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ABSTRACT

If G is a group and if the upper hypercenter, Z, of G is such that G/Z is finite then a recent theorem shows that G contains a finite normal subgroup L such that G/L is hypercentral. The purpose of the current paper is to obtain a version of this result for subgroups G of $GL(F, A)$, when A is an infinite dimensional F-vector space.

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Introduction

Let G be a group, R a ring, and A an RG-module. Then the set

$$\zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0 \text{ for each element } g \in G\} = C_A(G)$$

is an RG-submodule of A called *the RG-center of A*, analogous to the center of a group. The submodule analogous to the derived subgroup of a group is constructed as follows. Let ωRG be *the augmentation ideal of the group ring RG*, that is, the two-sided ideal generated by all elements of the form $g-1$, $g \in G$. The submodule $A(\omega RG)$ is said to be *the derived submodule* of A.

This paper is concerned with obtaining a linear version of the main theorem of the papers [3, 12]. To describe our work, we need some more terminology and notation.

The *upper RG-central series of A*,

$$\langle 0\rangle = \zeta_{RG,0}(A) \leq \zeta_{RG,1}(A) \leq \cdots \leq \zeta_{RG,\alpha}(A) \leq \zeta_{RG,\alpha+1}(A) \leq \cdots \leq \zeta_{RG,\gamma}(A)$$

is the series of RG-submodules defined by $\zeta_{RG,1}(A) = \zeta_{RG}(A)$, $\zeta_{RG,\alpha+1}(A)/\zeta_{RG,\alpha}(A) = \zeta_{RG}(A/\zeta_{RG,\alpha}(A))$ for every ordinal α, $\zeta_{RG,\lambda}(A) = \bigcup_{\mu<\lambda} \zeta_{RG,\mu}(A)$ for every limit ordinal λ, and $\zeta_{RG}(A/\zeta_{RG,\gamma}(A)) = \{0\}$. The last term $\zeta_{RG,\gamma}(A) = \zeta_{RG,\infty}(A)$ of this series is called *the upper RG-hypercenter of A*. The ordinal γ is said to be *the RG-central length of A* and will be denoted by $zl_{RG}(A)$. We note that $\zeta_{RG,\alpha+1}(A)(\omega RG) \leq \zeta_{RG,\alpha}(A)$ for every ordinal $\alpha < \gamma$. We say that A is *RG-hypercentral* if $\zeta_{RG,\gamma}(A) = A$, for some γ, and the RG-hypercentral module A is said to be *RG-nilpotent* if $zl_{RG}(A)$ is finite. Also, A is *RG-locally nilpotent* if the FH-submodule $M(FH)$ is FH-nilpotent for every finite subset M of A and every finitely generated subgroup H of G.

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If $\mathfrak{X}$ is a class of RG-modules and if A is an RG-module, we put

$$\mathrm{Res}_{\mathfrak{X}}(A) = \{B \mid B \text{ is an } RG\text{-submodule of } A \text{ such that } A/B \in \mathfrak{X}\}.$$

Then the intersection $A^{\mathfrak{X}}$ of all members of the family $\mathrm{Res}_{\mathfrak{X}}(A)$ is called *the $\mathfrak{X}$-residual of A*. If $\mathrm{Res}_{\mathfrak{X}}(A)$ has a least element L, then $L = A^{\mathfrak{X}}$ and $A/A^{\mathfrak{X}} \in \mathfrak{X}$, but this does not hold in general. If $\mathfrak{X}$ is the class of all RG-nilpotent modules, this definition gives us the *RG-nilpotent residual of A*, and if $\mathfrak{X}$ is the class of all RG-hypercentral modules, we obtain the *RG-hypercentral residual of A*. Likewise, when $\mathfrak{X}$ is the class of locally RG-nilpotent modules, then we obtain the *locally RG-nilpotent residual of A*.

Let F be a field. Our main results concern the relationship between the quotient module $A/\zeta_{FG,\infty}(A)$ and the locally FG-nilpotent residual of A, for certain types of subgroup G of $GL(F, A)$, the group of all F-automorphisms of A.

We recall that a group G has *finite special rank r* if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property. This rank is one of the most important numerical invariants of a group. We shall be concerned with certain other ranks here which we now discuss.

Let p be a prime. We say that a group G has *finite section p-rank $sr_p(G) = r$* if every elementary Abelian p-section U/V of G is finite of order at most p^r and there is an elementary Abelian p-section A/B of G such that $|A/B| = p^r$. Similarly, we say that a group G has *finite section 0-rank $sr_0(G) = r$* if every torsion-free Abelian section U/V of G satisfies $sr_{\mathbb{Z}}(U/V) \le r$ and there exists an Abelian torsion-free section A/B such that $sr_{\mathbb{Z}}(U/V) = r$. Here $sr_{\mathbb{Z}}(A)$ is the $\mathbb{Z}$-rank of the Abelian group A, the rank of A as a $\mathbb{Z}$-module. We note that if a group G has finite section p-rank for some prime p, then G has finite section 0-rank and $sr_0(G) \le sr_p(G)$. For, given a torsion-free Abelian section U/V of G, let S/V be a free Abelian subgroup of U/V such that U/S is periodic. Then $sr_{\mathbb{Z}}(U/V) = r_{\mathbb{Z}}(S/V)$. If $S/V = \mathrm{Dr}_{\lambda\in\Lambda}\langle d_\lambda\rangle$ say, then $(S/V)^p = \mathrm{Dr}_{\lambda\in\Lambda}\langle d_\lambda^p\rangle$ and so

$$(S/V)/(S/V)^p = (\mathrm{Dr}_{\lambda\in\Lambda}\langle d_\lambda\rangle)/(\mathrm{Dr}_{\lambda\in\Lambda}\langle d_\lambda^p\rangle) \cong \mathrm{Dr}_{\lambda\in\Lambda}\langle d_\lambda\rangle/\langle d_\lambda^p\rangle.$$

Since $sr_p(G) = r$ is finite, $(S/V)/(S/V)^p$ is a finite group that has order at most p^r. On the other hand, we have $|(S/V)/(S/V)^p| = p^{|\Lambda|}$ and so $sr_{\mathbb{Z}}(U/V) = r_{\mathbb{Z}}(S/V) = |\Lambda| \le r$. It readily follows that $sr_0(G) \le sr_p(G)$ as claimed.

It was proved in [12] that if G is a group, Z is the upper hypercenter of G and if G/Z is finite of order t, then G has a finite normal subgroup L, of order bounded in terms of t, such that G/L is hypercentral. A nonquantitative version of this result had earlier appeared in [3]. The main results of our paper are as follows. As will be evident, the two results are similar, but their proofs have some differences.

Theorem A. *Let F be a field of prime characteristic p, A an F-vector space and G a subgroup of $GL(F, A)$. If $\dim_F(A/\zeta_{FG,\infty}(A)) = d$ and $sr_p(G) = r$ are finite, then the locally FG-nilpotent residual L of A has finite dimension and A/L is FG-hypercentral. Moreover, there exists a function κ_4 such that $\dim_F(L) \le \kappa_4(r, d)$.*

Theorem B. *Let F be a field of prime characteristic 0, A an F-vector space and G a subgroup of $GL(F, A)$. If $\dim_F(A/\zeta_{FG,\infty}(A)) = d$ and $sr_0(G) = r$ are finite, then the locally FG-nilpotent residual L of A has finite dimension and A/L is FG-hypercentral. Moreover, there exists a function κ_9 such that $\dim_F(L) \le \kappa_9(r, d)$.*

The layout of the paper is as follows. In Section 1, we gather together some preliminary results. In Section 2, we discuss the positive characteristic case of our results and prove Theorem A. In Section 3, we discuss the characteristic zero case and prove Theorem B. Finally, in Section 4, we give an example of a subgroup G of $GL(F, A)$ of *infinite* 0-rank, for which $\dim_F(A/\zeta_{FG,\infty}(A))$ is finite but in which the locally nilpotent FG-nilpotent residual is infinite dimensional.

1. Hypercentral and Nilpotent modules

The following properties are immediate.

Lemma 1.1. *Let R be a ring, G a group, and A an RG-module. Suppose that A is RG-hypercentral (respectively, RG-nilpotent). Then we have*

(i) *If B is an RG-submodule of A, then B is RG-hypercentral (respectively, RG-nilpotent).*
(ii) *If H is a subgroup of G, then A is RH-hypercentral (respectively, RH-nilpotent).*
(iii) *If H is a subgroup of G and B is an RH-submodule of A, then B is RH-hypercentral (respectively, RH-nilpotent).*

An easy consequence of Lemma 1.1 is the following result.

Corollary 1.2. *Let R be a ring, G a group and A an RG-module. Suppose that A is locally RG-nilpotent. Then we have*

(i) *If B is an RG-submodule of A, then B is locally RG-nilpotent.*
(ii) *If H is a subgroup of G, then A is locally RH-nilpotent.*
(iii) *If H is a subgroup of G and B is an RH-submodule of A, then B is locally RH-nilpotent.*

From now on, we focus on the case in which the underlying ring $R = F$ is a field and will assume this notation from now on. The proof of our first result is analogous to the proof of [2, Lemma 2].

Lemma 1.3. *Let G be a finitely generated group and A a finitely generated FG-module. If B is an FG-submodule of A such that $\dim_F(A/B)$ is finite, then B is finitely generated as an FG-submodule.*

Proof. Let $M = \{g_1, \cdots, g_t\}$ be a subset of G such that $G = \langle M\rangle$. Without loss of generality, we can assume that $g_j^{-1} \in M$ for each $1 \le j \le t$. Choose a subset $V = \{a_1, \cdots, a_n\}$ of A such that $A = a_1FG + \cdots + a_nFG$. There exists a finite dimensional F-subspace C such that $A = B \oplus C$ and let $\{c_1, \cdots, c_k\}$ be a basis of C. Denote by p_B and p_C the canonical projections of A on B and C, respectively, and let E be the FG-submodule generated by the set

$$\{p_B(a_j), p_B(c_m g_s) \mid 1 \le j \le n,\ 1 \le m \le k,\ 1 \le s \le t\}.$$

By construction, $E \le B$. If $d \in E + C$, then $d = u + c$, where $u \in E$ and $c \in C$. We may write

$$c = \alpha_1 c_1 + \cdots + \alpha_k c_k$$

for suitable elements $\alpha_1, \cdots, \alpha_k \in F$. For each $1 \le j \le t$, we have

$$\begin{aligned} cg_j &= (\alpha_1 c_1 + \cdots + \alpha_k c_k)g_j = \alpha_1(c_1 g_j) + \cdots + \alpha_k(c_k g_j) \\ &= \alpha_1(p_B(c_1 g_j) + p_C(c_1 g_j)) + \cdots + \alpha_k(p_B(c_k g_j) + p_C(c_k g_j)) \\ &= \alpha_1 p_B(c_1 g_j) + \cdots + \alpha_k p_B(c_k g_j) + \alpha_1 p_C(c_1 g_j) + \cdots + \alpha_k p_C(c_k g_j)) \in E + C. \end{aligned}$$

It follows that $E + C$ is an FG-submodule of A. Since

$$a_j = p_B(a_j) + p_C(a_j) \in E + C, \text{ for all } 1 \le j \le n,$$

we have $E + C = A = B + C$. However, $E \le B$ and $B \cap C = \langle 0\rangle$ so that $B = E$. Thus, B is finitely generated as an FG-submodule. □

Lemma 1.4. *Let G be a finitely generated group and A a finitely generated FG-module. If $A/\zeta_{FG,\infty}(A)$ has finite dimension, then $zl_{FG}(A)$ is finite.*

Proof. Let

$$\langle 0\rangle = Z_0 \le Z_1 \le \cdots \le Z_\alpha \le Z_{\alpha+1} \le \cdots Z_\gamma = \zeta_{FG,\infty}(A) \le A$$

be the upper FG-central series of A. First, we remark that if Z_α is a finitely generated FG-module, then α cannot be a limit ordinal.

Since $\dim_F(A/\zeta_{FG,\infty}(A))$ is finite, Lemma 1.3 implies that $\zeta_{FG,\infty}(A)$ is finitely generated as an FG-submodule. Thus γ is not a limit ordinal, by our initial remark. Suppose that γ is infinite, so that $\gamma = \tau + n$ for some limit ordinal τ and positive integer n. Let $V = \{v_1, \cdots, v_n\}$ be a finite subset of A such that

$$Z_\gamma = v_1FG + \cdots + v_nFG.$$

Since $Z_\gamma/Z_{\gamma-1} = \zeta_{FG}(A/Z_{\gamma-1})$, we have

$$Z_\gamma/Z_{\gamma-1} = (v_1F + \cdots + v_nF)Z_{\gamma-1}/Z_{\gamma-1},$$

and, in particular, $Z_\gamma/Z_{\gamma-1}$ has finite dimension at most n. Again by Lemma 1.3, $Z_{\gamma-1}$ is finitely generated as an FG-submodule. Proceeding in this way, after finitely many steps, we deduce that the FG-submodule Z_τ is finitely generated. By our initial remark, we see that τ is not a limit ordinal, which is a contradiction. Hence γ must be finite, as required. □

This has an obvious consequence.

Corollary 1.5. *Let G be a finitely generated group and A a finitely generated FG-module. If A is FG-hypercentral, then A is FG-nilpotent.*

We need some more definitions. For the group G, the ring R and RG-module A the factor C/B of A are said to be *G-central* if $G = C_G(C/B)$; otherwise C/B is said to be *G-eccentric*. Also, A is said to be *G-hypereccentric*, if A has an ascending series of RG-submodules

$$\langle 0\rangle = A_0 \le A_1 \le \cdots \le A_\alpha \le A_{\alpha+1} \le \cdots \le A_\alpha = A$$

whose factors $A_{\alpha+1}/A_\alpha$ are G-eccentric simple FG-modules. We say that the RG-module A has *the Z-decomposition* if there is a direct decomposition

$$A = C \oplus E,$$

where C is the upper RG-hypercenter of A and E is a G-hypereccentric RG-submodule. We remark that if such decomposition exists then it is unique. We refer the reader to [11, Chapter 10] for further details.

We need a further lemma.

Lemma 1.6. *Let G be a group and suppose that A is a locally FG-nilpotent FG-module. If B is a finite dimensional FG-submodule of A, then there exists some $k \ge 1$ such that $B \le \zeta_{RG,k}(A)$.*

Proof. Let

$$\langle 0\rangle = Z_0 \le Z_1 \le \cdots \le Z_\alpha \le Z_{\alpha+1} \le \cdots \le Z_\gamma = A$$

be the upper FG-central series of A. We proceed by induction on $\dim_F(B)$, the case $k = 1$ is being clear. Since $B_1 = B \cap Z_1 \ne \langle 0\rangle$, $\dim_F(B/B_1) < \dim_F(B)$. We have

$$(B + Z_1)/Z_1 \cong B/(B \cap Z_1) = B/B_1.$$

In particular, $\dim_F((B+Z_1)/Z_1) = \dim_F(B/B_1) < \dim_F(B)$. By induction, there exists a positive integer k such that $(B + Z_1)/Z_1 \le Z_k/Z_1$ and then $B \le Z_k$ as required. □

Our next result generalizes [5, Corollary 2.3].

Lemma 1.7. *Let G be a group of finite special rank k and let A be an FG-module such that $zl_{FG}(A)$ is finite. If $\dim_F(A/\zeta_{FG,\infty}(A)) = d$, then the FG-nilpotent residual L of A has finite dimension at most $d(k+1)$. Moreover, the factor-module A/L is FG-nilpotent.*

Proof. Let

$$\langle 0\rangle = \zeta_{FG,0}(A) \le \zeta_{FG,1}(A) \le \zeta_{FG,2}(A) \le \cdots \le \zeta_{FG,t}(A) = Z$$

be the upper FG-central series of A. By Kaluzhnin's theorem [8], $G/C_G(Z)$ is nilpotent.

Let $C = C_G(Z)$ and $B = A(\omega FC)$. Then $Z \le \zeta_{FC}(A)$ so $\dim_F(A/\zeta_{FC}(A)) \le d$ and, by [5, Corollary 2.3], $\dim_F B \le dk$. Note that $C \le C_G(A/B)$ so that $G/C_G(A/B)$ is a nilpotent group. The factor-module $(A/B)/(ZB/B)$ has finite dimension over F, and so has a finite FG-composition series. By [9, Corollary 2.6], A/B has the Z-decomposition, that is

$$A/B = Y/B \oplus E/B,$$

where Y/B is the upper FG-hypercenter of A/B and E/B is an FG-hypereccentric FG-submodule. Since $ZB/B \le Y/B$, E/B has finite dimension and $\dim_F(E/B) \le d$. It follows that E has finite dimension and $\dim_F(E) \le dk + d = d(k+1)$. Since A/E is FG-nilpotent, $L \le E$. Hence $\dim_F(L) \le k(d+1)$. Also L is the intersection of all the FG-submodules X such that A/X is FG-nilpotent. Since E is finite dimensional, it is easy to see that

$$L = \cap_{i=1}^{r}\{E_i | A/E_i \text{ is } FG\text{-nilpotent}\},$$

for certain FG-submodules E_i. It follows that there is an embedding,

$$A/L \longrightarrow \operatorname*{Dr}_{i=1}^{r} A/E_i,$$

and from this, we deduce that A/L is FG-nilpotent, as required. □

Corollary 1.8. *Let G be a finitely generated group of finite special rank k. If A is a finitely generated FG-module such that $\dim_F(A/\zeta_{FG,\infty}(A)) = d$ is finite, then the FG-nilpotent residual L of A has finite dimension at most $d(k+1)$. Moreover, A/L is FG-nilpotent.*

Proof. By Lemma 1.4, $zl_{FG}(A)$ is finite and it suffices to apply Lemma 1.7. □

Corollary 1.9. *Let G be a finitely generated group of finite special rank k and let A be an FG-module. If $\dim_F(A/\zeta_{FG,\infty}(A)) = d$, then the locally FG-nilpotent residual L of A has finite dimension at most $d(k+1)$. Moreover, A/L is locally FG-nilpotent.*

Proof. Put $Z = \zeta_{FG,\infty}(A)$. Since A/Z has finite dimension, there exists a finite subset M such that $A = MF + Z$. Let $\mathcal{D}$ be the family of all finitely generated FG-submodules of A containing M. If $B \in \mathcal{D}$, then $Z \cap B \le \zeta_{FG,\infty}(B)$ and so $\dim_F(B/\zeta_{FG,\infty}(B)) \le d$. By Corollary 1.8, the FG-nilpotent residual $L(B)$ of B has finite dimension at most $d(k+1)$ and $B/L(B)$ is FG-nilpotent. Pick $C \in \mathcal{D}$ such that $B \le C$. Since $C/L(C)$ is FG-nilpotent, $B/(B \cap L(C))$ is FG-nilpotent, so $L(B) \le B \cap L(C)$, whence $L(B) \le L(C)$. Choose an FG-submodule $K \in \mathcal{D}$ such that $\dim_F L(K)$ is maximal. If $C \in \mathcal{D}$ and $K \le C$, we have $L(K) \le L(C)$, and the choice of K implies that $L(K) = L(C)$.

Let S be an arbitrary finite subset of A and put

$$T = \{M, S\}FG + K.$$

It follows that $T \in \mathcal{D}$, $K \le T$ and $L(T) = L(K)$, so $T/L(K)$ is FG-nilpotent. Then $A/L(K)$ is locally FG-nilpotent and hence $L \le L(K)$. It follows that L has finite dimension at most $d(k+1)$. As in the last part of the proof of Lemma 1.7, we deduce that A/L is locally FG-nilpotent. □

We now obtain the conclusion of our theorems in the case when our groups are of finite special rank. This key result will be very useful later.

Proposition 1.10. *Let G be a group of finite special rank k and let A be an FG-module. If $\dim_F(A/\zeta_{FG,\infty}(A)) = d$ is finite, then the locally FG-nilpotent residual L of A has finite dimension at most $d(k+1)$. Moreover, A/L is locally FG-nilpotent.*

Proof. Let $\mathcal{S}$ be the family of all finitely generated subgroups of G. If $H \in \mathcal{S}$, the locally FH-nilpotent residual L_H of A has finite dimension at most $d(k+1)$, by Corollary 1.9. Pick $K \in \mathcal{S}$ such that $H \leq K$. Since L_K is an FK-submodule of A, it is also an FH-submodule of A. By Corollary 1.9, A/L_K is locally FK-nilpotent, and Corollary 1.2 implies that A/L_K is also locally FH-nilpotent. Hence $L_H \leq L_K$. Choose a subgroup $T \in \mathcal{S}$ such that $\dim_F L_T$ is maximal. If $V \in \mathcal{S}$ and $T \leq V$, we have $L_T \leq L_V$, and the choice of T implies that $L_T = L_V$. In particular, L_T is an FV-submodule. If U is an arbitrary finitely generated subgroup of G, we have that L_T is $F\langle T, U\rangle$-invariant. Thus L_T is an FU-submodule of A. Since this holds for every finitely generated subgroup of G, L_T is in fact an FG-submodule.

Let H be an arbitrary finitely generated subgroup of G. Put $W = \langle H, T\rangle$. Since $L_T = L_W$, A/L_T is locally FW-nilpotent. Thus, A/L_T is locally FG-nilpotent and it follows that $L \leq L_T$ which means that L has finite dimension at most $d(k+1)$. As in the last part of the proof of Lemma 1.7, it follows that A/L is locally FG-nilpotent. □

2. The positive characteristic case

Lemma 2.1. *Let p be a prime and G a group. If G has finite section p-rank or finite section 0-rank, then no section of G contains a non-Abelian free group.*

Proof. Suppose the contrary, let V/U be a section of G that contains a non-Abelian free subgroup, say F/U. If the free rank of F/U is infinite, then F/U has a normal subgroup E/U such that F/E is a free Abelian group of infinite 0-rank. In this case, F/U has an infinite elementary Abelian p-factor-group, so that F/U must have infinite section p-rank. This gives us a contradiction. If F/U has finite free rank, $K/U = [F/U, F/U]$ is a free subgroup of countably infinite free rank [13, Section 36], and we again arrive at a contradiction. □

Let G be a finite group and suppose that

$$|G| = n = p_1^{k_1} \cdots p_m^{k_m}.$$

If H is a subgroup of G, then $|H| = p_1^{t_1} \cdots p_m^{t_m}$, where $t_j \leq k_j$, for $1 \leq j \leq m$. If P_j is a Sylow p_j-subgroup of H, then P_j has a subnormal series whose factors have order p, and it follows that P_j has at most t_j generators. Then H has at most $t_1 + \cdots + t_m$ generators. Since

$$\begin{aligned} t_1 + \cdots + t_m &= \log_{p_1}(p_1^{t_1}) + \cdots + \log_{p_m}(p_m^{t_m}) \leq \log_{p_1}(p_1^{k_1}) + \cdots + \log_{p_m}(p_m^{k_m}) \\ &\leq \log_2(p_1^{k_1}) + \cdots + \log_2(p_m^{k_m}) = \log_2 n, \end{aligned}$$

it follows that G has finite special rank at most $\log_2|G|$.

In the remainder of this section, we will assume throughout that F is a field of prime characteristic p. The next result is presumably well known.

Lemma 2.2. *Let G be a q-subgroup of $GL_n(F)$ for some prime $q \neq p$ and some positive integer n. Then G is almost Abelian and there exists a function κ_1 such that the special rank of G is at most $\kappa_1(n)$.*

Proof. Since $q \neq p$, G contains an Abelian normal subgroup H such that $|G/H| \leq \beta(n)$ for some function β [19, Corollary 9.4]. As we have seen above, G/H has special rank at most $\log_2\beta(n)$. By [5, Lemma 2.9], H has special rank at most n. Hence G has special rank at most $n+\log_2\beta(n) := \kappa_1(n)$. □

If α is a real number, then let $\iota(\alpha)$ denotes the greatest integer that is at most α.

Corollary 2.3. *Let G be a periodic subgroup of $GL_n(F)$ for some positive integer n and suppose that $sr_p(G) = r$ is finite. Then G is almost Abelian and there exists a function κ_2 such that the special rank of G is at most $\kappa_2(r, n)$.*

Proof. We note that G is locally finite by [19, Corollary 4.9]. Let P be a Sylow p-subgroup of G. The finiteness of $sr_p(P)$ yields that P has finite special rank r, by [1, Corollary 2.3], so P is Chernikov, by [13, Section 64]. On the other hand, P is a nilpotent group of finite exponent [19, 9.1]. It follows that P is finite, and hence G is almost Abelian, by [19, Corollary 9.7]. If $q \in \Pi(G)$ and $q \neq p$, then we may apply Lemma 2.2 to deduce that every Sylow q-subgroup of G has special rank at most $\kappa_1(n)$.

Now let H be an arbitrary finite subgroup of G. If $q \in \Pi(H)$ and $q \neq p$, then the Sylow q-subgroups of H have at most $\kappa_1(n)$ generators, whereas the Sylow p-subgroups of H have at most r generators. Hence H has at most $\max\{r, \iota(\kappa_1(n))\} + 1$ generators, by [14, Theorem 1] and it follows that G has special rank at most

$$\max\{r, \iota(\kappa_1(n))\} + 1 \leq r + \kappa_1(n) = r + n + \log_2\beta(n) := \kappa_2(r, n),$$

as required. □

We next require information concerning the subgroups G of $GL_n(F)$ in the case when $sr_p(G$ is finite. Such information is readily obtained using [1].

Corollary 2.4. *Let G be a subgroup of $GL_n(F)$ for some positive integer n and suppose that $sr_p(G) = r$ is finite. Then G has a finite series of normal subgroups*

$$T \leq L \leq V$$

such that

(i) *T is a periodic almost Abelian subgroup of finite special rank at most $\kappa_2(r, n)$, with finite Sylow p-subgroups;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *V/L is a free Abelian group;*
(iv) *G/V is finite.*

Moreover, $r(T) \leq \kappa_2(r, n)$, $sr_0(L/T) \leq r$, $sr_0(V/L) \leq r$, and there exists a function κ such that $|G/V| \leq \kappa(r)$. In particular, there exists a function κ_3 such that G has finite special rank at most $\kappa_3(r, n)$.

Proof. By Lemma 2.1, G has no non-Abelian free subgroups. It follows that G has a soluble normal subgroup S such that G/S is locally finite, by [19, Corollary 10.17], and we can apply [1, Theorem 2.15] directly to G. Then G has a series of normal subgroups

$$T \leq L \leq V$$

such that T is locally finite, L/T is torsion-free nilpotent, V/L is free Abelian and G/V is finite. Moreover, $sr_0(L/T) \leq r$, $sr_0(V/L) \leq r$, and there exists a function κ such that $|G/V| \leq \kappa(r)$. By Corollary 2.3, T is almost Abelian, the Sylow p-subgroups are finite, and there exists a function κ_2 such that the special rank of T is at most $\kappa_2(r, n)$, where

$$\kappa_2(r, n) = r + n + \log_2\beta(n).$$

It follows that G has finite special rank at most

$$r + n + \log_2\beta(n) + 2r + \log_2\beta(r) = 3r + n + \log_2(\beta(n)\kappa(r)) := \kappa_3(r,n),$$

as required. □

We now generalize Corollary 2.4 to infinite dimensional spaces as follows.

Lemma 2.5. *Let A be an FG-hypercentral vector space over F and let G be a subgroup of $GL(F,A)$. If $sr_p(G) = r$ is finite, then $\Pi(G) = \{p\}$ and G has a finite series of normal subgroups*

$$T \leq L \leq V$$

such that

(i) *T is a finite p-subgroup;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *V/L is a free Abelian group;*
(iv) *G/V is finite.*

Moreover, $sr_p(T) = r(T) \leq r$, $sr_0(L/T) \leq r$, $sr_0(V/L) \leq r$, and there exists a function κ such that $|G/V| \leq \kappa(r)$. In particular, G has finite special rank at most $3r + log_2\kappa(r)$.

Proof. Let

$$\langle 0\rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq Z_{\alpha+1} \leq \cdots \leq Z_\gamma = A$$

be the upper FG-central series of A. Since F has characteristic p, it follows that $G/C_G(Z_n)$ is a nilpotent p-group of exponent p^{n-1}, for every positive integer n. The section p-rank of G is finite, so $G/C_G(Z_n)$ must be finite. Moreover, $sr_p(G/C_G(Z_n)) = r(G/C_G(Z_n))$ by [1, Lemma 2.2]. Since $\cap_{n\geq 1}C_G(Z_n) = C_G(Z_\omega)$, Remak's theorem gives us the embedding

$$G_\omega := G/C_G(Z_\omega) \hookrightarrow \mathrm{Cr}_{n\geq 1}G/C_G(Z_n).$$

As we remarked above, $r(G/C_G(Z_n)) \leq r$ for each $n \geq 1$. By Lemma 2.1, G_ω contains no non-Abelian free subgroups, so it is locally almost soluble by [4, Theorem A]. It is not hard to see that the torsion elements of this group are p-elements. It follows that the maximal normal torsion subgroup $Tor(G_\omega) := T_\omega$ of G_ω is a p-subgroup. Being locally finite, it has finite special rank, by [1, Corollary 2.3]. Then T_ω is Chernikov [13, Section 64], and being residually finite, it is finite. Applying [1, Theorem 2.15], we deduce that G_ω has a finite series of normal subgroups $T_\omega \leq L_\omega \leq V_\omega$ such that T_ω is a finite p-group, L_ω/T_ω is a torsion-free nilpotent group, V_ω/L_ω is a free Abelian group, and G_ω/V_ω is finite. Moreover, $sr_p(T_\omega) = r(T_\omega) \leq r$, $sr_0(L_\omega/T_\omega) \leq r$, $sr_0(V_\omega/L_\omega) \leq r$, and there exists a function κ such that $|G_\omega/V_\omega| \leq \kappa(r)$. In particular, G_ω has finite special rank at most $3r + \log_2\kappa(r) := d$.

We now use transfinite induction. Let $G_\alpha := G/C_G(Z_\alpha), T_\alpha = Tor(G_\alpha)$ and suppose that we have already proved, for all ordinals $\alpha < \gamma$, that $\Pi(G_\alpha) = \{p\}$ and G_α is a locally almost soluble group of finite special rank at most d such that T_α is a finite p-subgroup of special rank at most r. If γ is a limit ordinal, then

$$\cap_{\alpha<\gamma}C_G(Z_\alpha) = C_G(Z_\gamma) = C_G(A) = \langle 1\rangle,$$

and then Remak's theorem gives us the embedding

$$G \hookrightarrow \mathrm{Cr}_{\alpha<\gamma}G_\alpha.$$

By transfinite induction, $r(G_\alpha) \leq d$ for each $\alpha < \gamma$. Since G contains no non-Abelian free subgroups, G is a locally almost soluble group by [4, Theorem A]. Since $\Pi(G_\alpha) = \{p\}$, it is not hard to prove that $\Pi(G) = \{p\}$. It follows that $Tor(G) := T$ is a p-subgroup and, reasoning as above, T is actually finite.

Again applying [1, Theorem 2.15], we deduce that G has a finite series of normal subgroups

$$T \leq L \leq V$$

such that T is a finite p-subgroup, L/T is a torsion-free nilpotent group, V/L is a free Abelian group, and G/V is finite. Moreover, $sr_p(T) = r(T) \leq r$, $sr_0(L/T) \leq r$, $sr_0(V/L) \leq r$, and there exists a function κ such that $|G/V| \leq \kappa(r)$. In particular, G has finite special rank at most $3r + \log_2\kappa(r)$.

Suppose now that $\gamma - 1$ exists. Each $x \in C_G(Z_{\alpha-1})$ acts trivially on the factors of the series

$$\{0\} \leq Z_{\gamma-1} \leq Z_\gamma = A.$$

Since the additive group of A is an elementary Abelian p-group, $C_G(Z_{\gamma-1})$ is also an elementary Abelian p-group. By the induction hypothesis, $T_{\gamma-1}$ is a finite p-group. If $Tor(G) := T$ then $C_G(Z_{\gamma-1}) \leq T$ and $T/C_G(Z_{\gamma-1}) = T_{\gamma-1}$ is a finite p-group. Hence T is a finite p-group and $r(T) \leq r$. Since $\Pi(G/C_G(Z_{\gamma-1})) = \{p\}$, we obtain that $\Pi(G) = \{p\}$. Finally, $G/T \cong G_{\gamma-1}/T_{\gamma-1}$ and the induction hypothesis applied to $G_{\gamma-1}$ gives the required result. □

Corollary 2.6. *Let A an F-vector space such that $dim_F A/\zeta_{FG,\infty}(A) = n$ is finite. Suppose that G is a subgroup of $GL(F, A)$ such that $sr_p(G) = r$ is finite. Then G has a finite series of normal subgroups*

$$T \leq L \leq V$$

such that

(i) *T is a periodic almost Abelian subgroup of finite special rank at most $\kappa_2(r, n)$ whose Sylow p-subgroups are finite;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *V/L is a free Abelian group;*
(iv) *G/V is finite.*

Moreover, $r(T) \leq \kappa_2(r, n)$, $sr_0(L/T) \leq r$, $sr_0(V/L) \leq r$, and there exists a function κ such that $|G/V| \leq \kappa(r)$. In particular, there exists a function κ_3 such that G has finite special rank at most $\kappa_3(r, n)$.

Proof. Put $Z = \zeta_{FG,\infty}(A)$. By Corollary 2.4, $H := G/C_G(A/Z)$ is almost soluble, $Tor(H)$ is almost Abelian with finite Sylow p-subgroups, and H has finite special rank. Moreover, every Sylow q-subgroup of $Tor(H)$, for every prime $q \neq p$ has special rank at most $\log_2\beta(n) + n$.

Let $C = C_G(A/Z)$. Then the FC-module A is FC-hypercentral. By Lemma 2.5, $Tor(C)$ is a finite p-subgroup and $C/Tor(C)$ is an almost soluble group of finite special rank. It follows that G is a generalized radical group, $T := Tor(G)$ is almost Abelian with finite Sylow p-subgroups and whose Sylow q-subgroups for primes $q \neq p$ have special rank at most $\log_2\beta(n) + n$. As in Corollary 2.3, we can prove that T has special rank at most $\kappa_2(r, n) = r + n + \log_2\beta(n)$.

Since the factor-group G/T is a generalized radical group of finite section p-rank, we can apply [1, Theorem 2.15] to this group. Proceeding as in the proof of Corollary 2.4, we obtain that G has finite special rank at most $\kappa_3(r, n)$, as required. □

Here and elsewhere we recall that a group G is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite.

Proof of Theorem A. Put $Z = \zeta_{FG,\infty}(A)$. By Corollary 2.6, G has finite special rank and moreover there exists a function $\kappa_3(r, d)$ such that $r(G) \leq \kappa_3(r, d)$. Now we apply Proposition 1.10 to deduce that L has finite dimension at most $\kappa_4(r, d) = d(\kappa_3(r, d) + 1)$, and A/L is locally FG-nilpotent.

Since the upper hypercenter of A/L contains $(Z+L)/L$, $(A/L)/\zeta_{FG,\infty}(A/L)$ has finite dimension. Since A/L is locally FG-nilpotent, $(A/L)/\zeta_{FG,\infty}(A/L)$ is FG-nilpotent. Hence A/L is FG-hypercentral. □

3. The characteristic zero case

In this section, our field F will have characteristic 0. Our result analogous to Lemma 2.2 is as follows.

Lemma 3.1. *Let G be a periodic subgroup of $GL_n(F)$. Then G is almost Abelian and there exists a function κ_5 such that the special rank of G is at most $\kappa_5(n)$.*

Proof. The group G has an Abelian normal subgroup H such that $|G/H| \leq \beta(n)$ for some function β, by [19, Corollary 9.4]. [5, Lemma 2.9] implies that every Abelian subgroup of G has special rank at most $k \leq n$. Thus G has finite rank, but we will obtain a different bound for the rank as follows. Let $p \in \Pi(G)$ and let P be an arbitrary finite p-subgroup of G. We choose a maximal Abelian normal subgroup C of P. Certainly P is nilpotent, and therefore $C = C_P(C)$. The factor-group $P/C_P(C)$ is known to be isomorphic to a p-subgroup of some $GL_k(\mathbb{Z}/p^m\mathbb{Z})$. We recall that a Sylow p-subgroup of the latter has special rank at most $\frac{1}{2}(5k-1)k$ [17, Lemma 7.44]. It follows that $P/C_P(C)$ has at most $\frac{1}{2}(5k-1)k$ generators. Hence the subgroup P has at most

$$\frac{1}{2}(5k-1)k + k = \frac{1}{2}(5k+1)k \leq \frac{1}{2}(5n+1)n$$

generators.

Let H be an arbitrary finite subgroup of G. If $p \in \Pi(H)$, then by the above argument, the Sylow p-subgroups of H have at most $\frac{1}{2}(5n+1)n$ generators. Hence H has at most $\frac{1}{2}(5n+1)n+1$ generators by [14, Theorem 1]. It follows that G has special rank at most $\frac{1}{2}(5n+1)n+1 := \kappa_5(n)$. □

We state our next result which is a special case of [6, Theorem E].

Proposition 3.2. *Let G be a generalized radical group such that $Tor(G) = \langle 1 \rangle$. Suppose that $sr_0(G) = r$ is finite. Then G has finite special rank and contains normal subgroups $L \leq V$ where*
(i) *L is a torsion-free nilpotent group;*
(ii) *V/L is a free Abelian group;*
(iii) *G/V is finite.*
Moreover, there are functions $\kappa_6(r)$ and $\kappa_7(r)$ such that $r(G) \leq \kappa_7(r)$ and $|G/V| \leq \kappa_6(r)$.

Corollary 3.3. *Let G be a subgroup of $GL_n(F)$ for some positive integer n. Suppose that $sr_0(G) = r$ is finite. Then G contains a finite series of normal subgroups*

$$T \leq L \leq V$$

such that
(i) *T is a periodic almost Abelian subgroup of finite special rank at most $\kappa_5(n)$;*
(ii) *L/T is a torsion-free nilpotent group;*
(iii) *V/L is a free Abelian group;*
(iv) *G/V is finite.*
Moreover G/T has finite special rank at most $\kappa_7(r)$, G has finite special rank at most $\kappa_8(r,n) := \kappa_5(n) + \kappa_7(r)$ and G/V has order at most $\kappa_6(r)$.

Proof. By Lemma 2.1, G contains no non-Abelian free subgroups. It follows that G has a soluble normal subgroup S such that G/S is finite [19, Corollary 10.17] so G is a generalized radical group. Now it suffices to apply Lemma 3.1 to the subgroup $T = Tor(G)$ and Proposition 3.2 to the factor-group G/T to obtain the required result. □

We recall that a group is called *polyrational* if it has a finite series each of whose factors is isomorphic with a subgroup of $\mathbb{Q}$.

Lemma 3.4. *Let A be an FG-hypercentral vector space over F and let G be a subgroup of GL(F, A). If $sr_0(G) = r$ is finite, then G is polyrational.*

Proof. Let

$$\langle 0\rangle = Z_0 \le Z_1 \le \cdots \le Z_\alpha \le Z_{\alpha+1} \le \cdots Z_\gamma = A$$

be the upper FG-central series of A. Since $\mathrm{char}(F) = 0$, an easy transfinite induction shows that $G/C_G(Z_\alpha)$ is torsion-free for each ordinal $\alpha \le \gamma$. We next prove using transfinite induction that G has a polyrational series of length at most the maximum of $\kappa_7(r)$ and $\frac{1}{2}r(r+1)$. Denote this maximum value by $\delta(r)$.

We first note that by Kaloujnine's theorem [8], $G/C_G(Z_n)$ is a nilpotent group for each $n \in \mathbb{N}$. Using [18, Theorem 7], we obtain that $G/C_G(Z_n)$ has finite special rank at most $\frac{1}{2}r(r+1)$ and nilpotency class at most $2r$. Being a torsion-free nilpotent group of finite special rank, the factor-group $G/C_G(Z_n)$ is polyrational. By Zaitsev theorem [20], the length of this series is at most $\frac{1}{2}r(r+1)$. It follows that there exists a positive integer m such that $C_G(Z_m) = C_G(Z_{m+n})$ for every $n \ge 1$. Then $C_G(Z_m) = C_G(Z_\omega)$ and $G/C_G(Z_\omega)$ is polyrational of polyrational length at most $\frac{1}{2}r(r+1)$.

Suppose inductively that $G/C_G(Z_\alpha)$ has polyrational length at most $\delta(r)$ for ordinals $\alpha < \beta$ and consider $G/C_G(Z_\beta)$. If $\beta - 1$ exists, then $G/C_G(Z_{\beta-1})$ has polyrational length at most $\delta(r)$. Also $C_G(Z_{\beta-1})/C_G(Z_\beta)$ is a torsion-free Abelian group, so $G/C_G(Z_\beta)$ has a polyrational series, and by Proposition 3.2, it follows that $G/C_G(Z_\beta)$ has finite special rank at most $\kappa_7(r)$. By Zaitsev's theorem [20], $G/C_G(Z_\beta)$ has a polyrational series of length at most $\delta(r)$.

Suppose next that β is a limit ordinal. Choose the least ordinal $\lambda < \beta$ such that $G/C_G(Z_\lambda)$ has maximal polyrational length $\delta(r)$. Then the polyrational length of $G/C_G(Z_\rho)$ is also $\delta(r)$ for $\lambda \le \rho < \beta$. However, $C_G(Z_\lambda)/C_G(Z_\rho)$ is torsion-free, so we must have $C_G(Z_\lambda) = C_G(Z_\rho)$, for all such ρ. Hence $C_G(Z_\lambda) = C_G(Z_\beta)$ and it now follows that $G/C_G(Z_\beta)$ has a polyrational series of length at most $\delta(r)$.

Setting $\beta = \gamma$ and noting that $C_G(Z_\gamma) = 1$, the result follows that G is polyrational of polyrational length at most $\delta(r)$. □

Corollary 3.5. *Let A be an F-vector space such that $dim_F A/\zeta_{FG,\infty}(A) = n$ is finite. Suppose that G is a subgroup of GL(F, A) such that $sr_0(G) = r$ is finite. Then $T = Tor(G)$ is a periodic almost Abelian subgroup of finite special rank at most $\kappa_5(n)$ and G/T is an almost soluble group of finite special rank at most $\kappa_7(r)$. In particular, G has finite special rank at most $\kappa_8(r, n) = \kappa_5(n) + \kappa_7(r)$.*

Proof. Put $Z = \zeta_{FG,\infty}(A)$. By Corollary 3.3, the factor-group $H := G/C_G(A/Z)$ is almost soluble and $Tor(H)$ is an almost Abelian subgroup of special rank at most $\kappa_5(n)$. Let $C = C_G(A/Z)$ so that the FC-module A is FC-hypercentral. By Lemma 3.4, C is a torsion-free soluble group. In particular, $T \cap C = \langle 1\rangle$, so

$$T = T/(T \cap C) \cong TC/C$$

is a subgroup of $Tor(H)$. It follows that T is an almost Abelian subgroup, of special rank at most $\kappa_5(n)$. Since G is almost soluble, G/T has finite special rank at most $\kappa_7(r)$, by Proposition 3.2. The result follows. □

We can now complete the proof of Theorem B.

Proof. Proof of Theorem B Put $Z = \zeta_{FG,\infty}(A)$. By Corollary 3.5, G has finite special rank and moreover there exists a function $\kappa_8(r, d)$ such that $r(G) \le \kappa_8(r, d)$. By Proposition 1.10, L has finite dimension at most $\kappa_9(r, d) := d(\kappa_8(r, d) + 1)$, and A/L is locally FG-nilpotent.

Since the upper hypercenter of A/L contains $(Z+L)/L$, $(A/L)/\zeta_{FG,\infty}(A/L)$ has finite dimension. Since A/L is locally FG-nilpotent, $(A/L)/\zeta_{FG,\infty}(A/L)$ is FG-nilpotent. Hence A/L is FG-hypercentral. □

4. An example

In this final section, we give an example to illustrate the limitations of our work. Our example illustrates that the rank conditions included in the hypotheses of our main theorems are necessary.

Let F be a field and let B be a vector space over F of F-dimension 3. Let $\{b_1, b_2, b_3\}$ be a basis of B. Let β be the automorphism of B defined (and extended linearly) by

$$\beta(b_1) = b_2 + b_3, \beta(b_2) = -b_1 - b_2, \beta(b_3) = b_3.$$

Then $C_B(\beta) = Fb_3$ and it is easy to see that B/Fb_3 is an irreducible $F\langle\beta\rangle$-module.

Let A be an infinite dimensional vector space over F with basis $\{a_n | n \in \mathbb{N}\}$ and let α be the automorphism of A defined (and extended linearly) by

$$\alpha(a_1) = a_1, \alpha(a_{n+1}) = a_{n+1} + a_n, \text{ for all } n \in \mathbb{N}.$$

It is clear that A is an $F\langle\alpha\rangle$-hypercentral module and that

$$Fa_1 \leq Fa_1 + Fa_2 \leq \cdots \leq Fa_1 + Fa_2 + \cdots Fa_n \leq \cdots$$

is the upper $F\langle\alpha\rangle$-hypercentral series of A. Let $C = A \oplus B$ and define an automorphism δ of C by

$$\delta(a, b) = (\alpha(a), \beta(b)), \text{ for all } (a, b) \in A \oplus B.$$

Let $D = C/F(b_3 - a_1)$. Then D has a basis $\{d_n | n \in \mathbb{N}\}$ and an automorphism γ such that

$$\gamma(d_1) = d_2 + d_3, \gamma(d_2) = -d_1 - d_2, \gamma(d_3) = d_3,$$
$$\gamma(d_{n+1}) = d_{n+1} + d_n, \text{ for all } n \geq 3.$$

Then γ has infinite order, the upper $F\langle\gamma\rangle$-hypercenter Z of D coincides with the subspace generated by $\{d_n | n \geq 3\}$ and D/Z is an irreducible $F\langle\gamma\rangle$-module. Furthermore, the $F\langle\gamma\rangle$-submodule generated by d_1 or d_2 coincides with $Fd_1 + Fd_2 + Fd_3$.

Let D_k, for $k \in \mathbb{N}$, be a vector space with basis $\{d_{k,n} | n \in \mathbb{N}\}$ and let γ_k be the automorphism of D_k defined by

$$\gamma_k(d_{k,1}) = d_{k,2} + d_{k,3}, \gamma_k(d_{k,2}) = -d_{k,1} - d_{k,2}, \gamma_k(d_{k,3}) = d_{k,3}$$
$$\gamma_k(d_{k,n+1}) = d_{k,n+1} + d_{k,n}, \text{ for all } n \geq 3.$$

Let Z_k denotes the $F\langle\gamma_k\rangle$-hypercenter of D_k, so that Z_k coincides with the subspace generated by $\{d_{k,n} | n \geq 3\}$.

Let $W = \underset{k\in\mathbb{N}}{\mathrm{Cr}}\, D_k$ denotes the Cartesian produce of the groups D_k and let $Y = \overset{r}{\underset{k=1}{\mathrm{Dr}}}\, \mathbb{N}/D_k$ denotes the corresponding direct product of these D_k. Then $G = \underset{k\in\mathbb{N}}{\mathrm{Dr}}\, \langle\gamma_k\rangle$ is a group of F-automorphisms of W. Clearly G is free Abelian of infinite 0-rank. Let $u_1 = (d_{k,1})_{k\in\mathbb{N}}$ and $u_2 = (d_{k,2})_{k\in\mathbb{N}}$. Consider the subspace $V = Y \oplus (Fu_1 + Fu_2)$. Clearly V is a G-invariant subspace of W, whose upper FG-hypercenter coincides with Y and V/Y is an irreducible FG-module of dimension 2. However, the locally FG-nilpotent residual of V coincides with $(\oplus_{k\in\mathbb{N}} Fd_{k,3}) \oplus (Fu_1 + Fu_2)$, which is of infinite dimension.

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