' may be denoted as p, q, p_s, p_t , etc. We begin with a technical lemma, which considers that the 1-PHAP is oriented: if $p_j p_k$ is an edge, point p_j is visited before point p_k .

Lemma 4 *Let $S = R \cup B$ be in convex position with $|R| = |B| \geq 1$ or $|R| = |B| + 1 \geq 2$, and let $\langle p_1, \dots, p_h \rangle$ be the clockwise circular order of its points. Suppose that P is an oriented 1-PHAP on S from $p = p_j$ to q , which contains a chord $p_j p_k$. If q is to the left of the oriented chord $p_j p_k$, then P visits the following points in the given order: (1) p_k , (2) all points of the set $\{p_{k+1}, \dots, p_{j-2}\}$ (regarded cyclically), (3) p_{j-1} , (4) either p_{j+1} or p_{k-1} .*

Proof. The oriented chord $p_j p_k$ splits $S \setminus \{p_j, p_k\}$ into two subsets: S_ℓ containing all points that are to the left of the chord, and $S_r = \{p_{k+1}, p_{k+2}, \dots, p_{j-1}\}$ whose points are to the right of the chord. Suppose that after p_k , path P visits a point in S_ℓ before than a point in S_r . Then, to reach the endpoint $q \in S_\ell$ we must cross twice the chord $p_j p_k$, which contradicts that P is 1-plane. Hence, P visits all points of S_r before crossing the chord $p_j p_k$ to go to a point in S_ℓ . We now prove that the last point visited in S_r is p_{j-1} .

Suppose on the contrary that $p' \in S_r \setminus \{p_{j-1}\}$ is the last visited point by P in S_r . Then, there is a chord $p' q'$ with $q' \in S_\ell$ that crosses $p_j p_k$ and splits S_r into two subsets: points to the left of the chord $p' q'$, and those that are to the right (here we include point p_k). Since P connects both sets (which are non-empty) before crossing the chord $p_j p_k$, there must be another crossing in chord $p' q'$ (besides the intersection point with $p_j p_k$); a contradiction. Thus, $p' = p_{j-1}$. An analogous argument proves that P visits either p_{j+1} or p_{k-1} just after p_{j-1} . \square

Theorem 4 *Let $S = R \cup B$ be in convex position with $|R| = |B| = n \geq 1$. Given $r \in R$ and $b \in B$, there exists a 1-plane Hamiltonian alternating path on S with endpoints r and b if and only if the configuration (S, r, b) is non-special.*

Proof. By Lemma 2, if the configuration (S, r, b) is non-special then the desired path exists. We now prove that there is no 1-PHAP for a special configuration (S, r, b) . Suppose on the contrary that this path, say P , does exist, and consider it to be oriented from r to b .

Following the notation of Lemma 4, P visits a blue point p_k just after $r = p_j$. Assume, without loss of generality, that $b = q$ is to the left of the oriented edge $p_j p_k$.

By Lemma 4, and considering the clockwise circular order $\langle p_1, \dots, p_{2n} \rangle$ of the points of S , path P visits first all points of $\{p_{k+1}, \dots, p_{j-2}\}$, then point p_{j-1} (which is red since (S, r, b) is special), and then point p_{k-1} that is blue (note that point p_{j+1} is also red).

Since the path from p_k to p_{j-1} has endpoints of different color, the set of points from p_k to p_{j-1} (clockwise) contains the same number of red and blue points. Thus, points p_k and p_{k-1} define a partition of $S \setminus \{r\}$ around $p_j = r$, as described in Lemma 1. Further, (S, r, b) is special and so $p_{k-1} = b$. Hence, path P reaches the endpoint b without visiting point p_{k-2} which, by Definition 1 of special configuration, exists and is blue. This contradicts that P is Hamiltonian. \square

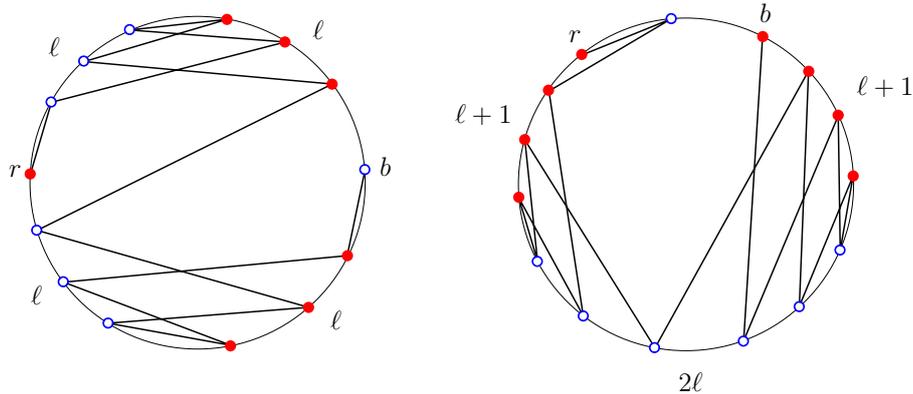


Figure 10: Left: A point configuration that has two runs of cardinality 1 and four runs of cardinality ℓ . It admits no 1-PHAP with endpoints r and b and less than $2\ell - 2$ crossings. Right: A point configuration that has one run of cardinality 1, two runs of cardinality $\ell + 1$, and one run of cardinality 2ℓ . It admits no 1-PHAP with endpoints r and b and less than $2\ell - 2$ crossings.

Remark 4 The bound of $n - \mathbf{r}(S)$ given in Lemma 2 for the number of crossings in a 1-PHAP is also tight when considering point sets S in convex position. Figure 10 illustrates two point configurations in which every 1-PHAP with endpoints r and b has at least $n - \mathbf{r}(S)$ crossings. Thus, by Lemma 2, at least one of those paths has exactly $n - \mathbf{r}(S)$ crossings.

We now turn to a natural problem for Hamiltonian alternating paths on non-special configurations: to determine whether being optimum implies being 1-plane. This is the analogous problem as that considered in Theorem 2(ii) for cycles, but there is a fundamental difference: the path will be assumed to have given endpoints.

We first state a well-known property based on the quadrangular property, and a consequence of it (whose straightforward proof is omitted); see also Figure 11.

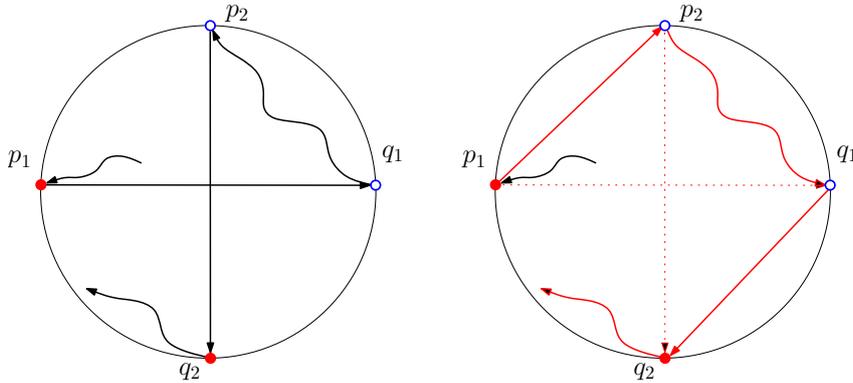


Figure 11: Replacing edges p_1q_1 and p_2q_2 by edges p_1p_2 and q_1q_2 .

Property 1 If G is a geometric graph whose edges pq and $p'q'$ intersect, then the geometric graph obtained by replacing those edges by pp' and qq' (or by pq' and qp') has fewer crossings than G . Even more, the graph resulting from applying this transformation several times has fewer crossings than G .

Corollary 3 Let $S = R \cup B$ be in convex position with $|R| = |B| \geq 1$ or $|R| = |B| + 1 \geq 2$, and let P be an oriented Hamiltonian alternating path on S from p to q . Suppose that p_1q_1 and p_2q_2 are two crossing edges of P such that q_1 and p_2 have the same color. Then, there exists a Hamiltonian alternating path on S from p to q with fewer crossings than P . See Figure 11.

Theorem 5 Let $S = R \cup B$ be in convex position with $|R| = |B| \geq 1$ or $|R| = |B| + 1 \geq 2$, and let $p, q \in S$ such that the configuration (S, p, q) is non-special. Then, every optimum Hamiltonian alternating path on S with endpoints p and q is 1-plane.

Proof. Let Opt be an optimum Hamiltonian alternating path with endpoints p and q . We can consider Opt as a directed path, where an edge $p_i q_j$ is directed from p_i to q_j . Further, to apply symmetry, we complete the abstract graph Opt to a cycle C by adding the final (directed) edge $e = qp$. Edge e is a dummy edge that is not crossed by any other edge (one can think on it as a curve from p to q placed outside $CH(S)$), and its endpoints can have the same color.

Suppose on the contrary that Opt is not 1-plane, i.e., there is at least one edge $e_1 = p_1 q_1$ in Opt that is crossed by two other edges, say $e_2 = p_2 q_2$ and $e_3 = p_3 q_3$.

Assume, without loss of generality, that C begins at point p_1 and then continues with: (1) edge e_1 , (2) a path P_1 from q_1 to p_2 , (3) edge e_2 , (4) a path P_2 from q_2 to p_3 , (5) edge e_3 , (6) a path P_3 from q_3 to p_1 . The dummy edge e may belong to P_1 , P_2 or P_3 .

Suppose that point p_1 is red, and so q_1 is blue (the argument is analogous otherwise). Since e_1 crosses edges e_2, e_3 and Opt is optimum, by Corollary 3, points p_2 and p_3 are also red, and thus q_2 and q_3 are blue. In particular, this implies that P_2 cannot consist of only one point.

The directed edge e_1 splits the points of $S \setminus \{p_1, q_1\}$ into two subsets, points to the left and points to the right of e_1 , respectively. Assume, without loss of generality, that point p_2 is to the left of e_1 (otherwise, totally symmetric cases to those below would appear).

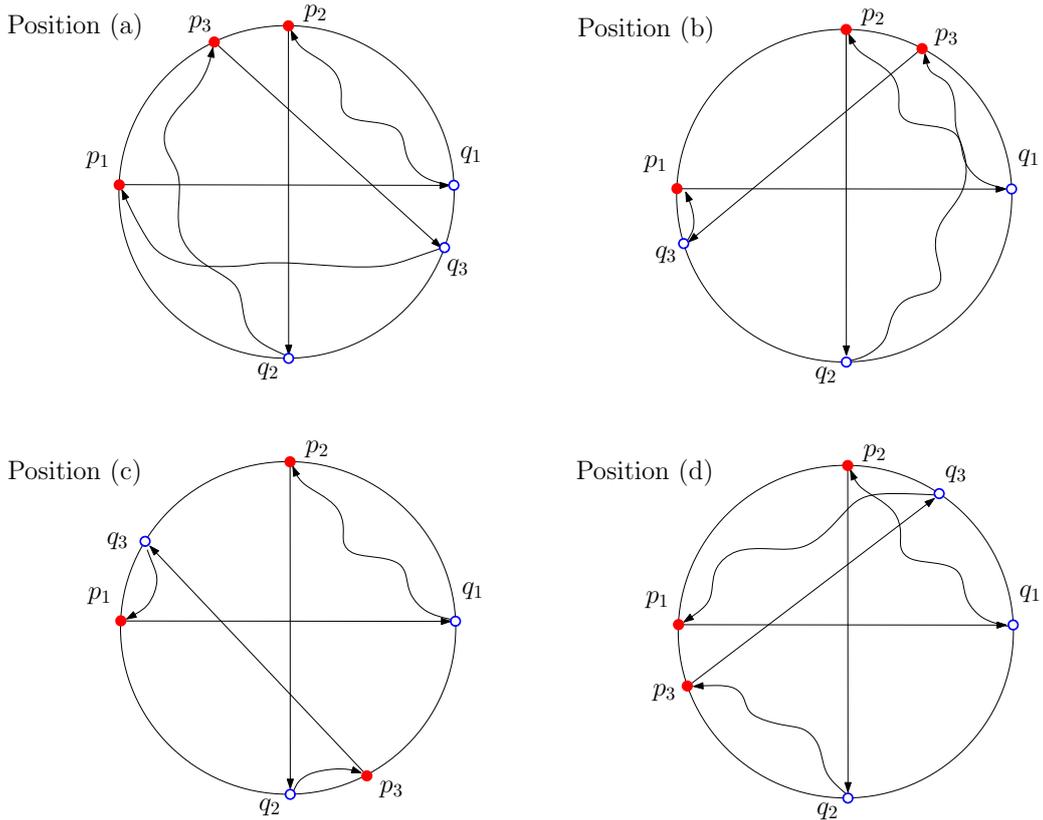


Figure 12: The four positions when e_2 crosses e_3 .

We next distinguish two cases depending on whether e_2 and e_3 intersect or not. Further, for each case, we shall consider four possibilities according to the position of p_3 with respect to p_1, p_2, q_1 and q_2 (clockwise).

- *Position (a)*: p_3 is located between p_1 and p_2 .

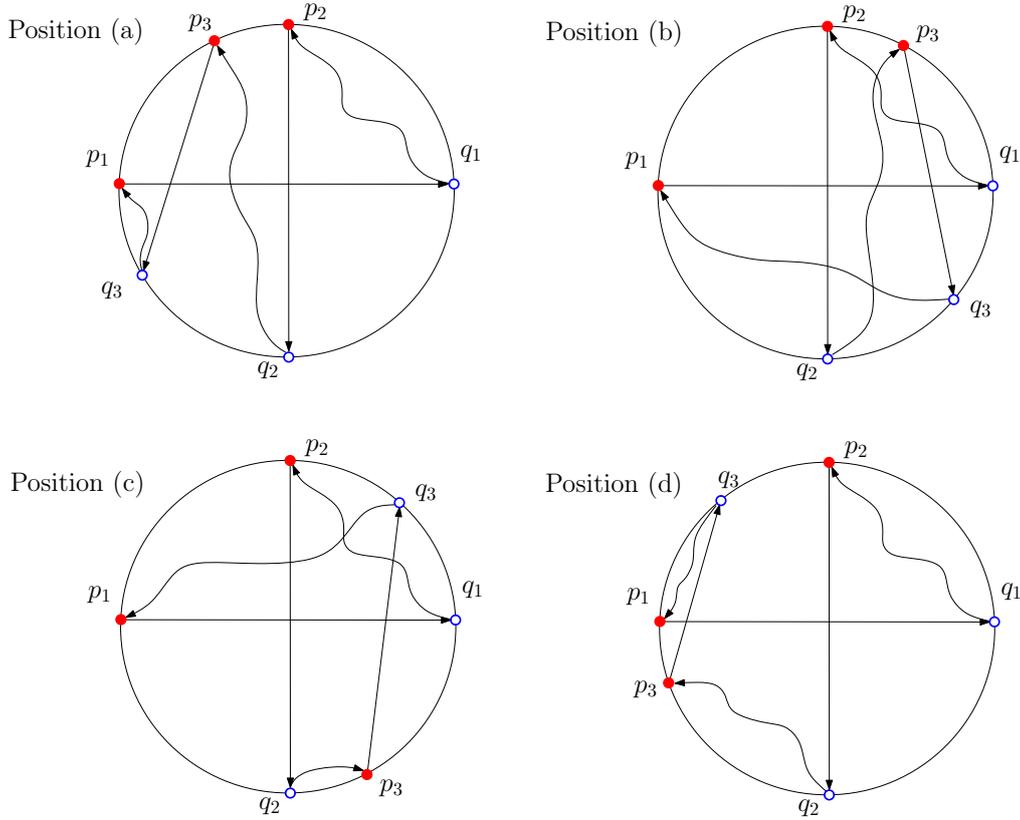


Figure 13: The four positions when e_2 does not cross e_3 .

- *Position (b)*: p_3 is located between p_2 and q_1 .
- *Position (c)*: p_3 is located between q_1 and q_2 .
- *Position (d)*: p_3 is located between q_2 and p_1 .

Figures 12 and 13 show the four positions in each of the two cases. The three paths P_1, P_2 and P_3 are illustrated as curves, although in fact they are polygonal lines connecting points of S .

Case 1: edges e_2 and e_3 cross.

In each of the four positions, by applying Property 1 two or three times, we obtain a Hamiltonian alternating cycle that uses all the edges of the paths P_1, P_2 and P_3 and has fewer crossings than C . This is a contradiction since, by removing edge e , there would be a path with fewer crossings than Opt . Therefore, Opt cannot contain three edges crossing each other.

Figure 14 shows the new cycles obtained for each position. The dotted edges are the original edges e_1, e_2 and e_3 , the dashed edges represent intermediate edges that appear after applying Property 1, and the thick edges are the final edges belonging to the new cycle. Further, numbers 1, 2 and 3 denote the intersection points of the edges to which we apply Property 1 and the order in which it is done. For example, for position (a), we first replace edges p_2q_2 and p_3q_3 by p_2q_3 and p_3q_2 . Then, p_1q_1 and p_3q_2 are replaced by p_1q_2 and p_3q_1 .

Case 2: edges e_2 and e_3 do not cross.

For positions (b) and (c) we reach the same contradiction as in case 1 by applying Property 1 (twice); see Figure 15. Thus, Opt contains no edges with such positions for their endpoints.

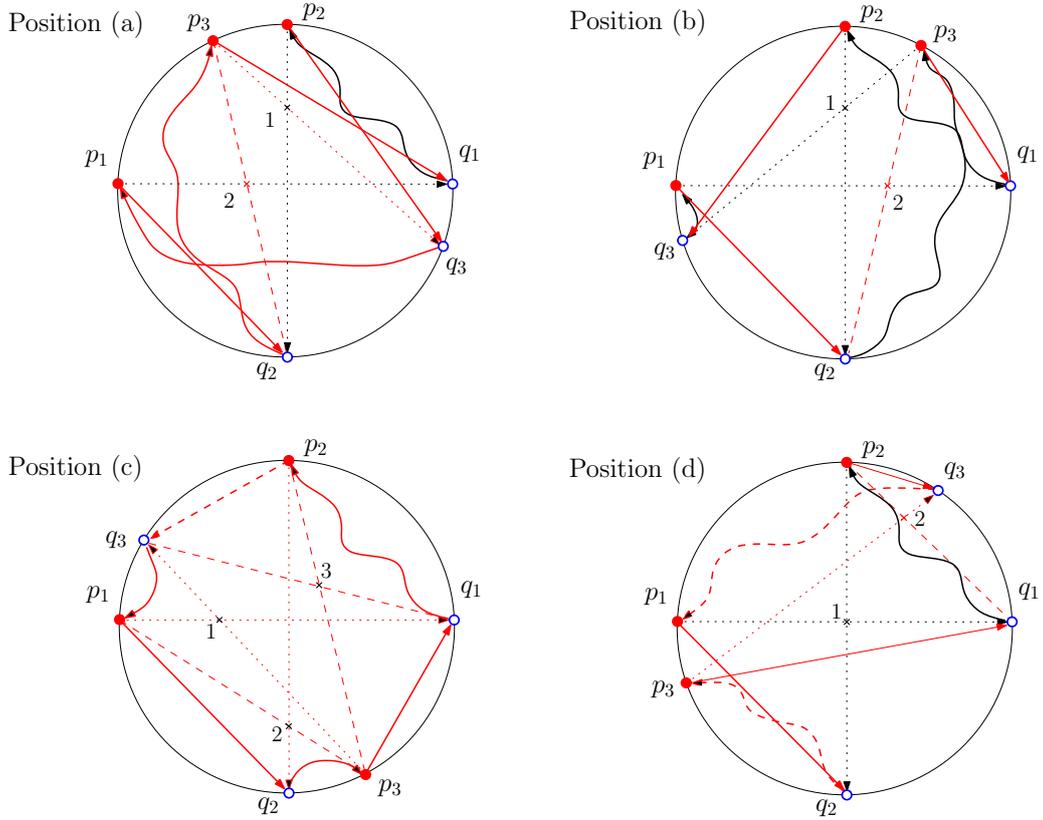


Figure 14: Constructing new paths with fewer crossings than Opt when e_2 crosses e_3 .

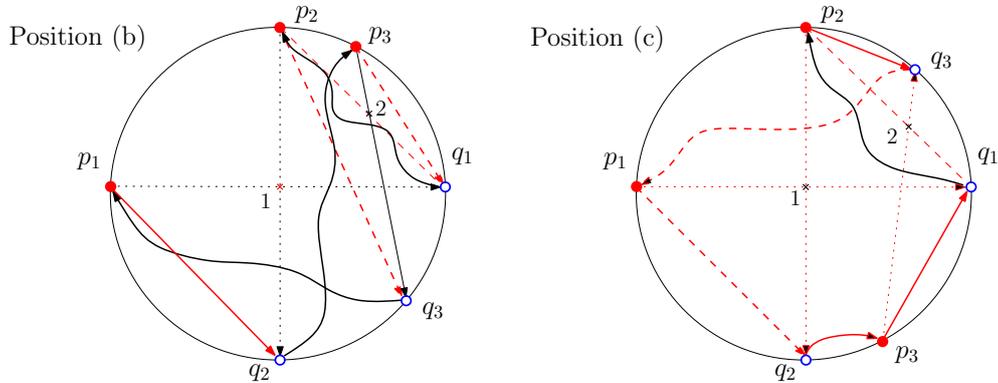


Figure 15: Alternative paths (or cycles) for positions (b) and (c).

Case 2.1: The endpoints of e_1, e_2, e_3 are in position (d).

There can be different pairs of non-crossing edges intersecting e_1 . Let us choose e_2 and e_3 such that the number of edges k of Opt with an endpoint between q_3 and p_2 (clockwise) and the other between q_2 and p_3 (again clockwise) is minimum. Suppose that $k > 0$, and let $e_d = p_d q_d$ be one of those k edges. By Corollary 3, point p_d is red and point q_d is blue.

Assume first that point p_d is to the left of edge e_1 (see Figure 16 left). Edge e_d belongs to neither P_1 nor P_3 ; otherwise e_1, e_d and e_2 would be in position (b) (for P_1) or e_1, e_3 and e_d would be in position (c) (for P_3). Further, if e_d belongs to P_2 , then edges e_1, e_d, e_3 are also in position (d), but the number of edges of Opt with one endpoint between q_3 and p_d , and the other between

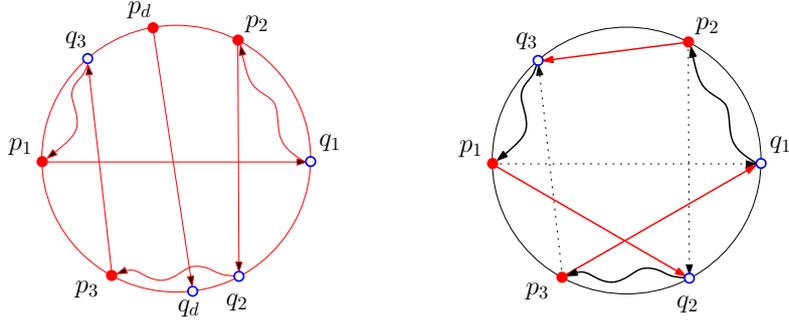


Figure 16: Analysis of position (d).

q_d and q_3 is smaller than k , which is a contradiction since k is minimum. Therefore, p_d is to the right of edge e_1 which also leads to a contradiction in all the possible situations: e_1, e_d, e_2 are in position (c) (edge $e_d \in P_1$); e_1, e_3, e_d are in position (b) (edge $e_d \in P_2$); e_1, e_2, e_d are in position (d), and the contradiction comes again from the minimum value k . Note that, in fact, the two first situations are not exactly positions (b) and (c), but their symmetric ones.

Hence, if position (d) occurs in Opt then $k = 0$.

We now replace in C edges e_1, e_2, e_3 by p_1q_2, p_2q_3, p_3q_1 (see Figure 16 right), and thus obtain a new cycle C' . Since there are no edges e_d with an endpoint between q_3 and p_2 and the other between q_2 and p_3 , one can check that an edge crossing one or two edges among p_1q_2, p_2q_3, p_3q_1 also crosses at least one or two of the replaced edges e_1, e_2, e_3 . Hence, $C' \setminus \{e\}$ is a path that has fewer crossings than Opt , which is a contradiction.

Thus, we can conclude that Opt contains no edges with endpoints in position (d).

Case 2.2: The endpoints of e_1, e_2, e_3 are in position (a).

Since positions (c) and (d) are forbidden, there is no edge in Opt crossed by two edges with opposite directions. This property, which shall be called the *non-opposite edges property*, has several implications that will be assumed in the remaining of the proof:

- There cannot exist edges in Opt with origin on the right of e_1 and extreme on its left.
- The dummy edge $e = qp \in C$ does not belong to Opt (and crosses no edge in Opt), and so it must be in path P_2 in order to go from q_2 to p_3 , with p to the left of e_1 and q to its right.
- All edges of P_3 must have both endpoints between q_3 and p_1 (clockwise); otherwise, either e_3 or e_1 would be crossed by edges in opposite directions. For the same reason, all edges of P_1 must have both endpoints between points p_2 and q_1 (clockwise).
- Point p cannot be placed between p_2 and q_1 (clockwise), and q cannot be a point between q_3 and p_1 (clockwise); otherwise, e_2 or e_3 would be crossed by edges in opposite directions.

An arbitrary straight-line intersecting C and not containing points of S is crossed by the same number of edges in direction left-right than in direction right-left. As a consequence, the difference between the number of edges crossing a given edge in one direction and in the opposite direction is at most 1. By the non-opposite edges property, an edge cannot be crossed by three different edges in Opt , but since one of the edges of C is the dummy edge e , there can be an edge crossed by two edges (both in the same direction). Therefore, e_1 is intersected exactly twice, and e_2 and e_3 can be crossed once or twice.

Let P' be the path from p to p_3 . We now distinguish four cases.

Case 2.2.1: Point p is placed between p_3 and p_2 , clockwise. See Figure 17 top left.

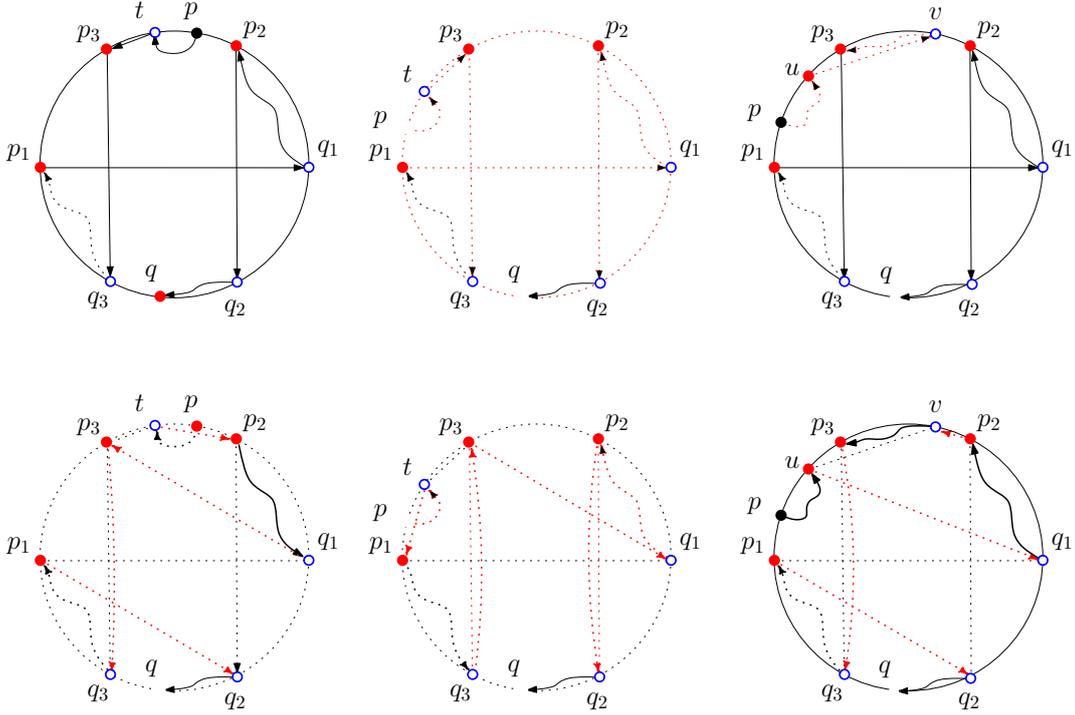


Figure 17: Alternative paths for cases 2.2.1, 2.2.2 and 2.2.3.

Both endpoints of all edges of path P' are between p_3 and p_2 as e_1 is intersected only by edges e_2 and e_3 , which cannot be crossed by two edges in opposite directions. Moreover, P_3 only visits points between q_3 and p_1 , P_1 only visits points between p_2 and q_1 , and the path from q_2 to q cannot cross e_1 . This implies that there are no points between p_1 and p_3 .

Let t be the previous point to p_3 in P' . A new path from p to q is obtained by replacing edges tp_3, e_1, e_2 by tp_2, q_1p_3, p_1q_2 , and removing edge e . Since there are no points between p_1 and p_3 , and e_1 is only crossed by e_2 and e_3 , an edge that intersects one of the new edges also crosses at least one of the removed edges (see Figure 17 bottom left). Thus, the new path has fewer crossings than Opt , which is a contradiction.

Case 2.2.2: Point p is placed between p_1 and p_3 , clockwise, and no edge of path P' crosses edge e_3 . See Figure 17 top center.

As in case 2.2.1, we again reach a contradiction by obtaining a new path from p to q with fewer crossings than Opt . To do this, it suffices to replace in C edges tp_3, e_1 by tp_1, p_3q_1 , and remove edge e (see Figure 17 bottom center).

Case 2.2.3: Point p is placed between p_1 and p_3 , clockwise, and exactly one edge uv of path P' crosses edge e_3 . See Figure 17 top right.

By the non-opposite edges property, uv crosses e_3 from right to left. Further, Corollary 3 implies that u is red and v is blue. Thus, edges uv, e_1 and e_2 in C can be replaced by uq_1, p_2v and p_1q_2 to obtain a new path after removing the dummy edge e . Since e_1 is only intersected by e_2 and e_3 , and e_3 is only crossed by e_1 and uv , one can easily check that this new path has fewer crossings than Opt (see Figure 17 bottom right), which is a contradiction.

Case 2.2.4: $p = p_3$

By symmetry and using q instead of p , the same analysis as above can be done to show that

q must coincide with q_2 . Thus, the two endpoints of Opt have distinct color which implies that $|R| = |B|$ and, by hypothesis, the configuration (S, p, q) is non-special.

Since P_2 consists of the dummy edge $e = qp$, path P_3 only visits points between q_3 and p_1 (clockwise), and P_1 only visits points between p_2 and q_1 (clockwise) it follows that $p = p_3$ is the unique point between p_1 and p_2 , and $q = q_2$ is the unique point between q_1 and q_3 . Thus, the neighbors of p and q on S are both red and blue, respectively.

Let R_1 and B_1 be, respectively, the sets of red and blue points that belong to path P_1 , which has endpoints of different color, and so $|R_1| = |B_1|$. Further, this path only visits points between p_2 and q_1 (clockwise) which implies that it can be completed to a cycle C_1 (with edge p_2q_1) adding no new crossings.

By Theorem 2, cycle C_1 (and hence P_1) has at least $|R_1| - \mathbf{r}(R_1 \cup B_1)$ crossings. Analogously, path P_3 has at least $|R_3| - \mathbf{r}(R_3 \cup B_3)$ crossings, where R_3 and B_3 are the sets of red and blue points that belong to that path. Therefore, Opt has at least

$$|R_1| - \mathbf{r}(R_1 \cup B_1) + |R_3| - \mathbf{r}(R_3 \cup B_3) + 2$$

crossings (the two crossings in edge e_1 are also counted). Since $n = |R| = |R_1| + |R_2| + 1$ (note that $R \setminus (R_1 \cup R_2) = \{p\}$) and $\mathbf{r}(R_1 \cup B_1) + \mathbf{r}(R_3 \cup B_3) = \mathbf{r}(S) + 1$ (the red run to which p belongs is counted twice), then the number of crossings in Opt is at least $n - \mathbf{r}(S)$.

Now, the configuration (S, p, q) is non-special and the neighbors of p and q have their same respective color so there must be a partition $S_1 \cup S_2$ of S , say the counterclockwise partition around p , such that $p_{i+1} \neq q$. Moreover, this partition is obtained by rotating a ray counterclockwise around p and S_1 has minimum size, which implies that p_{i+1} is located between q_3 and p_1 (clockwise). Then, we can argue as in case 2 of the proof of Lemma 2 to obtain a 1-PHAP with at most $n - \mathbf{r}(S) - 1$ crossings: connect the paths $P_{S_1 \cup \{p, p_{i+1}\}}(p, p_{i+1})$ and $P_{S_2}(p_{i+1}, q)$ (see Figure 3 right). This is a contradiction since such a path has fewer crossings than Opt .

Therefore, in all cases we reach a contradiction and so the path Opt is 1-plane. \square

For special configurations (S, r, b) , one cannot obtain Hamiltonian alternating paths with endpoints r and b that are 1-plane (Theorem 4), and thus, it is natural to ask about the minimum number of crossings that one might expect. The answer to this question is obtained as a consequence of Theorem 5. Indeed, consider a special configuration (S, r, b) and an optimum Hamiltonian alternating path Opt with endpoints r and b . We can argue as in the proof of Theorem 5, for all cases except the 2.2.4, to show that no edge of Opt can be crossed twice. In case 2.2.4, the hypothesis of the configuration being non-special is only used in the last paragraph, and the preceding construction can be used: Opt has $n - \mathbf{r}(S)$ crossings and consists of two optimum 1-plane paths P_1, P_3 and one edge e_1 , which is crossed twice. Further, by Theorem 3, P_1 and P_3 can be computed in linear time. Thus, we have the following corollary.

Corollary 4 *Let (S, r, b) be a special configuration for a point set $S = R \cup B$ in convex position with $|R| = |B| = n \geq 1$. Then, every Hamiltonian alternating path on S with endpoints r and b has at least $n - \mathbf{r}(S)$ crossings. Moreover, an optimum Hamiltonian alternating path on S with endpoints r and b can be computed in $O(n)$ time, has $n - \mathbf{r}(S)$ crossings, and all its edges are crossed at most once, except for one of them that is crossed exactly twice.*

Finally, we analyze the complexity of computing an optimum 1-PHAP on S in convex position, provided that the endpoints of the path are fixed and the configuration is non-special (otherwise, by Theorem 4, there does not exist such a path).

Let $\langle p_1, \dots, p_h \rangle$ be the clockwise circular order of the points of S . Recall the following notations from Section 3.1; here, we consider $|R| = |B| = n \geq 1$ or $|R| = |B| + 1 = n + 1 \geq 2$.

- sets $S[p_i, p_j] = \{p_i, p_{i+1} \dots, p_j\}$ (clockwise) and $S(p_i, p_j) = S[p_i, p_j] \setminus \{p_i, p_j\}$.
- point $p_{J(i)}$ is the first point of S (clockwise) such that $|S[p_i, p_{J(i)}] \cap R| = |S[p_i, p_{J(i)}] \cap B|$ for a given point p_i .

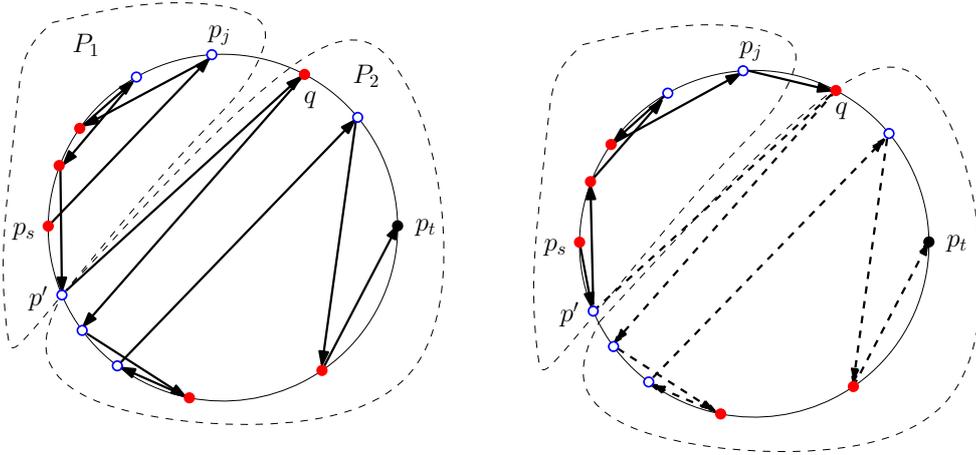


Figure 18: Point p' coincides with p_{s-1}

- point $p_{J'(i)}$ is the first point of S (counterclockwise) such that $|S[p_{J'(i)}, p_i] \cap R| = |S[p_{J'(i)}, p_i] \cap B|$ for a given point p_i .

Theorem 6 below describes an $O(n^2)$ time and space dynamic programming algorithm for computing an optimum 1-PHAP on S with fixed endpoints p_s and p_t . As a main tool, it uses the following lemma. Observe that in both results the paths are considered to be oriented.

Lemma 5 *Let $S = R \cup B$ be in convex position with $|R| = |B| \geq 1$ or $|R| = |B| + 1 \geq 2$, and let p_s, p_t be two non-consecutive points of S such that the configuration (S, p_s, p_t) is non-special if p_s, p_t have distinct color. Then, there exists an optimum 1-PHAP on S from p_s to p_t that contains either an optimum sub-path from p_s to $p_{J(s)}$, or an optimum sub-path from p_s to $p_{J'(s)}$.*

Proof. Consider an optimum 1-PHAP, say Opt , directed from p_s to p_t , and let p_j be the point of S visited after p_s in Opt . Assuming that p_j belongs to $S(p_s, p_t)$, we show that there exists an optimum 1-PHAP on S from p_s to p_t that contains an optimum sub-path from p_s to $p_{J(s)}$. A similar analysis can be done when p_j belongs to $S(p_t, p_s)$, to show that there exists an optimum 1-PHAP on S from p_s to p_t that contains an optimum sub-path from p_s to $p_{J'(s)}$.

If $p_j = p_{s+1}$, then p_j coincides with $p_{J(s)}$. Otherwise, by Lemma 4, path Opt visits the following points in the given order: point p_j , all points in $\{p_{s+2}, \dots, p_{j-1}\}$, point p_{s+1} , and point p' , where p' is either p_{j+1} or p_{s-1} . Thus, Opt consists of two optimum 1-plane paths that are connected at p' : path P_1 from p_s to p' on $S[p_s, p_j] \cup \{p'\}$, and path P_2 from p' to p_t visiting the remaining points.

If p_s and p_{s+1} have different colors, then p_s and p' have the same color and are consecutive on $S[p_s, p_j] \cup \{p'\}$. By Corollary 2 (ii), edge $p_s p_{s+1}$ must appear in P_1 , and so $p_j = p_{s+1}$ which coincides with $p_{J(s)}$. Hence, we may assume that the color of p_s is the same of p_{s+1} and different of the p' .

Suppose that $p' = p_{s-1}$, and let q be the point of S visited after p' in Opt . A new path from p_s to p_t can be drawn by removing edges $p_s p_j$ and $p' q$, by adding edges $p_s p'$ and $p_j q$, and by reversing the orientation of the path from p_j to p_{s+1} (see Figure 18). One can check that this new path has fewer crossings than Opt , a contradiction. Thus, $p' \neq p_{s-1}$ which implies that $p' = p_{j+1}$.

Since the endpoints of P_1 have different colors, necessarily $S[p_s, p_{j+1}]$ contains the same number of red and blue points, and so $p_{J(s)} \in S[p_s, p_{j+1}]$. By Corollary 2(i), path P_1 has

$$\frac{|S[p_s, p_{j+1}]|}{2} - r(S[p_s, p_{j+1}])$$

crossings. If $p_{J(s)} \neq p_{j+1}$, then P_1 can be replaced by a path that consists of two optimum 1-plane paths connected at $p_{J(s)}$: path P' on $S[p_s, p_{J(s)}]$ from p_s to $p_{J(s)}$, and path P'' on $S[p_{J(s)}, p_{j+1}]$

from $p_{J(s)}$ to p_{j+1} . By Corollary 2, one can check that P_1 and $P' \cup P''$ have the same number of crossings. \square

Theorem 6 *Let $S = R \cup B$ be in convex position with $|R| = |B| \geq 1$ or $|R| = |B| + 1 \geq 2$, and let $p_s, p_t \in S$. Then, an optimum Hamiltonian alternating path on S from p_s to p_t can be computed in $O(n^2)$ time and space.*

Proof. If p_s and p_t are consecutive points of S , by Theorem 3, an optimum path from p_s to p_t can be computed in linear time. By Corollary 4, the same happens when (S, p_s, p_t) is a special configuration. Thus, we may assume that p_s and p_t are non-consecutive, and that (S, p_s, p_t) is non-special, and so, by Theorem 5, the optimum path is 1-plane.

By Lemma 5, an optimum path can be decomposed into two optimum paths: one from p_s to $p_{J(s)}$ on $S[p_s, p_{J(s)}]$ (or from p_s to $p_{J'(s)}$ on $S[p_{J'(s)}, p_s]$) and the other from $p_{J(s)}$ to p_t on $S[p_{J(s)}, p_{s-1}]$ (or from $p_{J'(s)}$ to p_t on $S[p_{s+1}, p_{J'(s)}]$). By Theorem 3, the first path can be computed in linear time. Therefore, given $p_s \leq p_i < p_t$ and $p_t \leq p_j < p_s$, we have to compute an optimum 1-PHAP on $S[p_i, p_j]$ with endpoints p_t and either p_i or p_j .

We explore all subsets $S[p_i, p_j]$ containing point p_t (but not p_s) in increasing order of their size. Let $Cross_1[p_i, p_j]$ denote the number of crossings of an optimum 1-PHAP on $S[p_i, p_j]$ from p_i to p_t . Analogously, $Cross_2[p_i, p_j]$ is the number of crossings of an optimum 1-PHAP on $S[p_i, p_j]$ from p_j to p_t . Let $Cross[p_i, p_j]$ be the number of crossings of an optimum 1-PHAP on $S[p_i, p_j]$ from p_i to p_j . The value $Cross_1[p_i, p_j]$ can be computed by defining the following values c_1 and c_2 :

$$c_1 = \begin{cases} \infty & \text{if } p_{J(i)} \notin S(p_i, p_t) \\ Cross[p_i, p_{J(i)}] + Cross_1[p_{J(i)}, p_j] & \text{if } p_{J(i)} \in S(p_i, p_t) \end{cases}$$

$$c_2 = \begin{cases} \infty & \text{if } p_{J_1(i)} \notin S(p_t+1, p_j) \\ Cross'[p_i, p_j] + Cross_2[p_{i+1}, p_{J_1(j)}] & \text{if } p_{J_1(i)} \in S(p_t+1, p_j) \end{cases}$$

where $p_{J_1(j)}$ is the counterclockwise first point after p_j such that there are the same number of red and blue points in $\{p_i\} \cup S[p_{J_1(j)}, p_j]$, and $Cross'[p_i, p_j]$ is the number of crossings of an optimum 1-PHAP from p_i to $p_{J_1(j)}$ on $\{p_i\} \cup S[p_{J_1(j)}, p_j]$ (see Figure 19 right).

Thus, by Lemma 5, $Cross_1[p_i, p_j] = \min\{c_1, c_2\}$. Note that, when p_i and p_t are consecutive, by Corollary 2, $Cross_1[p_i, p_j]$ is the difference between the red (blue) points and the red (blue) runs of $S[p_i, p_j]$.

We next show that indices $J(i), J_1(i)$ can be pre-computed in $O(n)$ time, which implies that $Cross[p_i, p_j]$ and $Cross'[p_i, p_j]$ can be computed in constant time.

PROCEDURE *J-PAIRS* computes $p_{J(i)}$ in linear time. Moreover, if $J_1(i) \neq j$, then $p_{J_1(i)}$ is the first counterclockwise point from p_j such that the difference between the number of red and blue points in $S[p_{J_1(i)}, p_j]$ equals one. It is not difficult to check that a slight modification of step 2 of PROCEDURE *J-PAIRS* also computes $p_{J_1(i)}$ in linear time.

By numbering cyclically the runs of S we can keep, for each point, the number of the run to which it belongs. Using this information, the number of red (blue) runs of any set $S[p_i, p_j]$ can be computed in constant time. In addition, by Corollary 2, $Cross[p_i, p_{J(i)}] = |S[p_i, p_{J(i)}]|/2 - r(S[p_i, p_{J(i)}])$, and so $Cross[p_i, p_{J(i)}]$ can also be computed in constant time. The same reasoning applies to show that $Cross'[p_i, p_j]$ is computed in constant time.

A similar analysis can be done to compute $Cross_2[p_i, p_j]$. Therefore, each of the $|S(p_s, p_t)| \times |S(p_t, p_s)|$ values of $Cross_1[p_i, p_j]$ and $Cross_2[p_i, p_j]$ can be obtained in constant time. With these pre-computed data and using a standard dynamic programming algorithm, we can compute the number of crossings and the edges of an optimum 1-PHAP on S from p_s to p_t in $O(n^2)$ time and space. \square

To conclude this section, observe that $Cross_1[p_i, p_{i-1}]$ gives the minimum number of crossings of a Hamiltonian alternating path on S from p_i to p_t . Thus, the previous algorithm can be used

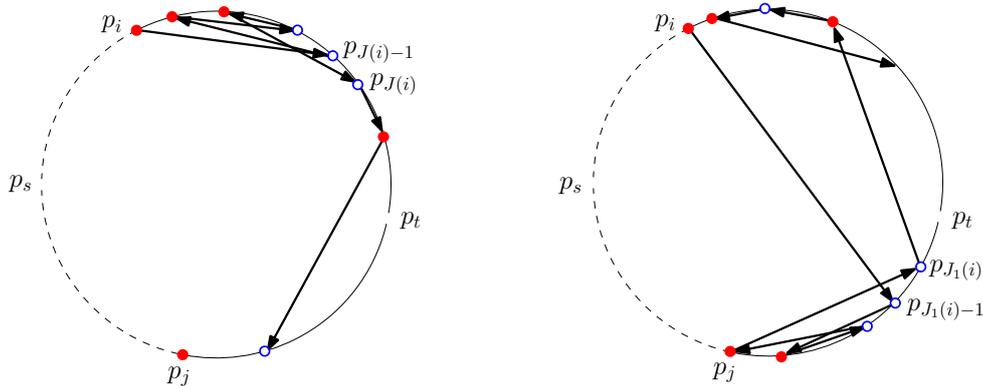


Figure 19: The two cases to compute $Cross_1[p_i, p_j]$.

to compute (in $O(n^2)$ time and space) all the optimum Hamiltonian alternating paths from any $p_i \neq p_t$ to p_t . Applying the algorithm n times for any possible endpoint p_t , one can obtain the all-pairs optimum Hamiltonian alternating paths in $O(n^3)$ time.

4 Open questions

We conclude this paper by formulating a number of problems, related to those here considered, that remain open.

Problem 1 *Characterize the special configurations (S, r, b) that do admit a 1-PHAP.*

Although for some point configurations S in general position, it is not difficult to find a 1-PHAC on S with minimum number of crossings, in general, computing such a path seems to be an NP-hard problem. Thus, it would be interesting to solve the following two problems.

Problem 2 *Among all Hamiltonian alternating cycles on S with minimum number of crossings, determine whether there is at least one that is 1-plane.*

Problem 3 *Decide whether it is possible to compute in polynomial time a 1-PHAC on S with minimum number of crossings.*

The construction of Remark 3 works because the number of runs in S is high. If S contains only one red run and one blue run, then there is a unique 1-PHAC on S . Thus, the following problem arises.

Problem 4 *Prove that the number of 1-plane Hamiltonian alternating cycles on S is exponential in $r(S)$ for S in convex position.*

Finally, we would like to improve the $O(n^2)$ running time of the dynamic programming algorithm of Theorem 6, and so the following problem should be addressed.

Problem 5 *Determine whether it is possible to reduce the number of explored subsets $S[p_i, p_j]$ to $o(n^2)$ (for example a linear subfamily, obtaining a linear time algorithm).*

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