Particular Solutions to the n–Body Problem

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February, 2019

Resumen

El *problema de n cuerpos* considera el movimiento en el espacio de un sistema mecánico de n cuerpos materiales (por lo general idealizados como partículas puntuales) bajo el efecto de sus influencias mutuas (formalizadas como fuerzas internas, esto es, fuerzas que satisfacen la tercera ley de Newton). Se trata de un problema dinámico de $3\,n$ grados de libertad gobernado por un sistema diferencial de orden $6\,n$.

Como primera consecuencia de la elección de este modelo de fuerzas, el centro de masas del sistema puede moverse a lo largo de una recta con velocidad constante o permanecer en estado de reposo. Las expresiones analíticas correspondientes a este hecho nos proporcionan seis integrales primeras escalares, funcionalmente independientes, del sistema de ecuaciones diferenciales del movimiento, las cuales se denominan *integrales del centro de masas*. Para muchos propósitos, se elije un sistema de referencia espacial cuyo origen coincide con el centro de masas del sistema de partículas (*sistema de referencia baricéntrico*), lo cual suele conducir a simplificaciones en la formulación y en el tratamiento del problema.

Muchas de las características más significativas del movimiento de un sistema mecánico no dependen de la ley de fuerzas concreta que se considere, sino del hecho de que la fuerza sea una fuerza *central*. Como consecuencia de esto, el momento angular total del sistema es un vector constante que nos ofrece tres nuevas integrales primeras, funcionalmente independientes, del sistema de ecuaciones, llamadas *integrales del momento angular*. Como el momento angular es un vector cuya dirección no varía, puede contemplarse como un vector característico de un plano que contiene al centro de masas del sistema de n partículas, denominado *plano invariable del sistema*. Si el momento angular se anula, entonces el plano invariable no existe.

Para concretar más, nos limitaremos a estudiar fuerzas centrales cuya magnitud obedece a leyes de fuerza proporcionales a una potencia de las distancias mutuas de la forma r^λ, siendo λ un número real. Al ser estas fuerzas conservativas, la energía total del sistema es una nueva cantidad escalar conservada que proporciona la *integral de la energía*, con la cual ya disponemos de diez integrales primeras del problema.

De este modo, el conjunto de ecuaciones diferenciales del movimiento del sistema de n partículas, que originalmente era un sistema diferencial de orden $6\,n$, tras encontrar estas 10 integrales primeras puede reducirse a un sistema diferencial de orden $6\,n - 10$, y no se pueden obtener nuevas integrales primeras funcionalmente independientes que puedan aplicarse en el problema de n cuerpos. Por ejemplo, en el caso $n = 3$, en el que necesitamos 18 integrales primeras para resolverlo, la falta de 8 de ellas impide encontrar la solución general del problema de 3 cuerpos. Las dificultades aumentan cuando se considera un número mayor de cuerpos ($n > 3$). Una solución completa del problema de n cuerpos requeriría $6\,n$ constantes de integración independientes, pero sólo disponemos de diez constantes del movimiento; por esta razón, la resolución completa del problema de n cuerpos no es viable y la búsqueda de ciertas soluciones particulares (configuraciones centrales, soluciones homográficas, órbitas periódicas,...) puede ser una alternativa para llegar a alcanzar un mejor conocimiento sobre el comportamiento del sistema.

Si la atracción gravitatoria es la única fuerza que actúa entre los cuerpos considerados, el problema se denomina *problema gravitatorio de n cuerpos*. Este caso constituye un modelo simplificado que

permite estudiar el movimiento planetario.

El caso más sencillo, cuando $n = 2$, es *el problema de dos cuerpos con fuerzas internas*, cuyo tratamiento puede separarse en dos subproblemas independientes que consideran el movimiento de un solo cuerpo: el problema del movimiento libre del centro de masas del sistema de dos cuerpos (un movimiento rectilíneo uniforme o un estado de reposo), y el problema del movimiento relativo de uno de los cuerpos con respecto al otro (que es un problema de un solo cuerpo en el seno de un campo de fuerzas central).

En particular, el problema gravitatorio de dos cuerpos es un sistema dinámico conservativo en el cual el subproblema del movimiento relativo, llamado "*problema de Kepler*", se aborda como el problema del movimiento de una partícula en torno a un centro fijo bajo la atracción newtoniana. Este problema puede ser completamente resuelto, y su resolución puede reducirse a, como mucho, dos cuadraturas.

De gran importancia histórica y conceptual son los resultados obtenidos por Euler y Lagrange sobre las soluciones particulares para ciertos casos concretos en los cuales el movimiento de tres partículas, sometidas a fuerzas de atracción gravitatoria, está confinado a un mismo plano. Uno de los casos corresponde a la configuración en la cual los tres cuerpos forman un triángulo equilátero que rota uniformemente, con respecto a un sistema de referencia inercial, con velocidad angular constante alrededor de un eje perpendicular al plano del movimiento; en este proceso los tres cuerpos siguen órbitas circulares coplanarias. En un segundo caso, los tres cuerpos yacen sobre una línea recta, y dicha recta gira con velocidad angular constante en el plano del movimiento. La posición relativa de los tres cuerpos a lo largo de la recta queda determinada por la única raíz, perteneciente al intervalo $(0, 1)$, de cierta ecuación de quinto grado. Tanto las configuraciones en triángulos equiláteros como en rectas se conocen como *soluciones de equilibrio relativo*. En ambos casos el sistema de tres cuerpos rota con velocidad angular constante alrededor del centro de masas y, por consiguiente, los cuerpos describen (con respecto a un sistema de referencia inercial) circunferencias en el plano del movimiento. En un último caso la solución permite que tres cuerpos también pueden formar un triángulo equilátero en el seno del plano del movimiento, pero la distancia entre ellos puede variar a lo largo del tiempo. En este caso los movimiento por separado de los tres cuerpos ya no son circulares, sino que describen cónicas con un foco en el centro de masas del sistema.

Estos conceptos y resultados, y el estudio de las principales propiedades y relaciones existentes entre ellos, constituyen parte del tema tratado en este trabajo. Además se considerará su generalización a otros tipos de fuerzas más generales (como ciertos modelos de fuerzas centrales conservativas) y a problemas de n cuerpos con $n > 3$.

En este Trabajo de Fin de Grado consideramos el movimiento de un sistema mecánico de n partículas en el cual las únicas fuerzas que intervienen son las acciones mutuas ejercidas por las partículas entre sí; dichas fuerzas actúan a lo largo de la recta que las une, su magnitud es proporcional a una potencia de la distancia que separa a las partículas, y no se consideran otras fuerzas externas actuando sobre este sistema. Bajo estas hipótesis, analizamos ciertas soluciones particulares del problema de n cuerpos: las *configuraciones centrales* (obtenidas cuando la fuerza que actúa sobre cada cuerpo es proporcional a su masa y a su vector de posición respecto al centro de masas del sistema; de este modo, bajo homotecias y rotaciones los n cuerpos mantienen sus posiciones relativas), y las *soluciones homográficas* (cuándo la configuración, con respecto a un sistema baricéntrico inercial, formada por los cuerpos en un instante t se mantiene semejante a sí misma cuando t varía). Estos conceptos y sus relaciones se analizan en el caso general, y a continuación se particularizan a los tres casos más sencillos ($n = 2, 3, 4$).

El número y tipo de soluciones homográficas son perfectamente conocidos y están completamente determinados para el problema gravitatorio de dos y tres cuerpos. Cuando $n \geq 4$ estas cuestiones aún no han sido resueltas, y hoy en día todavía constituyen problemas abiertos. Por este motivo nos limitamos únicamente a describir unos pocos casos, omitiendo las demostraciones de los enunciados correspondientes y dando las referencias bibliográficas pertinentes.

Los resultados y demostraciones presentados en este Trabajo de Fin de Grado han sido reformulados y adaptados a un lenguaje matemático moderno.

Los contenidos de este trabajo están organizados en un resumen en español, tres capítulos, una bibliografía y dos apéndices.

El Capítulo 1 ofrece el marco teórico y conceptual del problema de n cuerpos y el propósito de este trabajo.

El Capítulo 2 está dedicado a dar una clasificación geométrica de las soluciones del problema de n cuerpos, además de la definición y las propiedades más relevantes de las configuraciones centrales y de las soluciones homográficas, asi como la relación entre dichos conceptos.

En el Capítulo 3 se da una caracterización de las configuraciones centrales del problema general de n cuerpos en función de las distancias mutuas y, a partir de este resultado, se analizan los casos particulares de $n = 3$ y $n = 4$. El estudio del problema de tres cuerpos está completo y es exhaustivo. Sin embargo, en el problema de cuatro cuerpos la investigación está lejos de ser completa, y sólo se presentan algunos casos concretos de configuraciones centrales planas, dependiendo de las masas respectivas de los cuatro cuerpos.

En el Apéndice A se presenta la formulación del problema de n cuerpos, se introducen las principales magnitudes cinéticas y dinámicas de interés, y las 10 integrales primeras clásicas que admite este problema.

Finalmente, en el Apéndice B recogemos algunos conceptos y resultados matemáticos y cinemáticos utilizados a lo largo del trabajo.

Contents

Chapter 1

Context and Scope

Roughly speaking, *Mechanics* is the part of Physics concerned with the motion (Kinematics) and equilibrium (Statics) of material bodies, and the causes –namely, forces (Dynamics)– that produce motion and equilibrium. Throughout its development, Mechanics has motivated the emergence of concepts, methods and results that have enriched diverse fields of Mathematics, specially in Mathematical Analysis and related areas such as Classical Differential Geometry, Differential Equations, Dynamical Systems, Ergodic Theory, etc.

Conventionally, for may purposes, *material bodies* are usually idealized as *point particles*, or point masses, or mass points, i.e., purely geometric points (without size, without extension in space, without volume, without internal structure, without internal motions) endowed with *mass*, specially when the dimensions of the bodies are very small compared with the distances between them.

Celestial Mechanics is the branch of Mechanics devoted to the study of the motion of (natural and/or artificial) celestial bodies under the effect of various types of forces, the most important of which being their mutual gravitational attraction according to Newton's Law of Universal Gravitation.

In fact, starting from Isaac Newton himself, that gravitational attraction (described by an inverse–square law of the distances between the bodies) was the most prevalent force considered at the first stages of the historical development of Celestial Mechanics, as we conceive it nowadays. In this way, Celestial Mechanics evolved from Newton's laws of motion and the law of gravitation.

Originally the problems at the root of Celestial Mechanics were the general problem of the *motion of the known celestial bodies* (planets and their natural satellites) within the Solar System (modelled as an n–body problem), and the more specific problem of the *motion of the Moon around the Earth* (modelled as a 3– body system, Sun–Earth–Moon).

The *n–body problem*, also called the *many–body problem* and the *several–body problem*, considers the motion in space of a mechanical system of n material bodies under their mutual influence (formalized by *interaction forces*, or internal forces, that is, forces that satisfy Newton's Third Law, also known as the law of *action and reaction*). It is a dynamical problem with $3\,n$ degrees of freedom, governed by a differential system of order $6\,n$.

Needless to say that other types of forces not obeying Newton's Third Law, as well as external forces, can also be incorporated into the dynamical model under consideration, which results in a substantial additional increase in the mathematical complexity of the problem. *In this Undergraduate Dissertation we do not intend to consider such forces*. Accordingly, we restrict ourselves to the case in which the only forces acting on the particles are the mutual interaction forces, and there are no additional external forces.

As a first consequence of this choice of force model, the center of mass of the system either moves along a straight line at constant velocity or remains at rest. The analytical expressions corresponding to this fact provide us with six functionally independent scalar first integrals [1] of the differential system of

[1] Integrals of motion, constants of motion, motion integrals, preserved quantities, conserved quantities, conservation laws, theorems of conservation, principles of conservation, invariants, etc. are also words and expressions commonly used to refer to this notion.

equations of motion, the so–called *integrals of the center of mass* (see Appendix A, Proposition A.5). For many purposes, it is customary to choose spatial reference frames with the origin at the center of mass of the system of bodies (*barycentric reference frames*, or *barycentric coordinate systems*), which usually leads to simplifications in the formulation and treatment of the problem.

Many of the most significant features of the motion of such a mechanical system do not depend on the specific law of force describing the mutual interactions, but on the fact that the force is *central.* As a consequence of this fact, the total angular momentum $\boldsymbol{G}$ of the system is a constant vector quantity, which yields three new functionally independent scalar first integrals of the equations of motion, namely the *integrals of the angular momentum* (Appendix A, Proposition A.6). Since this vector $\boldsymbol{G}$ does not change its direction, it can be contemplated as a characteristic vector of a plane passing through the center of mass of the n–particle system, called the *invariable plane* (or *Laplace plane*) of the system (Appendix A, Definition A.3). If $\boldsymbol{G}$ vanishes, the invariable plane does not exist.

For the sake of concretion, we concentrate on central forces whose magnitude obeys *laws of force proportional to a power of the mutual distance*, i.e., of the form r^{λ}, the exponent λ being any real number. These forces turn out to be "conservative", the total energy $\mathcal{E}$ of the system is a new conserved scalar quantity, and we have a tenth functionally independent scalar first integral of the differential problem available, the *integral of the energy* (Appendix A, Proposition A.7).

Notice that these ten first integrals are algebraic functions of the position coordinates and velocities of the n bodies and on the time. They are also functionally independent of each other. In this way, the set of differential equations of motion of the system of n particles (Eqs. (A.1.1) and (A.1.2), Appendix A), originally a differential system of order $6n$ admitting these *ten classical first integrals*, can be reduced to order $6n - 10$.

Unfortunately, when dealing with this problem, *no new functionally independent first integrals can be found in general.* The possible existence of additional integrals for the gravitational n–body problem (see below) was analyzed by Bruns, Painlevé and Poincaré, who answered the question in the negative: they proved that these ten integrals are the only functionally independent integrals of the problem that can be expressed by means of simple known functions and their primitive functions (Wintner [14, Ch. II, §129, p. 97, Ch. V, §320bis–§321, pp. 240–242], Boccaletti and Pucacco [6, Ch. 3, §3.2, p. 187], Meirovitch [11, Ch. 11, §11.2], and references therein).

A complete solution of the n–body problem would require $6n$ independent constants of integration, but we have only ten known constants of motion at our disposal. For instance, even in the case of $n = 3$, in which 18 constants are required, the lack of 8 of them renders the general solution of the three–body problem unfeasible. The difficulties increase when the number of bodies is larger ($n > 3$). For this reason, the general solution of the n–body problem is not viable and the *search for certain particular solutions* (central configurations, equilibrium solutions, periodic orbits, quasi– periodic solutions, orbits on a torus, etc.) leading to a deeper insight into the problem, and a better understanding of the behaviour of the system, seems to be a promising strategy.

If the bodies are acted upon by their mutual gravitational attraction (say, $\lambda = -2$), we have the *gravitational n–body problem*, or the *gravitational many–body problem.* This case provides us with a simplified (but very useful) model for the problem of *planetary motion.*

The most simple case, when $n = 2$, is the *two–body problem with internal forces*, whose treatment can be separated into two independent, decoupled subproblems (Meirovitch [11, Ch. 1, §1.8, pp. 25–30], Boccaletti and Pucacco [6, Ch. 2, §2.1, pp. 126–136]) that are, indeed, two independent, fictitious one–body problems (two auxilary problems of a single particle):

- the problem of free motion of the center of mass of the two– body system (either a uniform rectilinear motion or a state of rest), and
- the problem of relative motion of one of the bodies with respect to the other (which is a one–body problem within a central force field).

In particular, the gravitational two–body problem is a conservative dynamical system in which the (sub)problem of relative motion, usually known as the *Kepler problem* and treated as the motion of a single particle about a fixed center under Newtonian attraction, can be completely solved, and its solution can be reduced to, at most, two quadratures (Meirovitch [11, Ch. 1, §1.9, pp. 30–37, Ch. 11, §11.1, pp. 409–413], Boccaletti and Pucacco [6, Ch. 2, §2.1, pp. 126–136, §2.4, pp. 147–156], Abad [1, Ch. 7, pp. 109–122, Ch. 8, pp. 123–140]).

At the beginning of the 17th century, on the basis of the heliocentric hypothesis postulated by Copernicus and using the observations of the positions of the planets in the Solar System collected by Tycho Brahe, Johannes Kepler had already achieved a *qualitative* description of planetary motion, summarized in three laws known as Kepler's Laws of Planetary Motion.

Once completely solved the gravitational two–body problem after Kepler and Newton, the study of the motion of three (or more) bodies under the effect of their mutual gravitational interaction attracted the attention of many eminent scholars [2] since the end of the seventeenth century, including Newton himself, who devoted some parts of his *"Principia"* to the investigation of the motion of the Moon. Although the general solution of the gravitational 3–body problem seemed to be an unattainable goal, and no general method for the *direct* integration of the differential equations of motion (leading to an explicit solution of the problem at issue) had been found, Euler and Lagrange were able to obtain *some exact, particular solutions* for certain special cases in which *the motion of the three bodies should be confined to the same plane* (Meirovitch [11, Ch. 11, §11.3, pp. 416–420]):

- One of the cases corresponds to a configuration in which *the three bodies form an equilateral triangle uniformly rotating with respect to an inertial reference frame with constant angular velocity around an axis perpendicular to the plane of motion*; in this process the three bodies follow *coplanar circular orbits.*

- In another case *the three bodies lie in a straight line, and this line rotates with constant angular velocity in the plane of motion.* The relative position of the three bodies along this line is determined from the only root of a certain quintic equation in the interval $(0, 1)$.

  These equilateral–triangle and straight–line configurations are known as the *relative—equilibrium solutions*, or *stationary solutions.* In both cases the 3–body system rotates with constant angular velocity around the center of mass and, consequently, the bodies describe *circumferences in the plane of motion* relative to an inertial reference frame.

- Another particular solution allows the three bodies to form *an equilateral triangle within the plane of motion, but the distance between them can change with time.* In this case the individual motions of the three bodies are not circular any more, but *general conic–sections with a focus at the center of mass of the system.*

These results, originally published in 1772 by Lagrange (who also recovered the straight–line solutions previously discovered by Euler), are at the root of the concepts of the *central configurations* and the *homographic solutions* in the 3– and n–body problem.

These concepts, and the study of some of their main properties and relationships, constitute the topic with which this Dissertation is concerned.

Cid [8] revisited the original Lagrange 1772 memoir and, with the help of tools of Vector Calculus, simplified the derivation performed by Lagrange, arriving at the same results and conclusions in an elegant and concise fashion.

The above *Lagrange results* concerning the gravitational 3– body problem *can be generalized* to the case of more general central–force models, and to n–body systems with $n > 3$.

In this Undergraduate Dissertation we consider the motion of a mechanical system of n particles such that the only forces are the mutual forces exerted by the particles on each other, such forces between each pair of particles act along the straight line connecting them, their magnitude is proportional to a

[2] Clairaut, D'Alembert, Euler, Lagrange, Laplace, Cauchy, Hamilton, Jacobi, Weierstrass, Painlevé, Poincaré, Hill, Sundman, Levi–Civita, Birkhoff, etc., just to mention but a few mathematicians whose research on this problem enriched Celestial Mechanics, Physics and Mathematics.

power of the mutual distance separating the particles, and there are no external forces acting upon the system.

Under these hypotheses, *we give an account of the theory of certain particular solutions to the n–body problem*: the so– called *central configurations* (obtained when the force acting on each body is proportional to its mass and to its position vector referred to the center of mass of the system; in this way, the n bodies maintain their relative position under homotheties and rotations) and the *homographic solutions* (when the configuration formed by the bodies at the instant t with respect to an inertial barycentric reference frame remains similar to itself as t varies). These concepts and their relationship will be analyzed in the general case, and then the most simple, special cases ($n = 2, 3, 4$) will be considered.

The number and types of the homographic solutions are perfectly known and completely determined for the gravitational 2– and 3–body problem. When $n \geq 4$ these questions have not been answered yet, and nowadays they still constitute open, unsolved problems. For this reason we restrict ourselves to describing just a few selected cases, omitting the proof of the respective statements, for which the corresponding bibliographical references (mainly journal articles) are given.

Most content of this Dissertation is based on a part of Chapter III ("Problema de n cuerpos") of the (unpublished) lecture notes *"Curso de Mecánica Celeste"*, [7], compiled by Professor Dr. Rafael Cid Palacios (University of Zaragoza) as teaching materials for his lectures in the optional subject "Celestial Mechanics" pertaining to the former, five–year Degree in Mathematics ("Licenciatura en Matemáticas") at the Faculty of Science of the University of Zaragoza in the 1970s and 1980s.

For the preparation of these lecture notes on the n–body problem Prof. Cid took his cue from Wintner [14], Ch. V ("The Problem of Several Bodies").

For a more modern presentation of some issues treated by Wintner, see Boccaletti & Pucacco [6], Ch. 3 ("The N–Body Problem"), and Ch. 4 ("The Three–Body Problem"). Also Meirovitch [11], Ch. 1, §1.7 ("Systems of Particles"), and Ch. 11, §11.2 ("The Many–Body Problem"), §11.3 ("The Three–Body Problem").

In some cases, the statements and/or proofs presented in this Undergraduate Dissertation are appropriately reformulated and adapted to a more modern mathematical language and presentation. In particular, the abstruse notations in Cid [7] and Wintner [14] have been translated into more easily readable vector and differential–operator notations.

The *content of this Dissertation* is organized into a Summary in Spanish language, three chapters, a bibliographical section, and two appendices.

The present *Chapter 1* offers the conceptual, theoretical framework of the n–body problem, and the purpose of our work.

Chapter 2 is devoted to a geometric classification of solutions to the n–body problem, the definition and most significant properties of central configurations and homographic solutions, and the relationship between these concepts.

In *Chapter 3* a characterization of central configurations in the general n–body problem, in terms of mutual distances between the bodies, is given, and then the result is applied to the particular cases of $n = 3$ and $n = 4$. The study in the case of the 3– body problem is complete and exhaustive. However in the gravitational 4–body problem the research is far from being complete, and only some special instances of planar central configurations, depending on the masses of the four bodies, are presented.

Appendix A contains the formulation of the several–body problem, as well as the definition of the main kinetic and dynamical quantities of interest, and the ten classical constants of the motion that this problem possesses.

Finally, in *Appendix B* we collect some mathematical and kinematical concepts and results that are used throughout the work.

Chapter 2

Central Configurations and Homographic Solutions in the n-Body Problem

In Appendix A a rigorous, general statement of the *n-body problem* is established, the most significant kinetic and dynamical quantities required throughout this study are defined, and the ten classical first integrals of the problem are summarized. Since finding a sufficient number of functionally independent first integrals that allows the complete integration of the problem is not possible, we will concentrate on *some particular solutions.*

In this chapter a geometric classification of solutions to the n-body problem is given, and some important properties of those solutions, along with some relationships between them, are collected. Besides two kinds of particular solutions are studied: the *central configurations* and the *homographic solutions.* The *equivalence* between both kinds of solutions is proved at the end of the chapter.

2.1 A Geometric Classification of the Solutions to the n-Body Problem.

A solution $\{\boldsymbol{r}_i = \boldsymbol{r}_i(t), i = 1, \ldots, n\}$ to the n-body problem is:

- *Rectilinear* if, for every t, the n bodies are placed in a fixed straight line, that is,

$$\boldsymbol{r}_i(t) = \rho_i(t)\,\boldsymbol{u}, \quad i = 1, \ldots, n, \tag{2.1.1}$$

  where $\boldsymbol{u}$ is a unit constant vector which defines the direction and $\rho_i(t)$ the magnitude of the position vector at each instant t.

- *Collinear* if, for every t, there exists a straight line containing the n bodies, although this line can change its position in space throughout time, that is,

$$\boldsymbol{r}_i(t) = \rho_i(t)\,\boldsymbol{u}(t), \quad i = 1, \ldots, n, \tag{2.1.2}$$

  using the same notation as before.

- *Planar* if the n bodies are placed in a fixed plane Π for all t.

- *Coplanar* (or *flat*) if the n bodies are placed in an instantaneous plane $\Pi(t)$ that can change its position at each t.

- *Spatial* if the n bodies are not contained neither in a straight line nor in a plane.

Proposition 2.1. *Rectilinear solutions do not admit an invariable plane. It means that* $\boldsymbol{G} = \boldsymbol{0}$.

Proof. Consider the position vectors from (2.1.1) and their derivatives. Introducing these expressions into (A.1.9) we obtain that $\boldsymbol{G} = \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i = \sum_i m_i\,(\rho_i\,\boldsymbol{u}) \times (\dot{\rho}\,\boldsymbol{u}) = \sum_i m_i\,\rho_i\,\dot{\rho}_i\,(\boldsymbol{u} \times \boldsymbol{u}) = \boldsymbol{0}$, since the cross product of any vector with itself always vanish. And therefore we obtain that $\boldsymbol{G} = \boldsymbol{0}$. ∎

Proposition 2.2. *A collinear solution is rectilinear if and only if* $\boldsymbol{G} = \boldsymbol{0}$.

Proof. The position vectors of a collinear solution are given by (2.1.2). Differentiating with respect to time t we have $\dot{\boldsymbol{r}}_i(t) = \dot{\rho}_i(t)\,\boldsymbol{u}(t) + \rho_i(t)\,\dot{\boldsymbol{u}}(t)$, and replacing it in (A.1.9), we obtain $\boldsymbol{G} = \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i = \sum_i m_i\,(\rho_i(t)\,\boldsymbol{u}(t)) \times (\dot{\rho}_i(t)\,\boldsymbol{u}(t) + \rho_i(t)\,\dot{\boldsymbol{u}}(t)) = \sum_i \left[m_i\,\rho_i^2\,(\boldsymbol{u}(t) \times \dot{\boldsymbol{u}}(t))\right] = [\boldsymbol{u}(t) \times \dot{\boldsymbol{u}}(t)] \sum_i m_i\,\rho_i^2$.

Since the factor $\sum_i m_i\,\rho_i^2 > 0$, then if $\boldsymbol{G} = \boldsymbol{0}$, we have $\boldsymbol{u}(t)\times\dot{\boldsymbol{u}}(t) = \boldsymbol{0}$. On the other hand, $\boldsymbol{u}(t)\cdot\boldsymbol{u}(t) = 1$, then $\boldsymbol{u}(t) \cdot \dot{\boldsymbol{u}}(t) = 0$, and the only possibility is $\dot{\boldsymbol{u}}(t) = 0$, i.e., $\boldsymbol{u}$ = const. (rectilinear solution).

Conversely, if the solution is rectilinear, then $\boldsymbol{u}$ = const. So, $\dot{\boldsymbol{u}} = \boldsymbol{0}$, and $\boldsymbol{G} = \boldsymbol{0}$. ■

Proposition 2.3. *Any collinear solution is planar.*

Proof. On the one hand, if $\boldsymbol{G} = \boldsymbol{0}$ the solution is rectilinear and therefore planar. On the other hand, if $\boldsymbol{G} \neq \boldsymbol{0}$, we only need to prove that the vector $\boldsymbol{u}$ is perpendicular to $\boldsymbol{G}$ for all t, which amounts to proving that $\boldsymbol{u} \cdot \boldsymbol{G} = 0$. To do it, it is necessary to take into account the equalities $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{u} \times \dot{\boldsymbol{u}} = \boldsymbol{0}$ as in Proposition 2.2.

Differentiating the equation (2.1.2) and forming the expression $\boldsymbol{r}_i \times m_i\dot{\boldsymbol{r}}_i$, we obtain

$$\boldsymbol{r}_i \times m_i\dot{\boldsymbol{r}}_i = \rho_i\,\boldsymbol{u} \times m_i\,(\dot{\rho}_i\boldsymbol{u} + \rho_i\dot{\boldsymbol{u}}) = (\rho_i\boldsymbol{u} \times m_i\dot{\rho}_i\boldsymbol{u}) + (\rho_i\boldsymbol{u} \times m_i\,\rho_i\dot{\boldsymbol{u}}) = m_i\,\rho_i\,\dot{\rho}_i\,(\boldsymbol{u} \times \boldsymbol{u}) + m_i\,\rho_i^2\,(\boldsymbol{u} \times \dot{\boldsymbol{u}}) = \boldsymbol{0},$$

and applying this to the product $\boldsymbol{u} \cdot \boldsymbol{G}$ we have

$$\boldsymbol{u} \cdot \boldsymbol{G} = \boldsymbol{u} \cdot \sum_{i=1}^{n} \boldsymbol{r}_i \times m_i\dot{\boldsymbol{r}}_i = \sum_{i=1}^{n} \boldsymbol{u} \cdot (\boldsymbol{r}_i \times m_i\,\dot{\boldsymbol{r}}_i) = \sum_{i=1}^{n} \boldsymbol{u} \cdot \boldsymbol{0} = 0.$$

Consequently, we obtain that the vector $\boldsymbol{u}$ is perpendicular to $\boldsymbol{G}$ for all t. And, since $\boldsymbol{G}$ is a characteristic vector of the invariable plane, we have that the position vectors remain in the invariable plane for all t. ■

Proposition 2.4. *If a planar solution admits an invariable plane ($\boldsymbol{G} \neq \boldsymbol{0}$), both planes coincide.*

Proof. Let a solution be planar; then the vectors $\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i$ for all $i = 1, \ldots, n$, are placed on this plane. Therefore, the vectors $m_i\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i$ and $\boldsymbol{G}$ are perpendicular to this plane, and, since $\boldsymbol{G}$ is a characteristic vector of the invariable plane, both planes coincide. ■

Proposition 2.5. *Every coplanar solution with* $\boldsymbol{G} = \boldsymbol{0}$ *is planar.*

Proof. Let $\boldsymbol{r}_i(t)$ be the position vector of the particle P_i in an inertial frame, and $\boldsymbol{s}_i(t)$ the position vector of P_i in a non-inertial frame such that its Oz axis is perpendicular to the plane of the motion. Let us suppose that at the instant $t = 0$ we have $\boldsymbol{r}_i^0 = \boldsymbol{r}_i(t = 0) = \boldsymbol{s}_i(t = 0)$, $\dot{\boldsymbol{r}}_i^0 = \dot{\boldsymbol{r}}_i(t = 0)$; then there exists a rotation matrix Q such that we have $\boldsymbol{r}_i^0 = Q\boldsymbol{s}_i$. From now on, we will apply the properties of a rotation matrix given in Section B.1 of Appendix B.

Since, by hypothesis, $\boldsymbol{G} = \boldsymbol{0}$, we also have $Q^{-1}\boldsymbol{G} = \boldsymbol{0}$, and then

$$Q^{-1}\boldsymbol{G} = Q^{-1}\,\sum_{i=1}^{n} m_i\boldsymbol{r}_i^0 \times \dot{\boldsymbol{r}}_i^0 = Q^{-1}\,\sum_{i=1}^{n} m_i\,(Q\boldsymbol{s}_i) \times \left(\dot{Q}\boldsymbol{s}_i + Q\dot{\boldsymbol{s}}_i\right) = \sum_{i=1}^{n} m_i\boldsymbol{s}_i \times \left(Q^{-1}\dot{Q}\boldsymbol{s}_i + \dot{\boldsymbol{s}}_i\right)$$

$$= \sum_{i=1}^{n} m_i\boldsymbol{s}_i \times Q^{-1}\dot{Q}\,\boldsymbol{s}_i + \sum_{i=1}^{n} m_i\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i = \sum_{i=1}^{n} m_i\boldsymbol{s}_i \times W\,\boldsymbol{s}_i + \sum_{i=1}^{n} m_i\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i = \boldsymbol{0}.$$

It means that

$$\sum_{i=1}^{n} m_i\boldsymbol{s}_i \times W\,\boldsymbol{s}_i = -\sum_{i=1}^{n} m_i\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i.$$

Consider a vector $\boldsymbol{r}$ perpendicular to the instantaneous plane $\Pi(t)$. Then $\boldsymbol{r}$ satisfies that $\boldsymbol{r} \cdot \boldsymbol{s}_i = 0$. In this case, if we form the cross product of both sides of the preceding expression with $\boldsymbol{r}$,

$$\sum_{i=1}^{n} \boldsymbol{r} \times (m_i\boldsymbol{s}_i \times W\,\boldsymbol{s}_i) = -\sum_{i=1}^{n} m_i\,\boldsymbol{r} \times (\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i) = \boldsymbol{0},$$

Figure 2.1: Left, central configuration. Right, non-central configuration.

since $\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i$ is perpendicular to the plane for all t, and therefore $\boldsymbol{r} \times (\boldsymbol{s}_i \times \dot{\boldsymbol{s}}_i) = \boldsymbol{0}$. Then, we obtain

$$\sum_{i=1}^{n} \boldsymbol{r} \times (m_i \boldsymbol{s}_i \times W\, \boldsymbol{s}_i) = \sum_{i=1}^{n} m_i\ [(\boldsymbol{r} \cdot W\, \boldsymbol{s}_i)\, \boldsymbol{s}_i - (\boldsymbol{r} \cdot \boldsymbol{s}_i)\ W \boldsymbol{s}_i] = \boldsymbol{0},$$

where we have used the identity $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{z})\, \boldsymbol{y} - (\boldsymbol{x} \cdot \boldsymbol{y})\, \boldsymbol{z}$. Consequently,

$$\sum_{i=1}^{n} m_i\, (\boldsymbol{r} \cdot W\, \boldsymbol{s}_i)\, \boldsymbol{s}_i = \boldsymbol{0},$$

from which we can deduce

$$\boldsymbol{r} \cdot W \left[\sum_{i=1}^{n} m_i\, (\boldsymbol{r} \cdot W\, \boldsymbol{s}_i)\, \boldsymbol{s}_i \right] = \boldsymbol{r} \cdot \sum_{i} m_i\, (\boldsymbol{r} \cdot W \boldsymbol{s}_i)\, W \boldsymbol{s}_i = \sum_{i=1}^{n} m_i\, [\boldsymbol{r} \cdot (\omega \times \boldsymbol{s}_i)]^2 = 0,$$

and this is only possible if

$$\boldsymbol{r} \cdot (\omega \times \boldsymbol{s}_i) = 0, \quad \text{or} \quad \omega \cdot (\boldsymbol{s}_i \times \boldsymbol{r}) = 0,$$

with $\boldsymbol{s}_i \times \boldsymbol{r} \in \Pi(t)$. Consequently, it is a planar solution. ■

2.2 Central Configurations

A *central configuration* of the n-body problem is a geometric configuration of n particles where the acceleration vector of each particle is a common scalar multiple of the corresponding position vector (Figure 2.1, Left).

Definition 2.1. Given n points, $P_1, \ldots, P_n$, with masses $m_1, \ldots, m_n$, and position vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ referred to a barycentric inertial reference frame, a *central configuration at an instant t* is a solution $\{\boldsymbol{r}_i, i = 1, \ldots, n\}$ that satisfies, for a scalar σ, independent of i, the following property:

$$m_i\, \ddot{\boldsymbol{r}}_i = \nabla_{\boldsymbol{r}_i} U = \sigma\, m_i\, \boldsymbol{r}_i, \quad i = 1, 2, \ldots, n. \tag{2.2.1}$$

After this definition, some remarks are in order.

- It is an *instantaneous concept.*
- *The scalar quantity σ is determined by U and J.* Indeed, using Definition 2.1 and Definition A.1 of J: $\sum_i \boldsymbol{r}_i \cdot \nabla_{\boldsymbol{r}_i} U = \sigma \sum_i m_i\, \boldsymbol{r}_i^2 = \sigma\, J$, and bearing in mind Equation (A.1.6) we obtain

$$\sigma = (\lambda + 1) \frac{U}{J}. \tag{2.2.2}$$

- If the points $P_1, \ldots, P_n$ form a central configuration, then the n points with the position vectors $c\, \boldsymbol{r}_1, \ldots, c\, \boldsymbol{r}_n$ and $Q\, \boldsymbol{r}_1, \ldots, Q\, \boldsymbol{r}_n$, where $c > 0$ is a scalar and Q is a rotation matrix, also form a central configuration themselves, as long as the definition of a central configuration only depends on their positions. In other words, *central configurations are independent of homotheties and rotations.*

Proposition 2.6. *The points $P_1, \ldots, P_n$ form a central configuration if and only if the following conditions are fulfilled:*

$$\nabla_{\boldsymbol{r}_i}\left(J^{-(\lambda+1)}U^2\right) = \mathbf{0}, \quad i = 1, \ldots, n. \tag{2.2.3}$$

This means that $P_1, \ldots, P_n$ form a central configuration if and only if $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ is a critical point of the scalar function $J^{-(\lambda+1)}\, U^2$.

Proof. Calculating the required gradient,

$$\nabla_{\boldsymbol{r}_i}\left(J^{-(\lambda+1)}U^2\right) = U\, J^{-(\lambda+1)}\left[2\,\nabla_{\boldsymbol{r}_i} U - (\lambda+1)J^{-1}\left(\nabla_{\boldsymbol{r}_i} J\right)\, U\right] = 2\, U\, J^{-(\lambda+1)}\left[\nabla_{\boldsymbol{r}_i} U - (\lambda+1)\frac{U}{J}\frac{\nabla_{\boldsymbol{r}_i} J}{2}\right].$$

The gradient of J, introduced in Definition A.1, Eqs. (A.1.3), is $\nabla_{\boldsymbol{r}_i} J = 2m_i\boldsymbol{r}_i$. Introducing this gradient and Eq. (2.2.2) into the preceding equality leads to

$$\nabla_{\boldsymbol{r}_i}\left(J^{-(\lambda+1)}U^2\right) = 2\, U\, J^{-(\lambda+1)}\left[\nabla_{\boldsymbol{r}_i} U - \sigma\, m_i\boldsymbol{r}_i\right] = \mathbf{0},$$

which holds true if and only if

$$\nabla_{\boldsymbol{r}_i} U = \sigma\, m_i\, \boldsymbol{r}_i.$$

■

2.3 Homographic Solutions to the n-Body Problem

Among all the particular solutions of the n-body problem, homographic solutions are *the most simple ones*. There are two important particular cases of homographic solutions: *homothetic solutions* and *relative-equilibrium solutions*.

Concepts and notations related to rotation matrices are presented in Appendix B.

2.3.1 Basic definitions

Definition 2.2. A *homographic solution* to the n-body problem is a solution $\{\boldsymbol{r}_i = \boldsymbol{r}_i(t),\ i = 1, \ldots, n\}$ in which the configuration formed by the n bodies remains *self-similar throughout time*. It is of the form

$$\boldsymbol{r}_i(t) = \rho(t)\, Q(t)\, \boldsymbol{r}_i^0, \quad i = 1, \ldots, n, \tag{2.3.1}$$

where $\boldsymbol{r}_i^0 = \boldsymbol{r}_i(t=0)$ are the so-called "*initial conditions*", $\rho(t) > 0$ is a real scaling factor, and $Q(t) \in SO(3)$ is a rotation matrix. This means that the ratios of the mutual distances between the bodies remain constant.

Definition 2.3. A *homothetic solution* to the n-body problem is a solution $\{\boldsymbol{r}_i = \boldsymbol{r}_i(t),\ i = 1, \ldots, n\}$ of the form

$$\boldsymbol{r}_i(t) = \rho(t)\, \boldsymbol{r}_i^0, \quad i = 1, \ldots, n, \tag{2.3.2}$$

which is the same definition as before, but considering $Q(t) = I_3$, which means that the angular velocity vector $\boldsymbol{\omega}$ (see Appendix B) is zero: $\boldsymbol{\omega} = \mathbf{0}$.

Geometrically speaking, these solutions are *dilatations* ($\rho > 1$) or *contractions* ($0 < \rho < 1$) of the initial configuration $\left\{\boldsymbol{r}_i^0,\ i = 1, \ldots, n\right\}$.

Definition 2.4. A solution to the n-body problem for which the configuration formed by the n bodies remains *self-congruent* is called a *relative equilibrium*. This particular case of homographic solutions is formalized as

$$\boldsymbol{r}_i(t) = Q(t)\, \boldsymbol{r}_i^0, \quad i = 1, \ldots, n, \tag{2.3.3}$$

which is the same definition as before, but considering $\rho(t) = 1$ for all t.

Geometrically speaking, these solutions are *rotations* of the initial configuration $\{\boldsymbol{r}_i^0,\ i = 1, \ldots, n\}$. This configuration is always the same, and this problem is considered to be equivalent to the problem of the rotation of a rigid body.

Let us note that (2.3.1) coincides with (B.2.1) with $\boldsymbol{s}(t) = \boldsymbol{s} = \boldsymbol{r}_i^0$ constant. Then, the expressions from (B.2.2) become

$$Q^{-1}\,\boldsymbol{r}_i = \rho \boldsymbol{r}_i^0, \quad Q^{-1}\,\dot{\boldsymbol{r}}_i = \mathcal{R}\,\boldsymbol{r}_i^0, \quad Q^{-1}\,\ddot{\boldsymbol{r}}_i = \mathcal{K}\,\boldsymbol{r}_i^0, \tag{2.3.4}$$

where the matrices $\mathcal{R}$ and $\mathcal{K}$ are given by (B.2.3), and the vector $\boldsymbol{\omega} = \omega\,\boldsymbol{e}_\omega$ (see Appendix B, Definition B.2) represents the instantaneous angular velocity vector.

2.3.2 Main Dynamical Quantities in Terms of Initial Conditions

The position of the bodies in a homographic solution can be expressed in terms of initial conditions. This can also be extended to the most usual dynamical quantities: U, J, σ, etc.

So, from now on, we use the superscript (^{0}) to particularize any dynamical expression at the initial instant. For example, $J^0 = \sum_i m_i(\boldsymbol{r}_i^0 \cdot \boldsymbol{r}_i^0)$.

Proposition 2.7. *For a homographic solution the following relations hold:*

$$U = \rho^{\lambda+1}\,U^0, \tag{2.3.5}$$
$$\nabla_{\boldsymbol{r}_i} U = \rho^{\lambda}\,Q\,\nabla_{\boldsymbol{r}_i^0} U^0, \tag{2.3.6}$$
$$J = \rho^2\,J^0, \tag{2.3.7}$$
$$2\,T = \left(\dot{\rho}^2 + \omega^2\,\rho^2\right) J^0 - \rho^2\,\omega^2 \sum_i m_i \left(\boldsymbol{r}_i^0 \cdot \boldsymbol{e}_\omega\right)^2, \tag{2.3.8}$$
$$Q^{-1}\,\boldsymbol{G} \cdot \boldsymbol{e}_\omega = \rho^2\,\omega \left[J^0 - \sum_i m_i \left(\boldsymbol{e}_\omega \cdot \boldsymbol{r}_i^0\right)^2 \right], \tag{2.3.9}$$
$$\ddot{\boldsymbol{r}}_i^0 = K(t)\,\boldsymbol{r}_i^0, \quad \text{with} \quad K(t) = \rho^{-\lambda}\,\mathcal{K}, \tag{2.3.10}$$
$$\sigma = \rho^{\lambda-1}\,\sigma^0, \tag{2.3.11}$$

where $\ddot{\boldsymbol{r}}_i^0 = \ddot{\boldsymbol{r}}_i(t = 0)$ represents the acceleration of the particle P_i at the initial instant $t = 0$.

Proof.

• The mutual distance $\|\boldsymbol{r}_{ik}\|$ satisfies

$$\|\boldsymbol{r}_{ik}\| = \|\boldsymbol{r}_k - \boldsymbol{r}_i\| = \|\rho\,Q\,(\boldsymbol{r}_k^0 - \boldsymbol{r}_i^0)\| = \|\rho\,Q\,\boldsymbol{r}_{ik}^0\| = \rho\,\|Q\,\boldsymbol{r}_{ik}^0\| = \rho\,\|\boldsymbol{r}_{ik}^0\|.$$

Then, applying the relation to the definition of U in (A.1.4) and the expression of $\nabla_{\boldsymbol{r}_i} U$ in (A.1.5) we prove (2.3.5) and (2.3.6).

• Using Proposition B.1 (see Appendix B) together with the expression of $Q^{-1}\,\boldsymbol{r}_i$ from (2.3.4), and taking into account that Q^{-1} is also an orthogonal matrix, we find

$$\boldsymbol{r}_i \cdot \boldsymbol{r}_i = \left(Q^{-1}\boldsymbol{r}_i\right) \cdot \left(Q^{-1}\boldsymbol{r}_i\right) = \left(\rho\,\boldsymbol{r}_i^0\right) \cdot \left(\rho\,\boldsymbol{r}_i^0\right) = \rho^2(\boldsymbol{r}_i^0)^2.$$

Applying this expression to Definition of J in (A.1.3) we prove (2.3.7).

• Using Proposition B.1 together with the expression of $Q^{-1}\,\dot{\boldsymbol{r}}_i$ from (2.3.4),

$$\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i = \left(Q^{-1}\dot{\boldsymbol{r}}_i\right) \cdot \left(Q^{-1}\dot{\boldsymbol{r}}_i\right) = \left(\mathcal{R}\boldsymbol{r}_i^0\right) \cdot \left(\mathcal{R}\boldsymbol{r}_i^0\right),$$

and using the general Proposition $\boldsymbol{x} \cdot M\boldsymbol{y} = M^T\boldsymbol{x} \cdot \boldsymbol{y}$ for any square matrix M, we obtain

$$\left(\mathcal{R}\boldsymbol{r}_i^0\right) \cdot \left(\mathcal{R}\boldsymbol{r}_i^0\right) = \left[\left(\mathcal{R}^T\,\mathcal{R}\right)\boldsymbol{r}_i^0\right] \cdot \boldsymbol{r}_i^0;$$

with the expression of $\mathcal{R}$ from (B.2.3) we arrive at

$$\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i = \dot{\rho}^2\boldsymbol{r}_i^{0^2} + \rho^2(W\,\boldsymbol{r}_i^0)^2.$$

We can expand $\left(W\boldsymbol{r}_i^0\right)^2$ taking into account that $W\boldsymbol{r}_i^0 = \boldsymbol{\omega}\times\boldsymbol{r}_i^0$ and $(\boldsymbol{x}\times\boldsymbol{y})^2 = ||\boldsymbol{x}||^2\,||\boldsymbol{y}||^2 - (\boldsymbol{x}\cdot\boldsymbol{y})^2$. Then,

$$\left(W\,\boldsymbol{r}_i^0\right)^2 = \left(\boldsymbol{\omega}\times\boldsymbol{r}_i^0\right)^2 = \omega^2\left(\boldsymbol{e}_\omega\times\boldsymbol{r}_i^0\right)^2 = \omega^2\left[\left(\boldsymbol{r}_i^0\right)^2 - \left(\boldsymbol{e}_\omega\cdot\boldsymbol{r}_i^0\right)^2\right],$$

and putting the last two expressions together in Definition A.1, Eqs. (A.1.3), of T we prove (2.3.8).

• Now, using the other statement in Proposition B.1 and the expressions of $Q^{-1}\,\boldsymbol{r}_i$ and $Q^{-1}\,\dot{\boldsymbol{r}}_i$ in (2.3.4), and remembering that Q^{-1} is a rotation matrix, we have

$$Q^{-1}\left(\boldsymbol{r}_i\times\dot{\boldsymbol{r}}_i\right) = \left(Q^{-1}\,\boldsymbol{r}_i\right)\times\left(Q^{-1}\,\dot{\boldsymbol{r}}_i\right) = \rho\,\boldsymbol{r}_i^0\times\mathcal{R}\,\boldsymbol{r}_i^0 = \rho^2\left(\boldsymbol{r}_i^0\times W\boldsymbol{r}_i^0\right) = \rho^2\left[\boldsymbol{r}_i^0\times(\boldsymbol{\omega}\times\boldsymbol{r}_i^0)\right] = \rho^2\omega\left[\boldsymbol{r}_i^0\times\left(\boldsymbol{e}_\omega\times\boldsymbol{r}_i^0\right)\right],$$

from which, using that $\boldsymbol{x}\times(\boldsymbol{y}\times\boldsymbol{z}) = (\boldsymbol{x}\cdot\boldsymbol{z})\,\boldsymbol{y} - (\boldsymbol{x}\cdot\boldsymbol{y})\,\boldsymbol{z}$,

$$Q^{-1}\left(\boldsymbol{r}_i\times\dot{\boldsymbol{r}}_i\right) = \rho^2\,\omega\left[\left(\boldsymbol{r}_i^0\cdot\boldsymbol{r}_i^0\right)\boldsymbol{e}_\omega - \left(\boldsymbol{r}_i^0\cdot\boldsymbol{e}_\omega\right)\boldsymbol{r}_i^0\right].$$

If we multiply this equality by m_i and adding up the resulting expressions over $i = 1,\ldots,n$,

$$Q^{-1}\sum_i m_i\left(\boldsymbol{r}_i\times\dot{\boldsymbol{r}}_i\right) = Q^{-1}\,\boldsymbol{G} = \rho^2\,\omega\sum_i m_i\left[\left(\boldsymbol{r}_i^0\cdot\boldsymbol{r}_i^0\right)\boldsymbol{e}_\omega - \left(\boldsymbol{r}_i^0\cdot\boldsymbol{e}_\omega\right)\boldsymbol{r}_i^0\right].$$

Taking the dot product of both sides with $\boldsymbol{e}_\omega$,

$$Q^{-1}\,\boldsymbol{G}\cdot\boldsymbol{e}_\omega = \rho^2\,\omega\sum_i m_i\left[\left(\boldsymbol{r}_i^0\cdot\boldsymbol{r}_i^0\right) - \left(\boldsymbol{r}_i^0\cdot\boldsymbol{e}_\omega\right)^2\right].$$

And remembering the Definition (A.1.3) of J^0,

$$Q^{-1}\,\boldsymbol{G}\cdot\boldsymbol{e}_\omega = \rho^2\,\omega\left[J^0 - \sum_i m_i\left(\boldsymbol{r}_i^0\cdot\boldsymbol{e}_\omega\right)^2\right],$$

which proves (2.3.9).

• Denoting the acceleration vector of P_i at the instant $t = 0$ as $\ddot{\boldsymbol{r}}_i^0$, we have

$$m_i\ddot{\boldsymbol{r}}_i^0 = m_i\,\ddot{\boldsymbol{r}}_i(t=0) = \nabla_{\boldsymbol{r}_i}U(t=0) = \nabla_{\boldsymbol{r}_i^0}U^0;$$

and then, applying (2.3.6),

$$m_i\ddot{\boldsymbol{r}}_i = \nabla_{\boldsymbol{r}_i}U = \rho^\lambda\,Q\,\nabla_{\boldsymbol{r}_i^0}U^0 = m_i\,\rho^\lambda\,Q\,\ddot{\boldsymbol{r}}_i{}^0;$$

and finally, in view of the last equation in (2.3.4),

$$\mathcal{K}\,\boldsymbol{r}_i^0 = Q^{-1}\,\ddot{\boldsymbol{r}}_i = \rho^\lambda\,\ddot{\boldsymbol{r}}_i^0 \iff \ddot{\boldsymbol{r}}_i^0 = \rho^{-\lambda}\,\mathcal{K}\,\boldsymbol{r}_i^0 = K(t)\,\boldsymbol{r}_i^0,$$

which proves Eq. (2.3.10).

• Considering Equalities (2.2.2), (2.3.5) and (2.3.7) we can write

$$\sigma = (\lambda+1)\,\frac{U}{J} = (\lambda+1)\,\frac{\rho^{\lambda+1}\,U^0}{\rho^2\,J^0} = \rho^{\lambda-1}\,\sigma^0,$$

which proves (2.3.11). ■

2.3.3 Some Properties of Homographic Solutions

Theorem 2.1. *All non-coplanar homographic solutions with $\lambda + 3 \neq 0$ are homothetic solutions.*

Proof. For any non-coplanar homographic solution, there exist three linearly independent vectors $\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0$. Then, $\boldsymbol{r}_i^0 \cdot \left(\boldsymbol{r}_j^0 \times \boldsymbol{r}_k^0\right) \neq 0$. Taking into account (2.3.10) we have

$$\ddot{\boldsymbol{r}}_i^0 = K(t)\,\boldsymbol{r}_i^0, \quad \ddot{\boldsymbol{r}}_j^0 = K(t)\,\boldsymbol{r}_j^0, \quad \ddot{\boldsymbol{r}}_k^0 = K(t)\,\boldsymbol{r}_k^0 \implies \left(\ddot{\boldsymbol{r}}_i^0, \ddot{\boldsymbol{r}}_j^0, \ddot{\boldsymbol{r}}_k^0\right) = K(t)\left(\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0\right),$$

where $(\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0)$ represents the matrix whose columns are the components of the vectors $\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0$, and $(\ddot{\boldsymbol{r}}_i^0, \ddot{\boldsymbol{r}}_j^0, \ddot{\boldsymbol{r}}_k^0)$ stands for the matrix whose columns are the components of the vectors $\ddot{\boldsymbol{r}}_i^0, \ddot{\boldsymbol{r}}_j^0, \ddot{\boldsymbol{r}}_k^0$. The inverse of matrix $(\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0)$ exists, since its determinant is $\boldsymbol{r}_i^0 \cdot (\boldsymbol{r}_j^0 \times \boldsymbol{r}_k^0) \neq 0$. Therefore,

$$K(t) = (\ddot{\boldsymbol{r}}_i^0, \ddot{\boldsymbol{r}}_j^0, \ddot{\boldsymbol{r}}_k^0)\,(\boldsymbol{r}_i^0, \boldsymbol{r}_j^0, \boldsymbol{r}_k^0)^{-1} = \text{const.}$$

The matrix K can be decomposed into a symmetric matrix B and an antisymmetric matrix, where, using (B.2.3) (see Appendix B), the constant matrix B is

$$B = \frac{1}{2}\left(K + K^T\right) = \rho^{-\lambda}\left(\ddot{\rho}\, I_3 + \rho\, W^2\right).$$

Subtracting pairs of diagonal elements of B we get $\rho^{-\lambda+1}(\omega_i^2 - \omega_j^2) = \text{const.}$ Furthermore, $\rho^{-\lambda+1}(\omega_i^2 \cdot \omega_j^2) = \text{const.}$ for $i, j = 1, 2, 3$, and it is easy to deduce that $\rho^{-\lambda+1}\,\omega^2 = \text{const.}$, and $\rho^{-\lambda+1}\,\omega_i^2 = \text{const.}$ for $i = 1, 2, 3$.

Moreover, adding the diagonal elements of B we get $3\rho^{-\lambda}\,\ddot{\rho} - 2\rho^{-\lambda+1}\,\omega^2 = \text{const.}$, and it is easy to deduce that

$$\rho^{-\lambda+1}\,\omega^2 = \text{const.}, \quad \rho^{-\lambda+1}\,\ddot{\rho} = \text{const.}, \quad \text{and} \quad B\,\boldsymbol{e}_\omega = \rho^{-\lambda}\left(\ddot{\rho} I_3 + \rho\, W^2\right)\boldsymbol{e}_\omega = \rho^{-\lambda}\,\ddot{\rho}\,\boldsymbol{e}_\omega, \tag{2.3.12}$$

because $W^2\,\boldsymbol{e}_\omega = \boldsymbol{0}$, see Eq. (B.1.6).

Since B and $\rho^{-\lambda}\,\ddot{\rho}$ are constants, it is deduced that $\boldsymbol{e}_\omega$ is constant. The vector $Q^{-1}\,\boldsymbol{G} \cdot \boldsymbol{e}_\omega$ given by (2.3.9) is also a constant vector, since it is the projection of a vector $Q^{-1}\,\boldsymbol{G}$, of constant norm, on the constant vector $\boldsymbol{e}_\omega$.

The term $J^0 - \sum_i m_i(\boldsymbol{e}_\omega \cdot \boldsymbol{r}_i^0)^2$ in (2.3.9) cannot vanish because it would require that $(\boldsymbol{r}_i^0)^2 = (\boldsymbol{e}_\omega \cdot \boldsymbol{r}_i^0)^2$, i.e, all the vectors $\boldsymbol{r}_i^0$ should have the same direction as $\boldsymbol{e}_\omega$, which contradicts the hypothesis of the theorem. Then $\rho^2\,\omega$ must be constant, and we have two possibilities: either $\omega = 0$ or $\omega \neq 0$.

- If $\omega \neq 0$, then dividing the constant $\rho^{-\lambda+1}\,\omega^2$ by $\rho^4\,\omega^2$ we find that $\rho^{-(\lambda+3)}$ is a constant. Therefore, considering $\lambda + 3 \neq 0$, we prove that ρ and $\boldsymbol{\omega}$ are constants and so is $K(t) = \rho^{-\lambda+1}\,W^2$. Since the result of multiplying W^2 by any vector gives a vector orthogonal to $\boldsymbol{e}_\omega$, then the initial configuration must be in a plane, which contradicts the hypothesis.
- If $\omega = 0$, then $Q = I_3$, and the configuration is homothetic. ■

Theorem 2.2. *Any coplanar homographic solution with $\lambda + 3 \neq 0$ is planar.*

Proof. In Proposition 2.3 we proved that all collinear solutions are planar; then these solutions can be excluded from this proof.

To more easily characterize the *planar solution* we define a barycentric inertial reference frame $\mathcal{S}_\omega$ in which the axis Oz is defined by $\boldsymbol{e}_\omega$, and the other axes are defined by two constant vectors in the plane of motion of the particles.

If we differentiate the expression $\ddot{\boldsymbol{r}}_i^0 = K(t)\,\boldsymbol{r}_i^0$ we obtain $\dot{K}(t)\,\boldsymbol{r}_i^0 = \boldsymbol{0}$. The third component of $\boldsymbol{r}_i^0$ will be zero in the reference frame $\mathcal{S}_\omega$, and the linear system $\dot{K}(t)\,\boldsymbol{r}_i^0 = \boldsymbol{0}$ gives the solution $\dot{k}_{11} = \dot{k}_{12} =$

$\dot{k}_{21} = \dot{k}_{22} = \dot{k}_{31} = \dot{k}_{32} = 0$. Besides the third component of $\ddot{\boldsymbol{r}}_i^0$ is zero, and from $\ddot{\boldsymbol{r}}_i^0 = K(t)\,\boldsymbol{r}_i^0$ we obtain $k_{31} = k_{32} = 0$. The remaining components of $K(t)$ are given by

$$\begin{aligned} k_{11} &= \rho^{-\lambda}\left[\ddot{\rho} - \rho\left(\omega_2^2 + \omega_3^2\right)\right] = \text{const.}\,, & k_{12} &= -\rho^{-\lambda}\left[2\dot{\rho}\omega_3 + \rho\left(\dot{\omega}_3 - \omega_1\omega_2\right)\right] = \text{const.}\,,\\ k_{22} &= \rho^{-\lambda}\left[\ddot{\rho} - \rho\left(\omega_1^2 + \omega_3^2\right)\right] = \text{const.}\,, & k_{21} &= -\rho^{-\lambda}\left[2\dot{\rho}\omega_3 + \rho\left(\dot{\omega}_3 + \omega_1\omega_2\right)\right] = \text{const.}\,,\\ k_{31} &= -\rho^{-\lambda}\left[-2\dot{\rho}\omega_2 - \rho\left(\dot{\omega}_2 - \omega_1\omega_3\right)\right] = 0\,, & k_{32} &= \rho^{-\lambda}\left[2\dot{\rho}\omega_1 + \rho\left(\dot{\omega}_1 + \omega_2\omega_3\right)\right] = 0\,. \end{aligned}$$

Calculating $k_{22} - k_{11}$ and $k_{12} + k_{21}$ we obtain $\rho^{-\lambda+1}(\omega_2^2 - \omega_1^2) = \text{const.}$, and $\rho^{-\lambda+1}\omega_1\omega_2 = \text{const.}$, which implies that

$$\rho^{-\lambda+1}\omega_1^2 = \text{const.}, \quad \rho^{-\lambda+1}\omega_2^2 = \text{const.},$$

since they are quantities whose difference and product are constant.

Introducing new scalars a_1, a_2, we rewrite the preceding expressions in the form

$$\omega_1 = a_1\,\rho^{\frac{\lambda-1}{2}}, \quad \omega_2 = a_2\,\rho^{\frac{\lambda-1}{2}}, \tag{2.3.13}$$

and differentiating with respect to t,

$$\dot{\omega}_1 = \frac{1}{2}(\lambda - 1)\,a_1\,\rho^{\frac{\lambda-3}{2}}\,\dot{\rho}, \qquad \dot{\omega}_2 = \frac{1}{2}(\lambda - 1)\,a_2\,\rho^{\frac{\lambda-3}{2}}\,\dot{\rho}.$$

Replacing these expressions of $\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2$ in the preceding formulas for k_{31} and k_{32} and simplifying the factor $\rho^{(\lambda-1)/2}$, we obtain a homogeneous linear system of two equations for the unknown quantities a_1 and a_2,

$$\begin{pmatrix} \rho\,\omega_3 & -\frac{1}{2}(\lambda+3)\,\dot{\rho} \\ \frac{1}{2}(\lambda+3)\,\dot{\rho} & \rho\,\omega_3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We study the compatibility of this system.

- This homogeneous system possesses non-trivial solutions if and only if the determinant of the matrix A of coefficients vanishes,

  $$\det A = \rho^2\,\omega_3^2 + \frac{1}{4}(\lambda+3)^2\,\dot{\rho}^2 = 0.$$

  Given that it is a sum of non-negative summands, this sum is zero if and only if each addend is zero, that is,

  $$\rho^2\,\omega_3^2 = 0, \quad \frac{1}{4}(\lambda+3)^2\,\dot{\rho}^2 = 0 \quad \Longrightarrow \quad \rho\,\omega_3 = 0, \quad \dot{\rho} = 0,$$

  since $\lambda + 3 \neq 0$ by hypothesis. Therefore,

  $$\omega_3 = 0, \quad \rho = \text{const.}$$

  Then, in view of (2.3.13), we also have that $\omega_1 = \text{const.}$ and $\omega_2 = \text{const.}$, since $\lambda + 3 \neq 0$ by hypothesis.

  Assume that $\rho = \text{const.}$ and W is a constant matrix; then, using (B.2.3), $K(t)$ is also a constant matrix.

  Accordingly, $K(t) = W^2$ and $\ddot{\boldsymbol{r}}_i^0 = W^2\,\boldsymbol{r}_i^0$. But the matrix W^2 transforms each vector $\boldsymbol{r}_i^0$ into a vector $\ddot{\boldsymbol{r}}_i^0$ contained in a plane perpendicular to the rotation axis (Proposition B.4, Appendix B), for which all vectors $\ddot{\boldsymbol{r}}_i^0$ are along a straight line. This would imply that the solution should be collinear, a case that has been excluded from the very outset.

- If the above homogeneous linear systems only admits the trivial solution $a_1 = a_2 = 0$, in view of (2.3.13) we have that $\omega_1 = \omega_2 = 0$, and the rotation would take place around the Oz axis. But the bodies were initially in the Oxy plane, then they will remain in that plane for any instant t, which implies that the solution is planar, and the statement is proved. ∎

Proposition 2.8. *A homographic solution is homothetic if and only if* $\boldsymbol{G} = \boldsymbol{0}$.

Proof. Homothetic solutions satisfy that $Q = I_3$, $\dot{Q} = 0_{3x3}$, and $W = Q^{-1}\dot{Q} = 0_{3x3}$. Therefore, $Q^{-1}\boldsymbol{G} = \boldsymbol{0}$, and $\boldsymbol{G} = \boldsymbol{0}$.

Conversely, if $\boldsymbol{G} = \boldsymbol{0}$ then $Q^{-1}\boldsymbol{G} = \boldsymbol{0}$, from which

$$Q^{-1}\boldsymbol{G}\cdot\boldsymbol{e}_\omega = \rho^2\,\omega\left[J^0 - \sum_i m_i(\boldsymbol{e}_\omega\cdot\boldsymbol{r}_i^0)^2\right] = 0.$$

Hence, either $\omega = 0$ (which means that we have a homothetic solution), or $J^0 = \sum m_i\,(\boldsymbol{e}_\omega\cdot\boldsymbol{r}_i^0)^2$, which is the same thing as saying that the solution is collinear with $\boldsymbol{G} = \boldsymbol{0}$, that is, the solution is homothetic. ∎

Proposition 2.9. *A planar homographic solution is a relative-equilibrium solution if and only if* ω *is constant. This means that the configuration rigidly rotates with constant angular velocity around the center of mass.*

Proof. Any relative-equilibrium solution must also be planar. To prove it, assume that it is not a coplanar solution. Then, it must be a homothetic solution (by Theorem 2.1), which is not possible with our hypothesis. But if it is coplanar, then it must be planar (by Theorem 2.2).

Then, assuming that it is a planar solution, the rotation must take place around an axis perpendicular to the plane, and we can write $\boldsymbol{\omega} = \omega\,\boldsymbol{e}_\omega$, with $\boldsymbol{e}_\omega$ a constant vector perpendicular to that plane. So, $\boldsymbol{G}$ and $\boldsymbol{e}_\omega$ must have the same direction, obtaining

$$\|\boldsymbol{G}\| = \|Q^{-1}\boldsymbol{G}\| = \|Q^{-1}\boldsymbol{G}\|\,\|\boldsymbol{e}_\omega\|\cos 0 = Q^{-1}\boldsymbol{G}\cdot\boldsymbol{e}_\omega = \rho^2\,\omega\left[J^0 - \sum_i m_i\,(\boldsymbol{e}_\omega\cdot\boldsymbol{r}_i^0)^2\right] = \rho^2\,\omega\,J^0, \quad (2.3.14)$$

from which we deduce that $\rho^2\omega = \text{const.}$, since $\|\boldsymbol{G}\|$ is constant. Furthermore, considering $\rho = 1$ we conclude that $\omega = \text{const.}$

Conversely, if the solution is planar and we have $\omega = \text{const.}$ in Equation (2.3.14), this shows that $\rho = \text{const.} = 1$, and the solution is a relative equilibrium. ∎

2.3.4 Determination of the Rotation and the Homothety of a Homographic Solution

According to Theorems 2.1 and 2.2, a homographic solution can only be either planar or homothetic.

In what follows we use the reference frame $\mathcal{S}_\omega$ introduced in the proof of Theorem 2.2. Then, the rotation is a rotation of angle $\theta = \theta(t)$ around the axis Oz. Associated to this rotation we have the matrices Q and W given by (B.1.3), and the angular velocity vector is $\boldsymbol{\omega} = (0, 0, \dot{\theta})$. The *homothetic solution* is characterized by a rotation vector $\boldsymbol{\omega} = \boldsymbol{0}$, which amounts to considering the angle $\theta = 0$.

From this, it follows that any homographic solution will be characterized by the functions $\theta = \theta(t)$ and $\rho = \rho(t)$ that represent, respectively, the rotation and the homothety.

Theorem 2.3. *A homographic solution with* $\lambda + 3 \neq 0$ *is determined by two scalar functions,* $\rho = \rho(t)$ *and* $\theta = \theta(t)$*, such that*

$$\ddot{\rho} - \rho\,\dot{\theta}^2 = \mu\,\rho^\lambda, \quad \rho^2\,\dot{\theta} = \Gamma, \quad (2.3.15)$$

where $\mu = \sigma^0$ *and* $\Gamma = \|\boldsymbol{G}\|/J^0$ *are constants.*

Proof. Applying the relations (2.3.5), (2.3.7) and (2.3.8) to Equation (A.1.7) we obtain

$$(\dot{\rho}^2 + \rho\,\ddot{\rho})\,J^0 = (\dot{\rho}^2 + \rho^2\omega^2)\,J^0 - \rho^2\omega^2\sum_i m_i\left(\boldsymbol{e}_\omega\cdot\boldsymbol{r}_i^0\right) + (\lambda+1)\,\rho^{\lambda+1}\,U^0. \quad (2.3.16)$$

Since a homographic solution with $\lambda + 3 \neq 0$ is planar or homothetic (Theorems 2.1 and 2.2), we study both cases.

• In the *planar case* there exists a vector $\boldsymbol{e}_\omega$ satisfying that $\boldsymbol{e}_\omega \cdot \boldsymbol{r}_i^0 = 0$. From (2.3.16), dividing by $\rho\, J^0$ and applying (2.2.2), we obtain the differential equation

$$\ddot{\rho} - \rho\,\dot{\theta}^2 = (\lambda+1)\frac{U^0}{J^0}\rho^\lambda = \sigma^0\,\rho^\lambda = \mu\,\rho^\lambda.$$

On the other hand, the expression (2.3.14), with $\boldsymbol{e}_\omega \cdot \boldsymbol{r}_i^0 = 0$, leads to

$$\rho^2\dot{\theta} = \frac{\|\boldsymbol{G}\|}{J^0} = \Gamma.$$

• In the *homothetic case*, $\boldsymbol{\omega} = \boldsymbol{0}$, then $\dot{\theta} = 0$, and applying (2.2.2) to (2.3.16) we have

$$\ddot{\rho} = \sigma^0\rho^\lambda = \mu\rho^\lambda.$$

In this case, the second equation of (2.3.15) is not necessary, since $\boldsymbol{\omega} = \boldsymbol{0}$ implies that $\dot{\theta} = 0$. ■

2.4 Relation between Homographic Solutions and Central Configurations

Lemma 2.1. *The matrix K of a homographic solution with $\lambda + 3 \neq 0$ is given by the expression*

$$K = diag(\sigma^0, \sigma^0, \sigma^0 + \rho^{1-\lambda}\dot{\theta}^2), \tag{2.4.1}$$

where diag(...) denotes a 3x3 diagonal matrix.

Proof. Differentiating the second equality in (2.3.15) we have $\rho\,\ddot{\theta} + 2\dot{\theta}\dot{\rho} = 0$, that together with the first expression of (2.3.15) converts (B.2.4) into

$$\mathcal{K} = diag(\sigma^0\rho^\lambda, \sigma^0\rho^\lambda, \sigma^0\rho^\lambda + \rho\dot{\theta}^2).$$

Finally, since $K = \rho^{-\lambda}\,\mathcal{K}$ (see Equation (2.3.10)), the lemma is proved. ■

Theorem 2.4. *A solution of the n-body problem is homographic if and only if the configuration of the n particles is central at any instant t.*

Proof. Let us suppose that we have a homographic solution of an n-body problem. Then

$$\ddot{\boldsymbol{r}}_i^0 = K(t)\,\boldsymbol{r}_i^0 = \sigma^0\,\boldsymbol{r}_i^0, \tag{2.4.2}$$

where we have used Equations (2.3.10), (2.3.11) and (2.5.1). The vector $\ddot{\boldsymbol{r}}_i^0$ is expressed in the reference frame $\mathcal{S}_\omega$ mentioned in the proof of Theorem 2.2, for which its third component is zero.

Applying successively (2.3.6), (2.5.2), (2.3.10) and (2.3.11) we obtain

$$m_i\,\ddot{\boldsymbol{r}}_i = \nabla_{\boldsymbol{r}_i} U = \rho^\lambda\,Q\,\nabla_{\boldsymbol{r}_i^0} U^0 = m_i\,\rho^\lambda\,Q\,\ddot{\boldsymbol{r}}_i^0 = m_i\,\rho^\lambda\,Q\,\sigma^0\,\boldsymbol{r}_i^0 = m_i\,\sigma^0\,\rho^{\lambda-1}\left(\rho\,Q\,\boldsymbol{r}_i^0\right) = m_i\,\sigma^0\,\rho^{\lambda-1}\,\boldsymbol{r}_i = \sigma\,m_i\,\boldsymbol{r}_i,$$

which, by virtue of Definition 2.3, Equation (2.3.2) and Definition 2.1, Equation (2.2.1), proves that the configuration is central at any instant t.

Conversely, we consider a central configuration. To prove that this configuration is a homographic solution it suffices to prove that every solution $\boldsymbol{r}_i = \rho\,Q\,\boldsymbol{r}_i^0$ is also a solution of the n-body problem, i.e., it satisfies the differential equations (A.1.5) of the problem.

Supposing that the configuration is central at the initial instant, $\nabla_{\boldsymbol{r}_i^0} U^0 = \sigma^0\,m_i\,\boldsymbol{r}_i^0$, see Equation (2.2.1), if we apply (2.3.6) we can write

$$m_i\,\ddot{\boldsymbol{r}}_i = \nabla_{\boldsymbol{r}_i} U = \rho^\lambda\,Q\,\nabla_{\boldsymbol{r}_i^0} U^0 = m_i\,\sigma^0\,\rho^\lambda\,Q\,\boldsymbol{r}_i^0 \quad\Longrightarrow\quad \rho^{-\lambda}\,Q^{-1}\,\ddot{\boldsymbol{r}}_i = \sigma^0\,\boldsymbol{r}_i^0.$$

On the other hand, in order to have a homograhic solution, the relations from Equation (2.3.4) must be satisfied, namely

$$\rho^{-\lambda}\,Q^{-1}\,\ddot{\boldsymbol{r}}_i = K(t)\,\boldsymbol{r}_i^0.$$

As a final step, it is only needed to verify the equality $K(t)\,\boldsymbol{r}_i^0 = \sigma^0\,\boldsymbol{r}_i^0$ which was proved before. ■

Chapter 3

The Particular Cases of the Three- and Four-Body Problem.

In the second chapter we have studied the definition and some properties of central configurations and homographic solutions, and their equivalence, in a general n-body problem with central forces of the type $f(r) = r^{\lambda}, \lambda \in \mathbb{R}$. In this chapter we intend to show how many central configurations there exist, and how they are, for the particular cases of $n \leq 4$.

In the case $n = 2$, taking into account that at any instant t the force always acts in the direction of the line joining both particles, any two-body geometric configuration of the particles is central and, consequently, all solutions are homographic. Accordingly, we will consider only the cases $n = 3$ and $n = 4$.

3.1 Mutual-Distance Characterization of Central Configurations.

If we have n particles $P_1, \ldots, P_n$, there are $q = \frac{1}{2}\, n\,(n-1)$ mutual distances ($\|\boldsymbol{r}_{ik}\|$ for $i, k = 1, \ldots, n$) between them. In particular, if $n = 3$, there are three mutual distances $\|\boldsymbol{r}_{12}\|$,$\|\boldsymbol{r}_{13}\|$,$\|\boldsymbol{r}_{23}\|$. If $n = 4$, there are six mutual distances $\|\boldsymbol{r}_{12}\|, \|\boldsymbol{r}_{13}\|, \|\boldsymbol{r}_{14}\|, \|\boldsymbol{r}_{23}\|, \|\boldsymbol{r}_{24}\|, \|\boldsymbol{r}_{34}\|$.

Depending on the geometric configuration of the particles in space, a certain number p of relations $R_j = R_j\,(\ldots \|\boldsymbol{r}_{ik}\| \ldots) = 0$, for $j = 1, \ldots, p$, can exist. For instance, if three particles are collinear then there is only one relation $R_1 = \|\boldsymbol{r}_{12}\| + \|\boldsymbol{r}_{23}\| - \|\boldsymbol{r}_{13}\| = 0$ (up to permutations).

The number p of relations between mutual distances plays an important role in the existence of central configurations. This role is a consequence of the next lemma.

Lemma 3.1. *The necessary and sufficient condition for the existence of a central configuration of n bodies is the existence of p constant multipliers, denoted μ_j, $j = 1, \ldots, p$, such that for each $\|\boldsymbol{r}_{ik}\|$*

$$\nabla_{\|\boldsymbol{r}_{ik}\|}\left(J^{-(\lambda+1)}\, U^2\right) + \sum_{j=1}^{p} \mu_j \nabla_{\|\boldsymbol{r}_{ik}\|}\, R_j\,(\ldots \|\boldsymbol{r}_{ik}\| \ldots) = \boldsymbol{0}, \quad 1 \leq i < k \leq n. \tag{3.1.1}$$

This means that central configurations are critical points of the scalar function $J^{-(\lambda+1)}\, U^2$ subject to the constraints $R_1 = 0, \ldots, R_p = 0$.

Proof. Suppose that n bodies form a central configuration and let q be the number of mutual distances that satisfy the p relations $R_j = R_j\,(\ldots \|\boldsymbol{r}_{ik}\| \ldots) = 0$. If we arrange these q distances such that the first $q - p$ ones are independent and the last p ones are determined by the relations $R_j = 0$, then we can define the expression

$$\mathcal{P} = J^{-(\lambda+1)}\, U^2 + \sum_{j=1}^{p} \mu_j\, R_j, \tag{3.1.2}$$

involving the p relations.

$\mathcal{P}$ admits a twofold interpretation: $\mathcal{P}$ can be considered as a function of the position vectors $\boldsymbol{r}_i$, or as a function of the mutual distances $\|\boldsymbol{r}_{ik}\|$.

• Differentiating $\mathcal{P}$ as a function of the position vectors $\boldsymbol{r}_i$ we obtain

$$d\mathcal{P} = \sum_{i=1}^{n} \nabla_{\boldsymbol{r}_i} \left(J^{-(\lambda+1)}\, U^2 \right) \cdot d\boldsymbol{r}_i, \tag{3.1.3}$$

because the differential of the summation $\sum_{j=1}^{p} \mu_j R_j$ on the right-hand side of (3.1.2) vanishes, since $\boldsymbol{r}_{ik} = \boldsymbol{r}_k - \boldsymbol{r}_i$. If the configuration is central, $\nabla_{\boldsymbol{r}_i} \left(J^{-(\lambda+1)}\, U^2 \right) = \boldsymbol{0}$, $i = 1, \dots, n$, by Proposition 2.6. Therefore, $d\mathcal{P} = 0$.

• On the other hand, differentiating $\mathcal{P}$ as a function of the mutual distances $\|\boldsymbol{r}_{ik}\|$ we have [1]

$$d\mathcal{P} = \sum_{1 \le i < k \le n} \left[\nabla_{\|\boldsymbol{r}_{ik}\|} \left(J^{-(\lambda+1)}\, U^2 \right) + \sum_{j=1}^{p} \mu_j\, \nabla_{\|\boldsymbol{r}_{ik}\|} R_j \right] d\|\boldsymbol{r}_{ik}\|. \tag{3.1.4}$$

In view of (3.1.4), and considering that the differentials $d\|\boldsymbol{r}_{ik}\|$ are independent, the bracketed coefficients in (3.1.4) must vanish, since $d\mathcal{P} = 0$, which requires the existence of appropriate multipliers $\mu_1, \dots, \mu_p$ satisfying the condition (3.1.1).

Conversely, if there exist p multipliers μ_j such that (3.1.1) is satisfied, then applying it to (3.1.4) we obtain $d\mathcal{P} = 0$. Inserting this expression into (3.1.3) we conclude that

$$\nabla_{\boldsymbol{r}_i} \left(J^{-(\lambda+1)}\, U^2 \right) = \boldsymbol{0}, \quad i = 1, \dots, n,$$

which, by Proposition 2.6, means that the configuration is central. ∎

In that follows we will study the possible central configurations with zero or only one relation between mutual distances.

3.1.1 Central Configurations with $p = 0$.

Proposition 3.1. *A geometric configuration of n bodies with $p = 0$ relations between their mutual distances is central if and only if the mutual distances are equal.*

Proof. The condition (3.1.1) for $p = 0$ becomes

$$\nabla_{\|\boldsymbol{r}_{ik}\|} \left(J^{-(\lambda+1)}\, U^2 \right) = \boldsymbol{0},$$

which means that

$$J^{-(\lambda+1)}\, U \left[-(\lambda+1) \frac{U}{J} \nabla_{\|\boldsymbol{r}_{ik}\|} J + 2\, \nabla_{\|\boldsymbol{r}_{ik}\|} U \right] = \boldsymbol{0},$$

and simplifying this, we obtain

$$\nabla_{\|\boldsymbol{r}_{ik}\|}\, U = \frac{\lambda+1}{2} \frac{U}{J} \nabla_{\|\boldsymbol{r}_{ik}\|}\, J. \tag{3.1.5}$$

Differentiating the formula (A.1.4) for U, and the formula (A.1.8) for the moment of inertia J expressed in terms of the mutual distances, with respect to $\|\boldsymbol{r}_{ik}\|$ and introducing the resulting expressions into (3.1.5), we obtain

$$(\|\boldsymbol{r}_{ik}\|)^{\lambda-1} = -\frac{\lambda+1}{2} \frac{U}{JM}, \quad i, k = 1, \dots, n,$$

---

[1] Here, the notation $\nabla_{\|\boldsymbol{r}_{ik}\|} f$ is an abbreviation for $\partial f / \partial \|\boldsymbol{r}_{ik}\|$.

which proves that the only central configuration for $p = 0$ is that in which the mutual distances are all equal.

Conversely, if all the mutual distances are equal, namely $\|\boldsymbol{r}_{ik}\| = c$, then

$$J^{-(\lambda+1)}\, U^2 = \frac{M^{\lambda+1}}{(\lambda+1)^2} \left(\sum_{1\neq i<k\neq n} m_i\, m_k \right)^{-(\lambda+1)} = \text{const.}\,, \tag{3.1.6}$$

and the configuration is central, by Proposition 2.6. ∎

3.1.2 Central Configurations with $p = 1$.

In particular we study the relation

$$R_1\left(\|\boldsymbol{r}_{ij}\|, \|\boldsymbol{r}_{jk}\|, \|\boldsymbol{r}_{ik}\|\right) = \|\boldsymbol{r}_{ij}\| + \|\boldsymbol{r}_{jk}\| - \|\boldsymbol{r}_{ik}\| = 0. \tag{3.1.7}$$

This is the only (up to permutations) possible relation in the three body problem (collinear case). But with a larger number of bodies there might be more relations that we do not consider here.

In this case Equations (3.1.1) take the form

$$\frac{\partial\left(J^{-(\lambda+1)}\, U^2\right)}{\partial\,\|\boldsymbol{r}_{ij}\|} + \mu_1 = 0, \quad \frac{\partial\left(J^{-(\lambda+1)}\, U^2\right)}{\partial\,\|\boldsymbol{r}_{jk}\|} + \mu_1 = 0, \quad \frac{\partial\left(J^{-(\lambda+1)}\, U^2\right)}{\partial\,\|\boldsymbol{r}_{ik}\|} - \mu_1 = 0,$$

which can be rewritten in the form

$$\begin{aligned} m_i\, m_j\, \|\boldsymbol{r}_{ij}\|^\lambda\, \alpha \; &+ m_i\, m_j\, \|\boldsymbol{r}_{ij}\|\, \beta \; + \gamma &= 0, \\ m_j\, m_k\, \|\boldsymbol{r}_{jk}\|^\lambda \alpha \; &+ m_j\, m_k\, \|\boldsymbol{r}_{jk}\|\, \beta + \gamma &= 0, \\ m_i\, m_k\, \|\boldsymbol{r}_{ik}\|^\lambda\, \alpha \; &+ m_i\, m_k\, \|\boldsymbol{r}_{ik}\|\, \beta \; - \gamma &= 0, \end{aligned}$$

which is a homogeneous system of three linear equations for the unknown quantities

$$\alpha = \frac{2\,U}{J^{\lambda+1}}, \quad \beta = -\,\frac{2\,(\lambda+1)\; U^2}{M\, J^{\lambda+2}}, \quad \gamma = \mu_1.$$

This system possesses non-trivial solutions if and only if

$$\begin{vmatrix} m_i\, m_j\, \|\mathbf{r}_{ij}\|^\lambda & m_i\, m_j\, \|\mathbf{r}_{ij}\| & 1 \\ m_j\, m_k\, \|\mathbf{r}_{jk}\|^\lambda & m_j\, m_k\, \|\mathbf{r}_{jk}\| & 1 \\ m_i\, m_k\, \|\mathbf{r}_{ik}\|^\lambda & m_i\, m_k\, \|\mathbf{r}_{ik}\| & -1 \end{vmatrix} = 0.$$

Denoting $\|\mathbf{r}_{jk}\| = \mu\, \|\mathbf{r}_{ij}\|$ we have $\|\mathbf{r}_{ik}\| = \|\mathbf{r}_{ij}\| + \|\mathbf{r}_{jk}\| = (1+\mu)\, \|\mathbf{r}_{ij}\|$, with $\mu \in (0, 1)$. From now on, we consider $\|\mathbf{r}_{ij}\| = 1$. Multiplying by m_k in the first row, by m_i in the second row, and by m_j in the third one, we obtain

$$\begin{vmatrix} 1 & 1 & m_k \\ \mu^{\,\lambda} & \mu & m_i \\ (1+\mu\,)^\lambda & 1+\mu & -m_j \end{vmatrix} = 0,$$

that is,

$$m_k(1+\mu)\,\mu\left[\mu^{\lambda-1} - (1+\,\mu)^{\lambda-1}\right] + m_i(1+\mu)^\lambda + m_j\,\mu^\lambda - m_i(1+\mu) - m_j\,\mu = 0, \tag{3.1.8}$$

which is the equation that determines μ and therefore the relative positions of the three points. In general, this equation has at least one real solution. In particular, for the gravitational 3-body problem this equation is a polynomial equation of degree 5 that has a unique positive root, which belongs to the interval $(0, 1)$.

3.2 Application to the Three-Body Problem

There are only two geometric configurations of three particles in $\mathbb{R}^3$: a triangle and a straight line.

The triangular configuration means that no relation between mutual distances exists ($p = 0$). The collinear configuration is the only possible configuration if the relation (3.1.7) is fulfilled, and then $p = 1$.

In Subsection 3.1.1 we have seen that the only central configuration (homographic solution) compatible with $p = 0$ satisfies that the three mutual distances are equal, which in this case corresponds to a configuration such that the particles are at the vertices of an equilateral triangle (left of Figure 3.1). This solution was originally discovered by *Lagrange* in the gravitational 3-body problem ($\lambda = -2$), and for this reason it bears his name. Let us note that three points are always in a plane, which means that

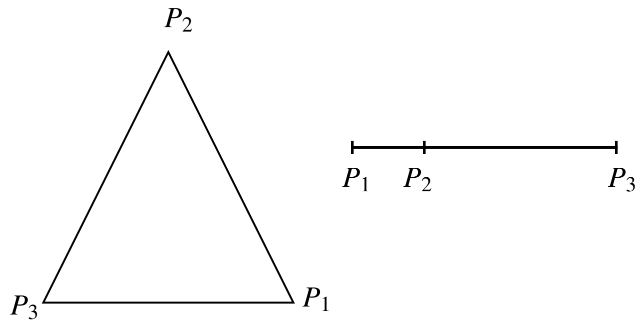


Figure 3.1: The two homographic solutions of the three-body problem. Left: Equilateral triangle (*Lagrange's solution*). Right: Collinear configuration (*Euler's Solution*).

the *Lagrange's solution* is coplanar and, due to the properties of the homographic solutions (Proposition 2.5), it is planar, i.e., the plane of the three bodies is a fixed plane.

The collinear configuration, with $p = 1$, gives rise to only one central configuration (see Subsection 3.1.2 and right of Figure 3.1) in which the relative distances are $\|\boldsymbol{r}_{12}\| = 1 - \mu$, $\|\boldsymbol{r}_{23}\| = \mu$, $\|\boldsymbol{r}_{13}\| = 1$, with μ fulfilling (3.1.8). In the gravitational case, this is the so-called *Euler's solution*.

In the gravitational problem, the functions ρ and θ given by (2.3.15) represent, for both types of solutions, a Keplerian orbit, which means that the three bodies move on conic sections, each one of them having the origin of the coordinate system as one of its foci (see Figures 3.2 and 3.3).

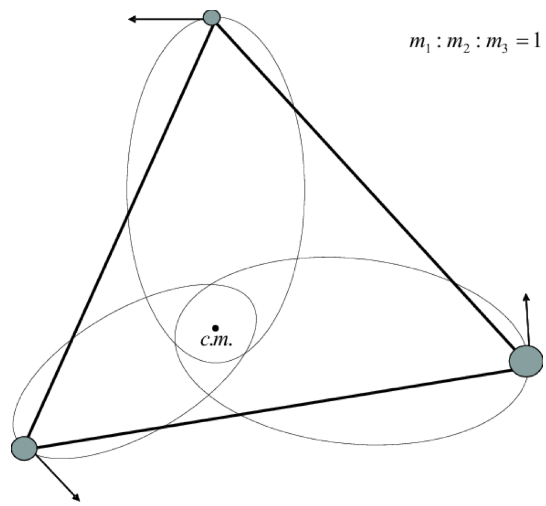


Figure 3.2: Lagrange's homographic solution for equal masses $m_1 = m_2 = m_3$.

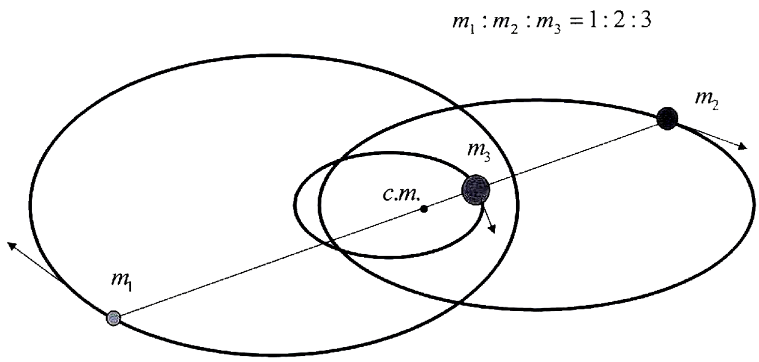


Figure 3.3: Euler's collinear solution to the three-body problem for the mass ratio $m_1 : m_2 : m_3 = 1 : 2 : 3$.

3.3 Application to the Four Body Problem

There are three possible geometric configurations of four particles $P_1, \ldots, P_4$ with masses $m_1, \ldots, m_4$ in $\mathbb{R}^3$:

- A tetrahedron (non-planar solution).
- A quadrilateral (coplanar solution).
- A straight line (collinear solution).

The first case corresponds to a configuration with $p = 0$ relations between the mutual distances. In Subsection 3.1.1 we have seen that the only central configuration (homographic solution) compatible with $p = 0$ satisfies that all mutual distances are equal, which in this case corresponds to a configuration in which the bodies are at the vertices of a regular tetrahedron. This is a non-planar solution and, consequently, a homothetic solution.

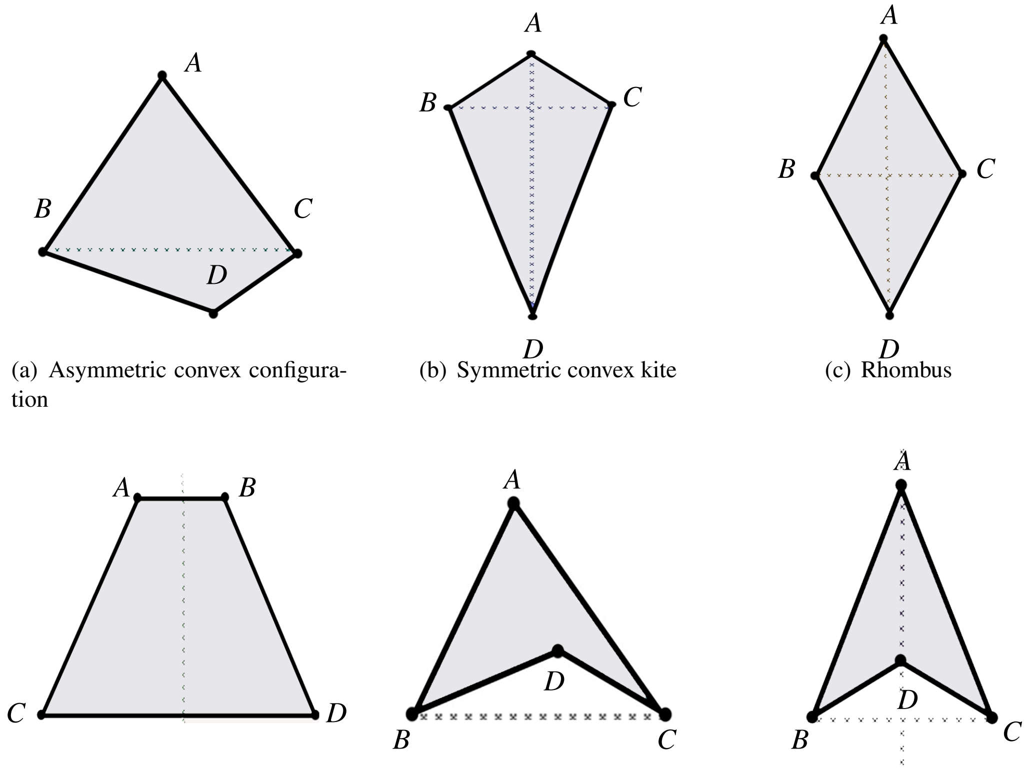
Regarding the collinear solution, Moulton [12] proved the existence of 12 collinear central configurations in the gravitational four-body problem, $\lambda = -2$. Later on, Woodlin and Xie [15] extended this result to any case $\lambda < 0$.

In the planar case the number and shape of the central configurations in the 4-body problem are not completely studied. In the next subsection we give (without proof) the main results concerning this question in the gravitational four-body problem.

3.3.1 Classification of Planar Geometric Configurations

Four points A, B, C, D, on a plane present one of the following geometric configurations (MacMillan and Bartky [10]).

- **Convex Configurations**. These are geometric configurations where the vertex D is outside the triangle formed by the vertices A, B and C. This group is further subdivided into
  - *Asymmetric*, when the configuration does not have any symmetry axis (Figure 3.4 a).
  - *Symmetric*, when the quadrilateral possesses at least one symmetry axis.
    - *Kite*, when the configuration has just one symmetry axis joining two non-adjacent vertices (Figure 3.4 b).
    - *Rhombus*, when the configuration has two symmetry axes joining pairs of non-adjacent vertices (Figure 3.4 c).



(a) Asymmetric convex configuration (b) Symmetric convex kite (c) Rhombus

(d) Isosceles trapezoid (e) Asymmetric concave configuration (f) Symmetric concave configuration

Figure 3.4: Planar geometric configurations in the four-body problem.

  - ∗ *Isosceles trapezoid*, when the configuration has just one symmetry axis which is the perpendicular bisector of the parallel sides of the trapezoid (Figure 3.4 d).

- **Concave Configurations**. These are configurations where the vertex D lies interior of the triangle formed by A, B and C. This group is further subdivided into
  - *Asymmetric*, when the configuration does not have any symmetry axis (Figure 3.4 e).
  - *Symmetric*, when the configuration has only one symmetry axis. It takes the form of an isosceles triangle where D is placed on the symmetry axis (Figure 3.4 f).

3.3.2 Some General Properties of Planar Central Configurations

Proposition 3.2. *Given four particles $P_1, \ldots, P_4$ of masses m_1, m_2, m_3, m_4, each one of them different from the others, there exists at least one convex central configuration (MacMillan and Bartky [10]).*

Proposition 3.3. *Given a convex central configuration the following statements hold (MacMillan and Bartky [10]):*

i) *The ratio between the lengths of the diagonals belongs to the interval $\left[1/\sqrt{3},\ \sqrt{3}\right]$.*

ii) *Every angle in the interior of the configuration belongs to the interval $[\pi/3, 2\pi/3]$.*

iii) *Every diagonal divides every angle in the interior of the configuration into two angles less than or equal to $\pi/3$.*

Proposition 3.4. *A convex central configuration is symmetric with respect to one of its diagonals if and only if the masses of the particles which are joined by the other diagonal are equal (Albouy et al. [4]).*

So, let us suppose that $m_1 = m_2$; then, there is a symmetry with respect to the diagonal $\overline{P_3P_4}$. In addition to this,

- *If $m_3 = m_4$, there are two symmetry axes, i.e., the particles are at the vertices of a rhombus (Figure 3.4 c, where P_1, P_2, P_3, P_4 are located at the vertices B, C, A, D, respectively).*
- *If $m_3 < m_4$, the particle P_3 is closer to the diagonal $\overline{P_1P_2}$ than P_4 (Figure 3.4 b, with the same distribution of particles and vertices as before).*

3.3.3 Mass-Based Classification of Planar Central Configuration.

Case 1. The 4-Body Problem with $\mathbf{m_1 = m_2 = m_3 = m_4}$.

Proposition 3.5. *Every planar central configuration of four bodies with masses $m_1 = m_2 = m_3 = m_4$ has, at least, one symmetry axis (Albouy [2]).*

Proposition 3.6. *Given four bodies with equal masses $m_1 = m_2 = m_3 = m_4$, there exist three different kinds of planar central configurations, one is convex and two are concave (Albouy [3]):*

i) P_1, P_2, P_3, P_4 form a square, special case of a rhombus with its four sides equal. (Figure 3.4 c, where the points P_1, P_2, P_3, P_4 are placed at vertices A, B, C, D).

ii) P_1, P_2, P_3 form an equilateral triangle, whereas P_4 is placed at their barycenter.

iii) P_1, P_2, P_3 from an isosceles triangle, whereas P_4 is somewhere on the axis, inside the triangle, but not necessarily at the barycenter (Figure 3.4 f, where the points P_1, P_2, P_3, P_4 are located at vertices A, B, C, D).

Case 2. The 4-Body Problem with $\mathbf{m_1 = m_2 = m_3 \neq m_4}$.

Proposition 3.7. *Given four bodies with masses m_1, m_2, m_3, m_4 such that $m_1 = m_2 = m_3 \neq m_4$, there exists just one convex central configuration (Long and Sun [9]).*

- *If $m_4 < m_3$, then we have Figure 3.4 b, where the points P_1, P_2, P_3, P_4 occupy the vertices B, C, D, A.*
- *If $m_4 > m_3$, then we have Figure 3.4 b, where the points P_1, P_2, P_3, P_4 are placed at the vertices B, C, A, D.*

Proposition 3.8. *Given four bodies with masses m_1, m_2, m_3, m_4 such that $m_1 = m_2 = m_3 \neq m_4$, there exist two possible central configurations which are concave (Long and Sun [9]):*

i) P_1, P_2, P_3 form an equilateral triangle with P_4 placed at the barycenter.

ii) P_1, P_2, P_3 form an isosceles triangle with P_4 placed on the symmetry axis of the triangle (Figure 3.4 f, where the points P_1, P_2, P_3, P_4 are located at the vertices B, C, A, D).

Case 3. The 4-Body Problem with $\mathbf{m_1 = m_2 \neq m_3 = m_4}$.

Proposition 3.9. *Given four bodies with masses m_1, m_2, m_3, m_4 such that $m_1 = m_2 \neq m_3 = m_4$, there exist two convex central configurations with at least one symmetry axis (Alvarez-Ramírez and Llibre [5]):*

i) A rhombus where the equal masses are at non-adjacent vertices (Figure 3.4 c, where the points P_1, P_2, P_3, P_4 occupy the vertices A, D, B, C).

ii) An isosceles trapezoid where P_1, P_2 and P_3, P_4 determine the parallel sides of the trapezoid. Remark that the small parallel side of the trapezoid corresponds to the two bodies of lower mass (Figure 3.4 d, where the points P_1, P_2, P_3, P_4 are located at the vertices A, B, C, D).

Proposition 3.10. *Given four bodies with masses m_1, m_2, m_3, m_4 such that $m_1 = m_2 \neq m_3 = m_4$, there exist two concave central configurations with a symmetry axis passing through P_4, which is placed inside the triangle formed by P_1, P_2, P_3 (Figure 3.4 f with the same distribution of masses as before). See Alvarez-Ramírez and Llibre [5].*

Case 4. The 4-Body Problem with $\mathbf{m_3 \neq m_1 = m_2 \neq m_4}$, and $\mathbf{m_3 \neq m_4}$.

Proposition 3.11. *Given four bodies with masses $m_3 \neq m_1 = m_2 \neq m_4$ and $m_3 \neq m_4$, there exists only one convex central configuration with the form of a kite (Perez-Chavela and Santoprete [13]).*

Conclusions

- From a geometric point of view, the solutions to the n-body problem can be classified into rectilinear, collinear, planar, coplanar and spatial. There exist certain relations between them, where the angular momentum plays a key role. For example, any collinear solution with $\boldsymbol{G} = \boldsymbol{0}$ is rectilinear, and likewise any coplanar solution with $\boldsymbol{G} = \boldsymbol{0}$ is planar.

- Two types of particular solutions have been studied: the central configurations and the homographic solutions.

  A *central configuration* of the n-body problem is a geometric configuration of n particles where the acceleration vector of each particle is a common scalar multiple of the corresponding position vector. A *homographic solution* to the n-body problem is a solution in which the configuration formed by the n bodies remains similar to itself throughout time. It depends on a scaling factor (*homothety*) and a *rotation*, apart from the initial conditions.

- From the definition of a homographic solution, some useful properties are proved. Except for the case of the law of force proportional to r^{-3}, any homographic solution is either homothetic (without rotation) or planar.

- A homographic solution is, in fact, a central configuration at any instant t. This intimate relationship between these two concepts allow us to use both expressions as synonyms.

- In the cases $n = 2$ and $n = 3$, the number and types of the homographic solutions are completely determined. On the contrary, for $n \geq 4$ these questions still constitute open problems.

- For the 2-body problem, all particular solutions are homographic solutions.

- For the 3-body problem, there are two kinds of homographic solutions: the collinear solutions which are called the *Euler's solutions*, and the triangular solutions which are called the *Lagrange's solutions.*

- In the 4-body problem, there exist only three possible kinds of homographic solutions: a regular tetrahedron, the collinear solutions, and the planar solutions. The latter can be classified into six different geometric figures.

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