Appendix A

Calculus of eigenvalues and eigenvectors

The eigenvalues of matrix $\widetilde{\mathbf{J}}_{\mathbf{n}}$ are the roots of the following polynomial:

$$(\widetilde{\lambda}^3 + a_1\,\widetilde{\lambda}^2 + a_2\,\widetilde{\lambda} + a_3) = 0 \tag{A.1}$$

whose coefficients a_1, a_2 and a_3 are given by:

$$a_1 = -2\widetilde{u} \quad a_2 = -p_{bz}\widetilde{B} - \widetilde{c}^2 + \widetilde{u}^2 \quad a_3 = -p_{bz}\widetilde{A} \tag{A.2}$$

Calling $Q = (3a_2 - a_1^2)/9$ and $R = (9a_1a_2 - 27a_3 - 2a_1^3)/54$, the eigenvalues are real if $Q^3 + R^2 < 0$, and are given by

$$\begin{aligned} \widetilde{\lambda}^2 &= 2\sqrt{-Q}\cos(\theta_p/3) - a_1/3 \\ \widetilde{\lambda}^3 &= 2\sqrt{-Q}\cos((\theta_p + 2\pi)/3) - a_1/3 \\ \widetilde{\lambda}^4 &= 2\sqrt{-Q}\cos((\theta_p - 2\pi)/3) - a_1/3 \end{aligned} \tag{A.3}$$

with $\theta_p = \arccos(R/\sqrt{-Q^3})$. The eigenvectors $\widetilde{\mathbf{e}}^m$, $m = 1, 2, 3$ associated to $\widetilde{\lambda}^m$, $m = 1, 2, 3$ are given by

$$\widetilde{\mathbf{e}}^m = \begin{pmatrix} 1 \\ \widetilde{\lambda}^m \\ \frac{-\widetilde{c}^2 + \widetilde{u}^2 + \widetilde{\lambda}^m(\widetilde{\lambda}^m - 2\widetilde{u})}{p_{bz}} \end{pmatrix} \tag{A.4}$$

Appendix B

Conservation of the numerical scheme

For the sake of clarity the necessity of the last term in the numerical scheme, B.1, is going to be explained. The lack of this term in the computed method may lead to a non conservative solution, which has been misunderstood by some authors as a diffusivity problem,

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \sum_{k=1}^{NE}\sum_{m=1}^{4}(\widetilde{\lambda}^-\alpha - \beta^-)_k^m \widetilde{\mathbf{e}}_{JI,k}^m l_k \frac{\Delta t}{A_i} - \sum_{k=1}^{NE} \delta \mathbf{E}_{Ii,k} \mathbf{n}_k l_k \frac{\Delta t}{A_i} \tag{B.1}$$

In case of having a set of equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}(\mathbf{U}, \mathbf{x}, \mathbf{y}) \tag{B.2}$$

which can be manipulated as follows

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{M_n}\left(\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{U}}{\partial y}\right) - \mathbf{H_n}\left(\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{U}}{\partial y}\right) = \mathbf{S_s}(\mathbf{U}, \mathbf{x}, \mathbf{y}) \tag{B.3}$$

where $\mathbf{M_n}$ is the flux normal to a direction given by the unit vector $\mathbf{n}$, $\mathbf{En} = \mathbf{F}n_x + \mathbf{G}n_y$, defined as

$$\mathbf{M_n} = \frac{\partial(\mathbf{En})}{\partial \mathbf{U}} \tag{B.4}$$

and $\mathbf{H_n}$ is the flux associated to the bed slope, projected onto the unit vector $\mathbf{n}$, $\mathbf{T_b n} = \mathbf{S_b} n_x + \mathbf{S_b} n_y$

$$\mathbf{H_n} = \frac{\partial(\mathbf{T_b n})}{\partial \mathbf{U}} \tag{B.5}$$

It is possible to define the following Jacobian matrix, $\mathbf{J_n}$, through the definitions in (B.4) and (B.5)

$$\mathbf{J_n} = \mathbf{M_n} - \mathbf{H_n} \tag{B.6}$$

which will allow us to define system (B.2) as belonging to the family of hyperbolic systems.

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{J_n}\left(\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{U}}{\partial y}\right) = \mathbf{S_s}(\mathbf{U}, \mathbf{x}, \mathbf{y}) \tag{B.7}$$

Due to the non linearity of the flow **En**, the Jacobian matrix has to be approximated in order to generate a local linearization. Roe (1986), proposed an approximated Jacobian matrix, $\widetilde{\mathbf{J}}_{\mathbf{n},k}$, for clean water, imposing that

$$\widetilde{\mathbf{J}}_{\mathbf{n}}(\mathbf{U}_i, \mathbf{U}_i) = \mathbf{J_n}(\mathbf{U}_i) \tag{B.8}$$

In Murillo & García-Navarro (2010) an augmented Roe solver is proposed to allow the inclusion of the sediment transport terms. Hence, it was developed the building of an approximate Jacobian matrix $\widetilde{\mathbf{J}}_{\mathbf{n},k}$ at each k edge of each cell combining the normal flux $\mathbf{E_n} = \mathbf{F}n_x + \mathbf{G}n_y$ with the bed slope source term $\mathbf{T}_{\mathbf{n},b}$ at each cell edge,

$$(\delta \mathbf{E} - \mathbf{T}_b)_k \mathbf{n}_k = (\widetilde{\mathbf{M}}_{\mathbf{n}} - \widetilde{\mathbf{H}}_{\mathbf{n}})_k \, (\mathbf{U}_j - \mathbf{U}_i) \tag{B.9}$$

$$(\delta \mathbf{E} - \mathbf{T_b})_k \mathbf{n}_k = \widetilde{\mathbf{J}}_{\mathbf{n},k} \delta \mathbf{U}_k \tag{B.10}$$

with $\delta(\mathbf{E_n})_k = (\mathbf{E}_j - \mathbf{E}_i)_{\mathbf{n}_k}$, $\delta \mathbf{U}_k = \mathbf{U}_j^n - \mathbf{U}_i^n$, and $\mathbf{U}_i^n$ and $\mathbf{U}_j^n$ the initial values at cells i and j sharing edge k. Figure B.1 collects the previous information in order to solve the Riemann problem in a 2D situation.

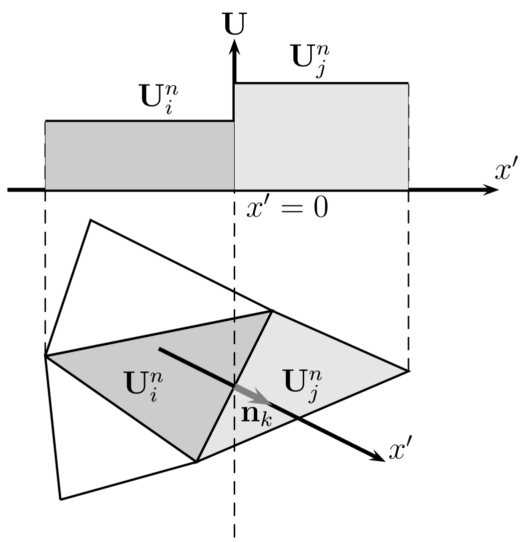

Figure B.1: Riemann problem in 2D along the normal direction to a cell side

On the other hand, the main difficulties in the definition of the approximate Jacobian matrix (B.9) for sediment transport that allows the construction of approximate solutions arises from the presence of a local A_g value, which is variable within each cell, see Figure B.2.

For this reason the net exchange of flow between internal walls must include the difference between the value of the cell in its centroid, $A_{g,i}$, and the value close to the wall within the cell $A_{g,I}$. This fact leads to new definitions of $\mathbf{E}$ at each pair of cells i and j, connected through edge k, that will be referred to as

$$\mathbf{E}_I = \mathbf{E}(\mathbf{U}_i, A_{g,k}) \qquad \mathbf{E}_J = \mathbf{E}(\mathbf{U}_j, A_{g,k}) \tag{B.11}$$

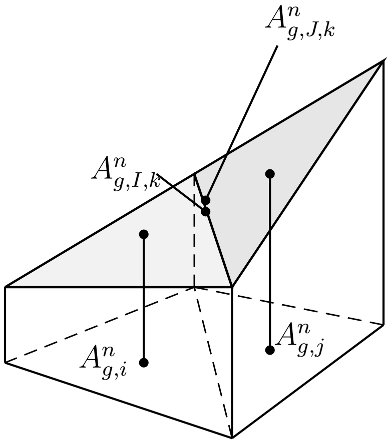

Figure B.2: Linear representation by cells.

Analyzing the flux term $\delta\mathbf{En}$ drives to

$$\begin{aligned}\delta\mathbf{En} &= E_j - E_i = E_J - E_I + (E_j - E_J) - (E_i - E_I) = \\ &= \delta E_{JI} + \delta E_{jJ} - \delta E_{iI} = \\ &= (\delta E_{JI} + \delta E_{Ii}) - \delta E_{Jj} = \\ &= (\delta E^-_{JI} + \delta E_{Ii}) + \delta E^+_{JI} - \delta E_{Jj} = \\ &= (\delta E^-_{JI} + \delta E_{Ii}) - (\delta E^-_{IJ} + \delta E_{Jj})\end{aligned} \tag{B.12}$$

As it can be appreciated not only a flux crossing the wall, δE^-_{JI}, is necessary to ensure a conservative numerical scheme. A flux with information of the variation of $Ag_{I,i}$ is also necessary and this is the additional term which appears in the numerical scheme.