Fitting functions of Jackson type for three-dimensional data

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ABSTRACT
We study some procedures for the approximation of three-dimensional data on a grid with a hypothesis of periodicity. The first part proposes a generalization of a discrete periodic approximation defined by Dunham Jackson. The functions used have the advantage of owning an analytical explicit expression in terms of the samples (specific values) of the original function or data. In the second part we describe a continuous approximation function for the same problem, defined through an integral. Some results of the rate of convergence and bounds of the approximation error are presented, with the single hypothesis of Hölder continuity or continuity of the original function.

KEYWORDS
Trigonometric approximation; trigonometric interpolation; smoothing; surface fitting; approximation on torus

1. Introduction

In 1885 Weierstrass proved that every continuous function defined in a compact interval is approximated by a polynomial with arbitrary precision. This fact motivated deeply the scientists of later generations and, specially, Dunham Jackson, American mathematician who published his work in the first decades of the twentieth century. This author wrote several books ([9, 11]) and numerous articles (see for instance [6–8, 10]) on polynomial and trigonometric approximation of continuous and discontinuous functions. His writings led to transcendental mathematical outcomes as for instance the inequalities named after him, which describe the degree of approximation of a continuous function by means of polynomials (algebraic and trigonometric).
He used approximation formulae of a periodic function f as for instance:

$$F(x) = h_m \int_{-\pi/2}^{\pi/2} f(x+2u) F_m(u) du$$

where $m \in \mathbb{N}$,

$$F_m(u) = \left(\frac{\sin(mu)}{m \sin(u)} \right)^4$$

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and

$$h_m^{-1} = \int_{-\pi/2}^{\pi/2} F_m(u)du,$$

that were introduced in his thesis (under the supervision of E. Landau in 1911). The author proved first the convergence of the approach (when m tends to infinity) for functions satisfying a Lispchitz condition, and thereafter for any continuous mapping. These results led to his important inequalities involving

$$d_n^*(f) = d(f, \mathcal{P}_n)$$

that represents the uniform distance from f to the space of polynomials of degree (order) at most n ([4]). The former theorems are generalized to the algebraic case by means of the substitution $x = \cos(\theta)$.
There are similar results for functions which are differentiable up to some order p ([4]). The author discussed the rate of convergence of Fourier and Legendre series as well (see for instance [6, 9, 11]). The approximation of discontinuous functions (in particular those of bounded variation) is studied in the second chapter of his book "The Theory of Approximation" ([9]). Some of his findings have been unfortunately neglected for a number of years. His Colloquium Lectures at the American Mathematical Society are a prodigy of clarity and elegance, that deserve to be reread.
A trigonometric approximant proposed by Jackson and far less known is defined by the formula for $f \in \mathcal{C}(2\pi)$ (continuous with period 2π):

$$\Sigma_m(x) = H_m \sum_{i=1}^{2m} f(x_i) \left(\frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right)^4,$$

where $x_{i+1} - x_i = \pi/m$; $i = 1, 2, \ldots, 2m - 1$, and

$$H_m^{-1} = \sum_{i=1}^{2m} \left(\frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right)^4. \tag{1}$$

H_m is a constant (not depending on x) and due to the fact that ([5], p. 340): For $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\left(\frac{\sin\left(\frac{nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right)^2 = n + 2\left((n-1)\cos(t) + (n-2)\cos(2t) + \ldots + \cos((n-1)t)\right), \tag{2}$$

Σ_m is a trigonometric sum of order $2(m-1)$ at most that fits f but does not interpolate it in general. This approach has the great advantage of its computation in an explicit form in terms of the samples (data or function values). Usually, the calculation of a fitting function requires the resolution of a system of equations coming from a least square process, or else the computation of integrals, in case of a function in closed form, in order to compute its sum with respect to some orthogonal system.
If f is Lipschitz with constant λ, Jackson obtained the following error bound for the

approximant Σ_m ([8]):

$$|f(x) - \Sigma_m(x)| \leq \frac{23\pi}{4}\frac{\lambda}{m},$$

that ensures the convergence when the partition is indefinitely refined. This result improves the ordinary trigonometric interpolation (denoted here by S_n), whose error is bounded by the expression ([8]):

$$|f(x) - S_n(x)| \leq \frac{21\lambda \log(n)}{n}.$$

For a merely continuous f, he obtained ([9]):

$$|f(x) - S_n(x)| \leq A\omega\left(\frac{2\pi}{n}\right)\log(n),$$

where ω is the modulus of continuity of f. We observe that the scheme is not necessarily convergent for an arbitrary function f.
Later on, several authors have studied the approximation properties of some trigonometric series coming from the Fourier sum of a periodic signal. For instance, the rate of p-convergence (in p-norm) of the following approximant:

$$\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k}S_k(x),$$

has been studied. This type of functions are used in the theory of Machines in Mechanical Engineering.
Taking the matrix $(a_{n,k})$ suitably, the model includes the Nörlund transform $N_n(f;x)$ or the weighted Riesz transform $R_n(f;x)$ ([14]). In the reference [3], the author provides sufficient conditions for the p-convergence of the Nörlund transform. He finds that if $f \in Lip(\alpha, p)$

$$\|f(x) - N_n(f;x)\|_p = \mathcal{O}(n^{-\alpha}),$$

and if $f \in Lip(1,1)$

$$\|f(x) - R_n(f;x)\|_1 = \mathcal{O}(n^{-1}).$$

Other authors have made important contributions to the topic. For instance, the references ([12, 14–16]) provide rates of p-convergence of τ_n for more general conditions on the infinite triangular matrix $(a_{n,k})$. Anyway these results have a rather theoretical character. From the practical point of view, there are few methods to handle (bi)periodic phenomena.

Hereafter we evoke the procedures used by Jackson, generalizing the approximants in the one and two dimensional cases. One of the formulae proposed is an explicit model in terms of the data on a two-dimensional grid. Some error bounds are deduced for more general exponents of the basic nodal functions, and the convergence is proved with the single hypotheses of periodicity and continuity on a compact interval

(Sections 2 and 3).

In Section 4 we present some results of the Jackson model along with other two-dimensional procedures as splines of radial basis or Shepard method. If the set of nodes is $X = \{x_1, x_2, \dots, x_n\}$, $(x_i \in \Omega \subset \mathbb{R}^2)$ a radial basis function is defined for $x \in \mathbb{R}^2$, as ([17])

$$s(f;x) = \sum_{j=1}^{n} \alpha_i \Phi(x, x_j) + p(x),$$

where $p(x)$ is a polynomial, and $\Phi : \Omega \times \Omega \to \mathbb{R}$ such that

$$\Phi(x,y) = \phi(\|x-y\|_2),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$. These functions are suitable for the approximation of scattered data and smooth functions. In general, they require some level of regularity for the functions to be approached. For instance, the following results can be found in [2]: For $f \in \mathcal{H}^{2k}(\Omega)$,

$$\|s-f\|_{p,\Omega} \leq C\|f\|_{\mathcal{H}^{2k}} h^{2k+\min\{\frac{n}{p}-\frac{n}{2},0\}},$$

where

$$h = \sup_{x\in\Omega} \inf_{\psi\in X} \|x-\psi\|,$$

or for $f \in \mathcal{C}^{2k}(\Omega)$,

$$\|s-f\|_{\infty,\Omega} \leq Ch^{2k}.$$

In both cases h must be such that $0 < h < 1$.
The computation of this type of splines involve usually the resolution of large systems of equations, and consequently they require computational resources which are not always accessible. The methods studied in this article own the advantage of an easy implementation (the formulae are explicit) and do not require the solution of equations. Unlike the radial basis methods, our theorems of convergence only require a minimal hypothesis of continuity. This fact makes them suitable to approximate functions on grids with a large number of nodes. However, the model requires the values on a mesh, and they cannot be used for scattered data.
Further the Jackson functions can be generalized to higher dimensions, and the periodicity in all the variables make them useful for the approximation of data or functions on tori.

2. Two-dimensional discrete approximant of Jackson type

In this Section we consider a trigonometric fitting mapping for functions or data defined on a grid on a two-dimensional interval $[-\pi,\pi] \times [-\pi,\pi]$, assuming periodicity in both variables. They are inspired (generalizing them) in approximations of Jackson, and they are defined explicitly in terms of the data (function values). Due to this fact, we name them discrete approximants, in order to distinguish them from other functions defined in later sections. Let T^1 represent the unit circle and let us consider $f \in \mathcal{C}(T^1 \times T^1)$ (continuous and periodic with period 2π in both variables), and an exponent $\gamma > 0$ ([8]). The approximation of f on a grid is defined by the expression:

$$\mathcal{J}_{mn\gamma}(f)(x,y) = K_{mn\gamma}(x,y) \sum_{i=1}^{2m} \sum_{j=1}^{2n} f(x_i, y_j) \left| \frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right|^\gamma \left| \frac{\sin\left(\frac{1}{2}n(y_j - y)\right)}{n \sin\left(\frac{1}{2}(y_j - y)\right)} \right|^\gamma, \tag{3}$$

where $x_{i+1} - x_i = \pi/m$; $i = 1, 2, \ldots, 2m-1$, $y_{j+1} - y_j = \pi/n$; $j = 1, 2, \ldots, 2n-1$,

$$K_{mn\gamma}^{-1}(x,y) = \sum_{i=1}^{2m} \sum_{j=1}^{2n} \left| \frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right|^\gamma \left| \frac{\sin\left(\frac{1}{2}n(y_j - y)\right)}{n \sin\left(\frac{1}{2}(y_j - y)\right)} \right|^\gamma. \tag{4}$$

In the case $\gamma = 4$, one has

$$K_{mn4}^{-1}(x,y) = \sum_{i=1}^{2m} \left(\frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right)^4 \sum_{j=1}^{2n} \left(\frac{\sin\left(\frac{1}{2}n(y_j - y)\right)}{n \sin\left(\frac{1}{2}(y_j - y)\right)} \right)^4 = H_m^{-1} H_n^{-1}. \tag{5}$$

H_m is a constant (not depending on the variable) of the one-dimensional case (1), such that ([8])

$$1/2 \leq H_m < 3/4.$$

Consequently, K_{mn4} is a constant such that

$$1/4 \leq K_{mn4} < 9/16.$$

The equality (2) implies that $\mathcal{J}_{mn4}(f)$ is a trigonometric polynomial of order at most $2(m-1)$ in x and $2(n-1)$ in y. If γ is a multiple of 2, the function is a trigonometric rational.

Lemma 2.1. *For all $m = 1, 2, \ldots$; $\gamma > 0$, and $v \in \mathbb{R}$:*

$$\left| \frac{\sin(mv)}{m \sin(v)} \right|^\gamma \leq 1. \tag{6}$$

Proof. Using the identity (2) for $n = m$ and $t = 2v$:

$$\left(\frac{\sin(mv)}{\sin(v)} \right)^2 = m + 2\left((m-1)\cos(2v) + (m-2)\cos(4v) + \ldots + \cos(2(m-1)v)\right).$$

Then

$$\left| \frac{\sin(mv)}{m \sin(v)} \right|^\gamma = \frac{1}{m^\gamma} \left| m + 2\left((m-1)\cos(2v) + (m-2)\cos(4v) + \ldots + \cos(2(m-1)v)\right) \right|^{\gamma/2}$$

$$\left|\frac{\sin(mv)}{m\sin(v)}\right|^{\gamma} \leq \frac{1}{m^{\gamma}}\left(m + 2\left(\frac{1+(m-1)}{2}\right)(m-1)\right)^{\gamma/2},$$

$$\left|\frac{\sin(mv)}{m\sin(v)}\right|^{\gamma} \leq \frac{1}{m^{\gamma}}(m^2)^{\gamma/2}.$$

□

Definition 2.2. Let f be a real continuous function defined in an n-dimensional compact interval I. The modulus of continuity of f is defined as

$$\omega(\delta) = \omega(\delta; f) = \sup_{\|x-x'\|\leq\delta} |f(x) - f(x')| \qquad (x, x' \in I).$$

Some of the properties of the modulus of continuity are ([1, 13]):

- $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$.
- If $\delta \leq \delta'$ then $\omega(\delta) \leq \omega(\delta')$.
- $\lim_{\delta\to 0}\omega(\delta) = 0$.
- $\omega(m\delta) \leq m\omega(\delta)$ for $m \in \mathbb{N}$.
- $\omega(\lambda\delta) \leq (\lambda+1)\omega(\delta)$ for $\lambda \geq 0$.
- If $0 < \delta_1 \leq \delta_2$ then $2\omega(\delta_1)/\delta_1 \geq \omega(\delta_2)/\delta_2$.

According to the definition of the approximant, $K_{mn\gamma}$, and the changes: $x_i = x+2u_i$, and $y_j = y + 2v_j$,

$$|\mathcal{J}_{mn\gamma}(f)(x,y) - f(x,y)| =$$

$$K_{mn\gamma}(x,y)\sum_{i=1}^{2m}\sum_{j=1}^{2n}(f(x+2u_i, y+2v_j) - f(x,y))\left|\frac{\sin(mu_i)}{m\sin(u_i)}\right|^{\gamma}\left|\frac{\sin(nv_j)}{n\sin(v_j)}\right|^{\gamma}, \tag{7}$$

where due to periodicity, we can assume $u_i \in [-\pi/2, \pi/2]$, $v_j \in [-\pi/2, \pi/2]$. According to the definition and properties of the modulus ω:

$$|f(x+2u_i, y+2v_j) - f(x,y)| \leq \omega\left(\sqrt{4u_i^2 + 4v_j^2}\right) \leq 2\omega\left(\sqrt{u_i^2 + v_j^2}\right) \leq 2\left(\omega(|u_i|) + \omega(|v_j|)\right). \tag{8}$$

Let us denote the approximation error as:

$$E_{mn\gamma}(x,y) = \mathcal{J}_{mn\gamma}(f)(x,y) - f(x,y).$$

Considering increasing order in $|u_i|$, $|v_j|$ and denoting them by $\overline{u}_i$, $\overline{v}_j$, for $i = 0, 1, \ldots, 2m-1$ and $j = 0, 1, \ldots, 2n-1$ one has ([8]):

$$|E_{mn\gamma}(x,y)| \leq 2K_{mn\gamma}(x,y)\sum_{i=0}^{2m-1}\sum_{j=0}^{2n-1}(\omega(\overline{u}_i) + \omega(\overline{v}_j))\left|\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)}\right|^{\gamma}\left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^{\gamma}, \tag{9}$$

where ([8])

$$\frac{\pi i}{4m} \leq \overline{u}_i \leq \frac{\pi(i+1)}{4m} \leq \frac{\pi}{2}, \tag{10}$$

for $i = 0, 1, \ldots, 2m-1$, and

$$\frac{\pi j}{4n} \leq \overline{v}_j \leq \frac{\pi(j+1)}{4n} \leq \frac{\pi}{2}, \tag{11}$$

for $j = 0, 1, \ldots, 2n-1$. Consequently,

$$\omega(\overline{u}_i) \leq (i+1)\omega\left(\frac{\pi}{4m}\right), \tag{12}$$

$$\omega(\overline{v}_j) \leq (j+1)\omega\left(\frac{\pi}{4n}\right). \tag{13}$$

For $i \geq 1$ and $j \geq 1$ using the inequality (due to the concavity of $\sin(v)$ in the interval $[0, \pi/2]$)

$$\sin(v) \geq \frac{2v}{\pi}, \tag{14}$$

for $v \in [0, \pi/2]$, and the inequalities (10), (11),

$$m\sin(\overline{u}_i) \geq m\sin\left(\frac{\pi i}{4m}\right) \geq m2\frac{i}{4m} \geq \frac{i}{2},$$

and

$$n\sin(\overline{v}_j) \geq \frac{j}{2}.$$

As a consequence, for $i \geq 1$ and $j \geq 1$

$$\left|\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)}\right|^\gamma \leq \left(\frac{2}{i}\right)^\gamma, \tag{15}$$

and

$$\left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^\gamma \leq \left(\frac{2}{j}\right)^\gamma. \tag{16}$$

We resume now the sum:

$$S = \sum_{i=0}^{2m-1}\sum_{j=0}^{2n-1}(\omega(\overline{u}_i) + \omega(\overline{v}_j))\left|\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)}\right|^\gamma \left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^\gamma = S_{0y} + S_{xy},$$

where

$$S_{0y} = \sum_{j=0}^{2n-1} (\omega(\overline{u}_0) + \omega(\overline{v}_j)) \left|\frac{\sin(m\overline{u}_0)}{m\sin(\overline{u}_0)}\right|^{\gamma} \left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^{\gamma},$$

and

$$S_{xy} = \sum_{i=1}^{2m-1} \sum_{j=0}^{2n-1} (\omega(\overline{u}_i) + \omega(\overline{v}_j)) \left|\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)}\right|^{\gamma} \left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^{\gamma}.$$

In turn,

$$S_{0y} = (\omega(\overline{u}_0)+\omega(\overline{v}_0)) \left|\frac{\sin(m\overline{u}_0)}{m\sin(\overline{u}_0)}\right|^{\gamma} \left|\frac{\sin(n\overline{v}_0)}{n\sin(\overline{v}_0)}\right|^{\gamma} + \sum_{j=1}^{2n-1} (\omega(\overline{u}_0)+\omega(\overline{v}_j)) \left|\frac{\sin(m\overline{u}_0)}{m\sin(\overline{u}_0)}\right|^{\gamma} \left|\frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)}\right|^{\gamma}.$$

Applying (10), (7), (13) and (16),

$$S_{0y} \leq \left(\omega\left(\frac{\pi}{4m}\right) + \omega\left(\frac{\pi}{4n}\right)\right) + \sum_{j=1}^{2n-1} \left(\omega\left(\frac{\pi}{4m}\right) + (j+1)\omega\left(\frac{\pi}{4n}\right)\right) \left(\frac{2}{j}\right)^{\gamma}.$$

The second property of the modulus of continuity implies that

$$S_{0y} \leq 2\omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right) + \omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right) \sum_{j=1}^{2n-1} (2+j) \left(\frac{2}{j}\right)^{\gamma},$$

and finally,

$$S_{0y} \leq \omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right) \left(2 + 2^{\gamma+1}\psi(\gamma) + 2^{\gamma}\psi(\gamma-1)\right) \tag{17}$$

if $\gamma > 2$ and ψ is the Riemann function defined as:

$$\psi(\gamma) = \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}}.$$

Let us consider now

$$S_{xy} = S_{x0} + \overline{S}_{xy},$$

where

$$S_{x0} = \sum_{i=1}^{2m-1} (\omega(\overline{u}_i) + \omega(\overline{v}_0)) \left|\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)}\right|^{\gamma} \left|\frac{\sin(n\overline{v}_0)}{n\sin(\overline{v}_0)}\right|^{\gamma},$$

$$\overline{S}_{xy} = \sum_{i=1}^{2m-1} \sum_{j=1}^{2n-1} (\omega(\overline{u}_i) + \omega(\overline{v}_j)) \left| \frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)} \right|^\gamma \left| \frac{\sin(n\overline{v}_j)}{n\sin(\overline{v}_j)} \right|^\gamma .$$

Arguing as before,

$$S_{x0} \le \omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right)\left(2 + 2^{\gamma+1}\psi(\gamma) + 2^\gamma \psi(\gamma-1)\right),$$

and

$$\overline{S}_{xy} \le 2^{2\gamma}\omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right) \sum_{i=1}^{2m-1} \sum_{j=1}^{2n-1} (2+i+j) \left(\frac{1}{i}\right)^\gamma \left(\frac{1}{j}\right)^\gamma ,$$

the last sum is $T_1 + T_2 + T_3$ where

$$T_1 = 2(\psi(\gamma))^2,$$

$$T_r \le \psi(\gamma)\psi(\gamma-1),$$

for $r = 2, 3$.
Let us bound now the factor $K_{mn\gamma}(x,y)$ of (3):

$$K_{mn\gamma}^{-1}(x,y) = H_{m\gamma}^{-1}(x) H_{n\gamma}^{-1}(y),$$

where

$$H_{m\gamma}^{-1}(x) = \sum_{i=0}^{2m-1} \left| \frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)} \right|^\gamma > \left| \frac{\sin(m\overline{u}_0)}{m\sin(\overline{u}_0)} \right|^\gamma + \left| \frac{\sin(m\overline{u}_1)}{m\sin(\overline{u}_1)} \right|^\gamma . \tag{18}$$

Since $m\overline{u}_i \le \pi/2$ for $i = 0, 1$ ((10), (14))

$$\frac{\sin(m\overline{u}_i)}{m\sin(\overline{u}_i)} \ge \frac{2m\overline{u}_i}{m\pi\sin(\overline{u}_i)} \ge \frac{2}{\pi},$$

and thus (18),

$$H_{m,\gamma}^{-1}(x) > 2\left(\frac{2}{\pi}\right)^\gamma , \tag{19}$$

and

$$H_{m,\gamma}(x) < \frac{1}{2}\left(\frac{\pi}{2}\right)^\gamma . \tag{20}$$

Collecting all the inequalities:

$$|\mathcal{J}_{mn\gamma}(f)(x,y) - f(x,y)| \le \omega\left(\frac{\pi}{4m} + \frac{\pi}{4n}\right) C'(\gamma) \le \omega\left(\frac{1}{m} + \frac{1}{n}\right) C(\gamma),$$

where $C'(\gamma)$, $C(\gamma)$ do not depend on m, n. If $\gamma > 2$, the values $\psi(\gamma), \psi(\gamma - 1)$ in (17) are finite, and the rate of convergence of the error when the partition is indefinitely refined is that of the modulus $\omega(\frac{1}{m} + \frac{1}{n})$ (it does not depend on the exponent γ).

Thus, we have proved the following result.

Theorem 2.3. *For any continuous function $f \in \mathcal{C}(T^1 \times T^1)$ and $\gamma > 2$, the approximant $\mathcal{J}_{mn\gamma}(f)$ converges uniformly to f as m and n tend to infinity.*

Remark 1. The uniform convergence on the compact interval implies the convergence in the p-norm for any $1 \leq p < \infty$.

Remark 2. If f satisfies a Lipschitz condition of order q, $(0 < q \leq 1)$, there exists a positive constant k such that $\omega(\overline{u}_i) \leq k\overline{u}_i^q$. Then, according to (10),

$$k\overline{u}_i^q \leq k\left(\frac{\pi(i+1)}{4m}\right)^q \leq k\left(\frac{\pi}{4m}\right)^q (i^q + 1).$$

This bound would provide a wider range of convergence values: $\gamma > 1 + q$. In this case, when $\gamma = 2$ the approximant is convergent as well.

Let us denote by

$$\|\mathcal{J}_{mn\gamma}\|$$

the norm of the operator $\mathcal{J}_{mn\gamma}$ with respect to the uniform (supremum) norm $\|\cdot\|_\infty$ in $\mathcal{C}([-\pi, \pi] \times [-\pi, \pi])$.

Theorem 2.4. *For $m, n \in \mathbb{N}$ and $\gamma > 2$, $\|\mathcal{J}_{mn\gamma}\| = 1$.*

Proof.

$$|\mathcal{J}_{mn\gamma}(f)(x,y)| \leq K_{mn\gamma}(x,y) \sum_{i=1}^{2m} \sum_{j=1}^{2n} |f(x_i, y_j)| \left| \frac{\sin\left(\frac{1}{2}m(x_i - x)\right)}{m \sin\left(\frac{1}{2}(x_i - x)\right)} \right|^\gamma \left| \frac{\sin\left(\frac{1}{2}n(y_j - y)\right)}{n \sin\left(\frac{1}{2}(y_j - y)\right)} \right|^\gamma,$$

and thus, according to the definition (4),

$$|\mathcal{J}_{mn\gamma}(f)(x,y)| \leq \|f\|_\infty.$$

Consequently

$$\|\mathcal{J}_{mn\gamma}(f)\|_\infty \leq \|f\|_\infty,$$

and

$$\|\mathcal{J}_{mn\gamma}\| \leq 1.$$

For the other side, let us think that the constant functions are fixed points of the operator $\mathcal{J}_{mn\gamma}$ according to its definition. Then, for $f(x,y)=k$,

$$1=\frac{\|\mathcal{J}_{mn\gamma}(f)\|_\infty}{\|f\|_\infty}$$

and the definition of the norm of the operator implies that

$$1\le\|\mathcal{J}_{mn\gamma}\|.$$

$\square$

Remark 3. The operator $\mathcal{J}_{mn\gamma}$ is linear and bounded and it does not amplify the errors in the z-values since: Let $f,\widetilde{f}$ be the functions corresponding to data (x_i,y_j,z_{ij}) and $(x_i,y_j,\widetilde{z}_{ij})$. Considering the linearity and the norm of the approximation operator:

$$\|\mathcal{J}_{mn\gamma}(f)-\mathcal{J}_{mn\gamma}(\widetilde{f})\|_\infty\le\|f-\widetilde{f}\|_\infty.$$

3. Continuous approximants of Jackson

3.1. One-dimensional case

We consider now an integral approximant of the continuous and periodic function $f\in\mathcal{C}(2\pi)$. This model was proposed by Jackson in the article "On approximation by trigonometric sums and polynomials" ([7]). We consider a more general case, where the exponent 4 appearing in the paper, is replaced by any positive exponent $\gamma>0$:

$$\mathcal{F}_{m\gamma}(f)(x)=h_{m\gamma}\int_{-\pi/2}^{\pi/2}f(x+2u)\left|\frac{\sin(mu)}{m\sin(u)}\right|^\gamma du, \tag{21}$$

where

$$h_{m\gamma}^{-1}=\int_{-\pi/2}^{\pi/2}\left|\frac{\sin(mu)}{m\sin(u)}\right|^\gamma du. \tag{22}$$

If $\gamma=4$, $\mathcal{F}_{m\gamma}(f)$ is a trigonometric polynomial of order at most $2(m-1)$ ([7]). In the general case the kernels

$$\left|\frac{\sin(mu)}{m\sin(u)}\right|^\gamma$$

have an "order" of $\gamma(m-1)/2$ (we admit here non-integer values of the exponent). If $\gamma=2q$, where $q\in\mathbb{N}$, $\mathcal{F}_{m\gamma}(f)$ is a trigonometric polynomial of order at most $q(m-1)$.

Lemma 3.1. *If $p<-1$ and $\gamma>-(p+1)$ then*

$$K_{p\gamma}=\int_0^{+\infty}u^p|\sin(u)|^\gamma du\le\frac{1}{(p+\gamma+1)}-\frac{1}{(p+1)}. \tag{23}$$

If $\gamma > 2$,

$$i^1_{m\gamma} = \int_0^{\pi/2} u \left| \frac{\sin(mu)}{m\sin(u)} \right|^\gamma du \leq \frac{1}{m^2} \left(\frac{\pi}{2}\right)^\gamma \left(\frac{1}{2} - \frac{1}{2-\gamma}\right), \tag{24}$$

and if $\gamma > 1$,

$$i^0_{m\gamma} = \int_0^{\pi/2} \left| \frac{\sin(mu)}{m\sin(u)} \right|^\gamma du \leq \frac{1}{m} \left(\frac{\pi}{2}\right)^\gamma \left(1 - \frac{1}{1-\gamma}\right). \tag{25}$$

If $\gamma > 0$,

$$h_{m\gamma} \leq \frac{m}{2} C_\gamma, \tag{26}$$

where

$$C_\gamma^{-1} = \int_0^{\pi/2} \left(\frac{\sin(u)}{u}\right)^\gamma du. \tag{27}$$

Proof. For (23)

$$K_{p\gamma} = \int_0^1 u^p |\sin(u)|^\gamma du + \int_1^{+\infty} u^p |\sin(u)|^\gamma du = k_1 + k_2.$$

$$k_1 = \int_0^1 u^p |\sin(u)|^\gamma du = \int_0^1 u^{p+\gamma} \frac{|\sin(u)|^\gamma}{u^\gamma} du \leq \int_0^1 u^{p+\gamma} du = \frac{1}{p+\gamma+1},$$

whenever $p + \gamma + 1 > 0$.

$$k_2 = \int_1^{+\infty} u^p |\sin(u)|^\gamma du \leq \int_1^{+\infty} u^p du \leq -\frac{1}{(p+1)},$$

if $p + 1 < 0$. As a consequence,

$$K_{p\gamma} \leq \frac{1}{p+\gamma+1} - \frac{1}{(p+1)},$$

if $p < -1$ and $\gamma > -(p+1)$. For the inequality (24) let us consider that

$$i^1_{m\gamma} = \int_0^{\pi/2} u \left| \frac{\sin(mu)}{m\sin(u)} \right|^\gamma du = \frac{1}{m^\gamma} \int_0^{\pi/2} u^{1-\gamma} \frac{u^\gamma}{(\sin(u))^\gamma} |\sin(mu)|^\gamma du,$$

$$i^1_{m\gamma} \leq \frac{1}{m^\gamma} \left(\frac{\pi}{2}\right)^\gamma \int_0^{\pi/2} u^{1-\gamma} |\sin(mu)|^\gamma du.$$

In the last expression, the following inequality is used: for $0 < u \leq \frac{\pi}{2}$

$$\frac{\sin(u)}{u} \geq \frac{2}{\pi}.$$

With the change of variable: $mu = \widehat{u}$

$$i^1_{m\gamma} \leq \frac{1}{m^2}\left(\frac{\pi}{2}\right)^{\gamma}\int_0^{m\pi/2} \widehat{u}^{1-\gamma}|\sin(\widehat{u})|^{\gamma} d\widehat{u} \leq \frac{1}{m^2}\left(\frac{\pi}{2}\right)^{\gamma}\int_0^{+\infty} \widehat{u}^{1-\gamma}|\sin(\widehat{u})|^{\gamma} d\widehat{u} =$$

$$\frac{1}{m^2}\left(\frac{\pi}{2}\right)^{\gamma} K_{1-\gamma,\gamma}$$

and

$$i^1_{m\gamma} \leq \frac{1}{m^2}\left(\frac{\pi}{2}\right)^{\gamma}\left(\frac{1}{2} - \frac{1}{2-\gamma}\right),$$

in the case $\gamma > 2$. Following the same steps we obtain the inequality (25) (concerning $i^0_{m\gamma}$).

For the expression (26) let us think that

$$h^{-1}_{m\gamma} = \int_{-\pi/2}^{\pi/2} \left|\frac{\sin(mu)}{m\sin(u)}\right|^{\gamma} du \geq 2\int_0^{\pi/2} \left|\frac{\sin(mu)}{mu}\right|^{\gamma} du.$$

with the change $\widehat{u} = mu$

$$h^{-1}_{m\gamma} \geq \frac{2}{m}\int_0^{m\pi/2} \left|\frac{\sin(\widehat{u})}{\widehat{u}}\right|^{\gamma} d\widehat{u}$$

and

$$h^{-1}_{m\gamma} \geq \frac{2}{m}\int_0^{\pi/2} \left|\frac{\sin(\widehat{u})}{\widehat{u}}\right|^{\gamma} d\widehat{u},$$

from which the result is obtained. □

Hereafter we will bound the approximation error of the function $\mathcal{F}_{m\gamma}(f)$ (21). Since

$$f(x) = h_{m\gamma}\int_{-\pi/2}^{\pi/2} f(x)\left|\frac{\sin(mu)}{m\sin(u)}\right|^{\gamma} du,$$

then

$$\mathcal{F}_{m\gamma}(f)(x) - f(x) = h_{m\gamma}\int_{-\pi/2}^{\pi/2} (f(x+2u) - f(x))\left|\frac{\sin(mu)}{m\sin(u)}\right|^{\gamma} du,$$

and

$$|\mathcal{F}_{m\gamma}(f)(x) - f(x)| \leq h_{m\gamma} \int_{-\pi/2}^{\pi/2} |f(x+2u) - f(x)| \left| \frac{\sin(mu)}{m\sin(u)} \right|^{\gamma} du. \tag{28}$$

If $E_{m\gamma}(f)(x)$ represents the absolute value of the error:

$$E_{m\gamma}(f)(x) \leq h_{m\gamma} \int_{-\pi/2}^{\pi/2} 2\omega(|u|) \left| \frac{\sin(mu)}{m\sin(u)} \right|^{\gamma} du \leq 4h_{m\gamma} \int_{0}^{\pi/2} \omega(u) \left| \frac{\sin(mu)}{m\sin(u)} \right|^{\gamma} du. \tag{29}$$

Now we consider that, for any $m, u > 0$, the properties of the modulus imply

$$\omega(u) = \omega\left(mu\frac{1}{m}\right) \leq (mu+1)\omega\left(\frac{1}{m}\right).$$

Then,

$$E_{m\gamma}(f)(x) \leq 4\omega\left(\frac{1}{m}\right) h_{m\gamma}(mi^1_{m\gamma} + i^0_{m\gamma}),$$

where $i^1_{m\gamma}$ and $i^0_{m\gamma}$ where defined in Lemma 3.1:

$$E_{m\gamma}(f)(x) \leq 4\omega\left(\frac{1}{m}\right) h_{m\gamma}\frac{1}{m}\left(\frac{\pi}{2}\right)^{\gamma}\left(\frac{3}{2} - \frac{1}{2-\gamma} - \frac{1}{1-\gamma}\right),$$

whenever $\gamma > 2$. The expression (26) provides

$$E_{m\gamma}(f)(x) \leq 2\omega\left(\frac{1}{m}\right) C_{\gamma}\left(\frac{\pi}{2}\right)^{\gamma}\left(\frac{3}{2} - \frac{1}{2-\gamma} - \frac{1}{1-\gamma}\right).$$

Thus, we have proved the following result.

Theorem 3.2. *If f is a periodic and continuous function, the approximant $\mathcal{F}_{m\gamma}(f)$ is convergent to f for any $\gamma > 2$ when m tends to infinity. The rate of convergence is that of $\omega(1/m)$ and it does not depend on γ.*

Remark 4. For a Lispchitz function of order q, the inequality (29) is replaced by

$$E_{m\gamma}(f)(x) \leq 2^{q+1} h_{m\gamma} \int_{0}^{\pi/2} u^q \left| \frac{\sin(mu)}{m\sin(u)} \right|^{\gamma} du,$$

and the range of the convergence values extends to $\gamma > 1 + q$.

3.2. Two-dimensional case

Let us consider now a biperiodic function $f \in \mathcal{C}(I_x \times I_y)$ or $f \in \mathcal{C}(T^1 \times T^1)$ and for

$$2(p-1) \leq m \leq 2p, \quad 2(q-1) \leq n \leq 2q, \tag{30}$$

define the operator

$$\mathcal{G}_{pq\gamma}(f)(x,y) = H_{pq\gamma}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(x+2u, y+2v)G_{pq\gamma}(u,v)dudv,$$

where

$$G_{pq\gamma}(u,v) = \left|\frac{\sin(pu)\sin(qv)}{p\sin(u)q\sin(v)}\right|^{\gamma}, \tag{31}$$

and

$$H_{pq\gamma}^{-1} = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} G_{pq\gamma}(u,v)dudv = h_{p\gamma}^{-1}h_{q\gamma}^{-1}, \tag{32}$$

where $h_{p\gamma}^{-1}, h_{q\gamma}^{-1}$ are defined by the expression (22).

Hereafter we are going to bound the uniform error committed in the approximation. For it, we present the following Lemmas.

Lemma 3.3. *For p, q, m, n as in (30),*

$$u \geq \frac{1}{p} \Rightarrow \omega(u) \leq 4pu\omega\left(\frac{1}{m}\right),$$

$$v \geq \frac{1}{q} \Rightarrow \omega(v) \leq 4qv\omega\left(\frac{1}{n}\right).$$

Proof. The last property of the modulus implies that if $u \geq \frac{1}{p}$,

$$\frac{2\omega\left(\frac{1}{p}\right)}{\frac{1}{p}} \geq \frac{\omega(u)}{u}.$$

Since $1/p \leq 2/m$

$$\omega(u) \leq 2pu\omega\left(\frac{2}{m}\right) \leq 4pu\omega\left(\frac{1}{m}\right).$$

□

Lemma 3.4. *Let us consider the following integrals:*

$$C_{pq\gamma}^{0} = \int_{0}^{\pi/2}\int_{0}^{\pi/2} G_{pq\gamma}(u,v)dudv,$$

$$C_{pq\gamma}^{1u} = \int_{0}^{\pi/2}\int_{0}^{\pi/2} uG_{pq\gamma}(u,v)dudv,$$

$$C^{1v}_{pq\gamma} = \int_0^{\pi/2}\int_0^{\pi/2} vG_{pq\gamma}(u,v)dudv,$$

where $G_{pq\gamma}$ is defined by the expression (31). Then, for $\gamma > 2$,

$$C^0_{pq\gamma} = H^{-1}_{pq\gamma}/4,$$

$$pC^{1u}_{pq\gamma}H_{pq\gamma} < \frac{1}{2}\left(\frac{\pi}{2}\right)^\gamma \left(\frac{1}{2} - \frac{1}{2-\gamma}\right) C_\gamma,$$

$$qC^{1v}_{pq\gamma}H_{pq\gamma} < \frac{1}{2}\left(\frac{\pi}{2}\right)^\gamma \left(\frac{1}{2} - \frac{1}{2-\gamma}\right) C_\gamma,$$

where C^{-1}_γ is defined in (27) and $H_{pq\gamma}$ in (32).

Proof. The inequality concerning $C^0_{pq\gamma}$ is a direct result of the definition of $H^{-1}_{pq\gamma}$ (32) and the symmetry of $G_{pq\gamma}$.
For the second, let us consider that

$$C^{1u}_{pq\gamma} = h^{-1}_{q\gamma} i^1_{p\gamma},$$

where $h^{-1}_{q\gamma}$ is defined as in (22), and $i^1_{p\gamma}$ in (24). Then

$$pC^{1u}_{pq\gamma}H_{pq\gamma} = pC^{1u}_{pq\gamma}h_{p\gamma}h_{q\gamma} = pi^1_{p\gamma}h_{p\gamma} \leq \frac{1}{2}\left(\frac{\pi}{2}\right)^\gamma \left(\frac{1}{2} - \frac{1}{2-\gamma}\right) C_\gamma,$$

bearing in mind (24) and (26). $\square$

The definition of $H_{pq\gamma}$ (32) implies the inequality

$$|\mathcal{G}_{pq\gamma}(f)(x,y) - f(x,y)| \leq H_{pq\gamma}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} |f(x+2u, y+2v) - f(x,y)|G_{pq\gamma}(u,v)dudv.$$

Using the definition of modulus of continuity of a two-variable function and its properties:

$$|\mathcal{G}_{pq\gamma}(f)(x,y) - f(x,y)| \leq 2H_{pq\gamma}\int_{-\pi/2}^{\pi/2} \omega\left(\sqrt{u^2+v^2}\right) G_{pq\gamma}(u,v)dudv$$

$$|\mathcal{G}_{pq\gamma}(f)(x,y) - f(x,y)| \leq 8H_{pq\gamma}\int_0^{\pi/2}\int_0^{\pi/2} \omega(u+v)G_{pq\gamma}(u,v)dudv.$$

$$|\mathcal{G}_{pq\gamma}(f)(x,y) - f(x,y)| \leq 8H_{pq\gamma}\int_0^{\pi/2}\int_0^{\pi/2} (\omega(u)+\omega(v))G_{pq\gamma}(u,v)dudv.$$

Let us divide the last domain in four rectangles induced by the intervals $[0,1/p]$, $[1/p,\pi/2]$, $[0,1/q]$, $[1/q,\pi/2]$. Let the last integral be computed as a sum of four subintegrals:

$$\int_0^{\pi/2}\int_0^{\pi/2}(\omega(u)+\omega(v))G_{pq\gamma}(u,v)dudv = I_1+I_2+I_3+I_4.$$

being

$$I_1=\int_0^{1/p}\int_0^{1/q}\left(\omega(u)+\omega(v)\right)G_{pq\gamma}(u,v)dudv \le \left(\omega\left(\frac{1}{p}\right)+\omega\left(\frac{1}{q}\right)\right)C^0_{pq\gamma},$$

where

$$C^0_{pq\gamma}=\int_0^{\pi/2}\int_0^{\pi/2}G_{pq\gamma}(u,v)dudv.$$

Since $1/p\le 2/m$, $1/q\le 2/n$ and $\omega(1/m)\le\omega(1/m+1/n)$,

$$I_1\le 2\left(\omega\left(\frac{1}{m}\right)+\omega\left(\frac{1}{n}\right)\right)C^0_{pq\gamma}\le 4\omega\left(\frac{1}{m}+\frac{1}{n}\right)C^0_{pq\gamma}. \tag{33}$$

The second integral is defined and bounded as:

$$I_2=\int_0^{1/q}\int_{1/p}^{\pi/2}(\omega(u)+\omega(v))G_{pq\gamma}(u,v)dudv.$$

Using Lemma 3.3

$$I_2\le\int_0^{1/q}\int_{1/p}^{\pi/2}\left(4pu\omega\left(\frac{1}{m}\right)+\omega\left(\frac{1}{q}\right)\right)G_{pq\gamma}(u,v)dudv.$$

Since $1/q\le 2/n$

$$I_2\le\int_0^{1/q}\int_{1/p}^{\pi/2}\left(4pu\omega\left(\frac{1}{m}\right)+2\omega\left(\frac{1}{n}\right)\right)G_{pq\gamma}(u,v)dudv.$$

The properties of the modulus imply

$$I_2\le 2\omega\left(\frac{1}{m}+\frac{1}{n}\right)\int_0^{1/q}\int_{1/p}^{\pi/2}(2pu+1)G_{pq\gamma}(u,v)dudv.$$

Then

$$I_2\le 2\omega\left(\frac{1}{m}+\frac{1}{n}\right)(2pC^{1u}_{pq\gamma}+C^0_{pq\gamma}), \tag{34}$$

according to the definitions of $C^0_{pq\gamma}$ and $C^{1u}_{pq\gamma}$ in Lemma 3.4. The third integral is

$$I_3 = \int_{1/q}^{\pi/2} \int_{1/p}^{\pi/2} (\omega(u) + \omega(v)) G_{pq\gamma}(u,v) du dv.$$

Using Lemma 3.3

$$I_3 \leq \int_{1/q}^{\pi/2} \int_{1/p}^{\pi/2} \left(4pu\omega\left(\frac{1}{m}\right) + 4qv\omega\left(\frac{1}{n}\right)\right) G_{pq\gamma}(u,v) du dv,$$

$$I_3 \leq 4\omega\left(\frac{1}{m} + \frac{1}{n}\right)\left(pC^{1u}_{pq\gamma} + qC^{1v}_{pq\gamma}\right). \tag{35}$$

For the last integral

$$I_4 = \int_{1/q}^{\pi/2} \int_{0}^{1/p} (\omega(u) + \omega(v))\, G_{pq\gamma}(u,v) du dv.$$

Arguing as before

$$I_4 \leq \int_{1/q}^{\pi/2} \int_{0}^{1/p} \left(2\omega\left(\frac{1}{m}\right) + 4qv\omega\left(\frac{1}{n}\right)\right) G_{pq\gamma}(u,v) du dv,$$

and finally

$$I_4 \leq 2\omega\left(\frac{1}{m} + \frac{1}{n}\right)(C^0_{pq\gamma} + 2qC^{1v}_{pq\gamma}). \tag{36}$$

The inequalities (33), (34), (35) and (36) provide a bound for the error $E_{pq\gamma}(x)$ of the approximation $\mathcal{G}_{pq\gamma}(f)$

$$E_{pq\gamma}(x) \leq 64 H_{pq\gamma}\omega\left(\frac{1}{m} + \frac{1}{n}\right)(C^0_{pq\gamma} + pC^{1u}_{pq\gamma} + qC^{1v}_{pq\gamma}).$$

Using Lemma 3.4 we obtain a uniform error bound (provided $\gamma > 2$)

$$E_{pq\gamma}(x) \leq K_\gamma \omega\left(\frac{1}{m} + \frac{1}{n}\right).$$

Thus we deduce the following convergence result.

Theorem 3.5. *If $\gamma > 2$, the two-dimensional approximant $\mathcal{G}_{pq\gamma}(f)$ is convergent to $f \in \mathcal{C}(T^1 \times T^1)$ as m, n tend to infinity. The rate of convergence is that of $\omega\left(\frac{1}{m} + \frac{1}{n}\right)$.*

Remark 5. For functions satisfying a Lipschitz condition of order β, the range of convergence values of γ extends to $\gamma > \beta + 1$ as in the discrete case.

Remark 6. The statements about the norm of the operator (Theorem 2.4 and Remark 3) are valid for $\mathcal{G}_{pq\gamma}$ as well.

4. Numerical examples

In this section we present some numerical computations of the discrete model proposed for different functions, grids and exponents:
Figure 1 displays the three-dimensional representation of the function $f(x,y) = \sqrt{|\sin(x)\cos(y)|}$ on the square $[-\pi,\pi] \times [-\pi,\pi]$. Figure 2 shows the surface concerning its discrete Jackson approximant $\mathcal{J}_{mn\gamma}(f)(x,y)$ computed for $m = n = 10$ (grid of 20×20 points) and $\gamma = 4$. Figure 3 represents the approximant surface for $m = n = 10$ and $\gamma = 5$ on the same rectangle.

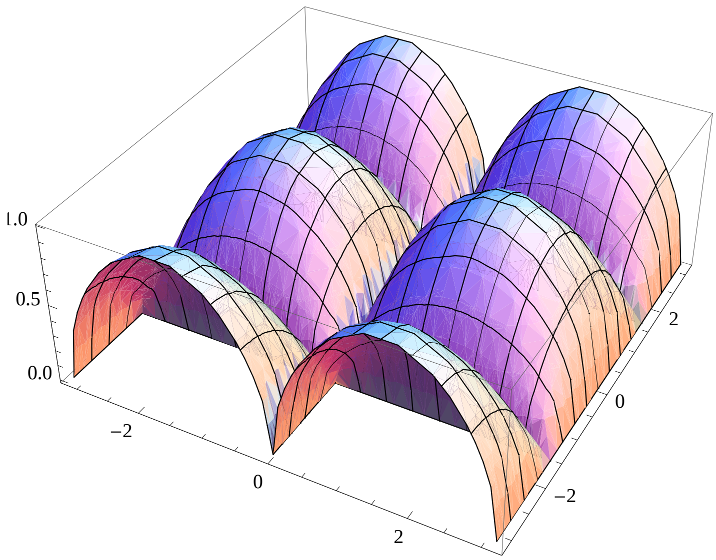

Figure 1. Graph of the function $f(x,y) = \sqrt{|\sin(x)\cos(y)|}$ in the square $[-\pi,\pi] \times [-\pi,\pi]$.

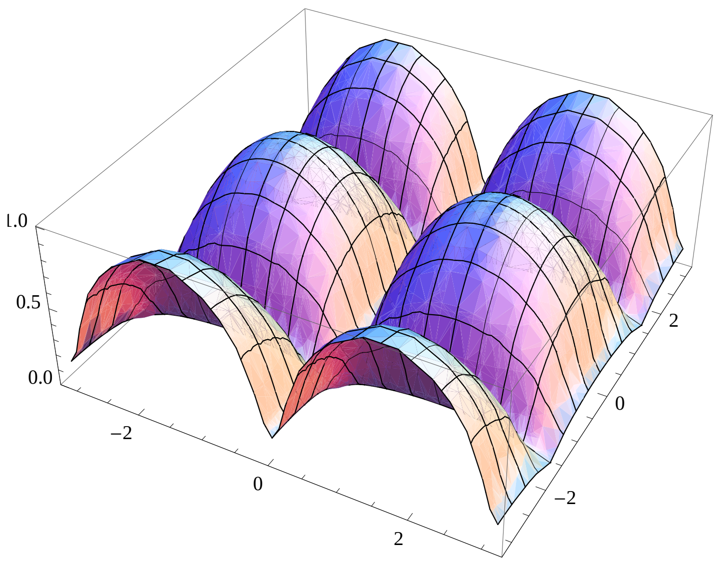

Figure 2. Graph of the discrete approximant of $f(x,y) = \sqrt{|\sin(x)\cos(y)|}$ for $m = 10$, $n = 10$ and $\gamma = 4$ in the square $[-\pi,\pi] \times [-\pi,\pi]$.

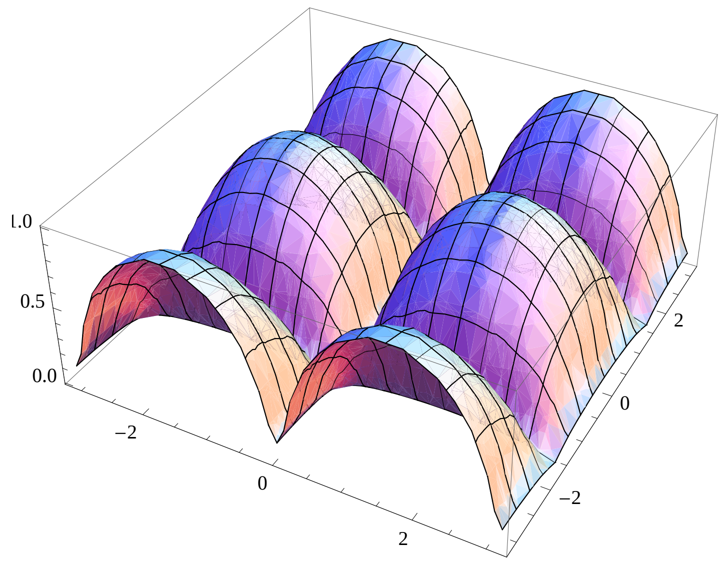

Figure 3. Graph of the discrete approximant of $f(x,y) = \sqrt{|\sin(x)\cos(y)|}$ for $m = 10$, $n = 10$ and $\gamma = 5$ in the square $[-\pi, \pi] \times [-\pi, \pi]$.

Table 1 collects the errors in the approximation of the functions $f_1(x,y) = |\sin(x)\cos(y)|$ (rows 1, 2, 5, 6) and $f_2(x,y) = \sqrt{|\sin(x)\cos(y)|}$ (rows 3, 4, 7, 8) at the points $(\pi/3, \pi/6)$ and $(\pi/3, 4\pi/9)$, using the discrete model for $m = n = 5$ or $m = n = 10$, and the values $4, 4.5, 5$ of the exponent γ. The column $\gamma = 4$ shows the errors of the classical case. We observe that in several cases, nearby values of γ provide a better result.

Table 1. Approximation errors of function values for different choices of m, n and γ, using the discrete model.

		$\gamma = 4$	$\gamma = 4.5$	$\gamma = 5$
$m = n = 5$	$\|\sin(\pi/3)\cos(\pi/6)\|$	0.0781323	0.0700892	0.0637317
$m = n = 10$	$\|\sin(\pi/3)\cos(\pi/6)\|$	0.0214228	0.0213215	0.0226161
$m = n = 5$	$\sqrt{\|\sin(\pi/3)\cos(\pi/6)\|}$	0.0572732	0.0507959	0.0458076
$m = n = 10$	$\sqrt{\|\sin(\pi/3)\cos(\pi/6)\|}$	0.0150111	0.0143546	0.0147956
$m = n = 5$	$\|\sin(\pi/3)\cos(4\pi/9)\|$	-0.1242290	-0.1202180	-0.1177190
$m = n = 10$	$\|\sin(\pi/3)\cos(4\pi/9)\|$	0.0006592	0.0014973	0.0013869
$m = n = 5$	$\sqrt{\|\sin(\pi/3)\cos(4\pi/9)\|}$	-0.1277870	-0.1255740	-0.1243110
$m = n = 10$	$\sqrt{\|\sin(\pi/3)\cos(4\pi/9)\|}$	0.1006510	0.1004220	0.0990830

Table 2 collects the errors obtained by three different methods of data approximation on a grid. We evaluated the functions $f_3(x,y) = 1 + \log(\cos^2(x)\sin^2(y))$, $f_4(x,y) = |\sin(x)\cos(y)|^{1/16}$ and $f_5(x,y) = |\sin(x)\cos(y)|^{1/8}$ on the square $[-\pi, \pi] \times [-\pi, \pi]$ at the points $(\pi/3, \pi/6)$ and $(\pi/3, 4\pi/9)$. The last column of the table collects the errors of a Jackson approximation function for $m = n = 5$ and $\gamma = 4.5$. The third column shows the differences between the exact and the computed values by means of a Shepard function on a grid with the same nodes. The second column displays the errors obtained by a radial basis two-dimensional spline such that $\Phi(x, y) = (x^2 + y^2)\log(x^2 + y^2)$. The number of nodes was chosen such that the computational cost was similar to the other procedures.

Table 2. Approximation errors of function values for three different methods.

	RB Splines	Shepard	Jackson
$\log(1 + \cos^2(\pi/3)\sin^2(\pi/6))$	0.5043500	-0.0587742	-0.0180435
$\log(1 + \cos^2(\pi/3)\sin^2(4\pi/9))$	-0.1203860	0.0385543	0.0070074
$\lvert\sin(\pi/3)\cos(\pi/6)\rvert^{1/16}$	0.0803814	0.0516266	0.0096613
$\lvert\sin(\pi/3)\cos(4\pi/9)\rvert^{1/16}$	-0.0492603	-0.0089789	-0.0296620
$\lvert\sin(\pi/3)\cos(\pi/6)\rvert^{1/8}$	0.1531250	0.0668364	0.0174424
$\lvert\sin(\pi/3)\cos(4\pi/9)\rvert^{1/8}$	-0.0901880	-0.0470231	-0.0548085

5. Conclusions

- The discrete method proposed is a cheap procedure to perform function and values approximation of periodic multivariate data on grids. Its computational cost is much lower than other methods as two-dimensional splines, and more precise than other algorithms.
- The nodal Jackson approximants are explicit in terms of the data, unlike other functions that require the solution of large systems of equations as, for instance, the least-squares fits.
- The method proposed generalizes the functions of Jackson, providing a family of approximants that may be used when an additional problem is imposed to the initial approximation issue. The exponential parameter provides a variety of functions suitable for a best choice when the problem requires a dose of flexibility.
- The order of smoothness of the mappings is wider than in the classical case. For instance, if the exponent γ is lower than one, the nodal functions belong to $\mathcal{C}^0$ class, but they are not in the $\mathcal{C}^1$ class.
- The procedure is one of the few methods of periodic approximation (in both variables), thus the methods may be used for the approximation of data and functions on tori. These geometric objects appear in many fields of the scientific knowledge like physics, engineering, etc.
- The functions proposed may be useful for the resolution of (fractional) differential equations. They may also served to find degrees of approximation to more general spaces of functions.
- In the article, we have deduced error bounds for the generalized discrete and continuous models. According to the theorems, and exponent γ greater than 2 in the nodal functions ensures the convergence of the approximants when the number of nodes in both variables tend to infinity. The order of convergence is that of the modulus of continuity of the original function and it does not depend

on γ (although the multiplying constants change with this parameter).

- In principle, the convergence result requires only the continuity of the function f to be approximated. If f belongs to a Lipschitz class with exponent q ($0 < q \leq 1$), then the range of convergence values of the exponent is extended to $\gamma > 1 + q$.
- In the continuous model, we have obtained similar results along with a previous convergence theorem for the single dimensional case.

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