

GROWTH ORDERS AND ERGODICITY FOR ABSOLUTELY CESÀRO BOUNDED OPERATORS

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ABSTRACT. In this paper, we extend the concept of absolutely Cesàro boundedness to the fractional case. We construct a weighted shift operator belonging to this class of operators, and we prove that if T is an absolutely Cesàro bounded operator of order α with $0 < \alpha \leq 1$, then $\|T^n\| = o(n^\alpha)$, generalizing the result obtained for $\alpha = 1$. Moreover, if $\alpha > 1$, then $\|T^n\| = O(n)$. We apply such results to get stability properties for the Cesàro means of bounded operators.

1. INTRODUCTION

Let X be a complex Banach space and $T \in \mathcal{B}(X)$, the Banach algebra of all bounded linear operators defined on X . We denote by \mathcal{T} the discrete semigroup given by the natural powers of the operator T , that is, $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$. Recall ([1, 2, 9, 10, 15, 22]) that the *Cesàro sum* of order $\alpha \geq 0$ of T is the family of operators $(\Delta^{-\alpha}\mathcal{T}(n))_{n \in \mathbb{N}_0} \subset \mathcal{B}(X)$ given by

$$\Delta^{-\alpha}\mathcal{T}(n)x := (k^\alpha * \mathcal{T})(n)x = \sum_{j=0}^n k^\alpha(n-j)T^j x, \quad x \in X, n \in \mathbb{N}_0,$$

and the *Cesàro mean* of order $\alpha \geq 0$ of T is the family of operators $(M_T^\alpha(n))_{n \in \mathbb{N}_0}$ given by

$$M_T^\alpha(n)x := \frac{1}{k^{\alpha+1}(n)} \Delta^{-\alpha}\mathcal{T}(n)x, \quad x \in X, n \in \mathbb{N}_0,$$

where $(k^\alpha(n))_{n \in \mathbb{N}_0}$ is the *Cesàro kernel* of order α . It is given by

$$k^\alpha(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \quad n \in \mathbb{N}_0, \alpha > 0,$$

where Γ is the Gamma function, and $k^0(n) = \delta_{n,0}$ for $n \in \mathbb{N}_0$, where $\delta_{n,j}$ is the Kronecker delta, i.e., $\delta_{n,j} = 1$ if $j = n$ and 0 otherwise. We attach some of its properties that we will use along the paper. It satisfies that $k^\alpha * k^\beta = k^{\alpha+\beta}$, for $\alpha, \beta > 0$. Asymptotically, the behaviour of k^α is

$$(1) \quad k^\alpha(n) := \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(1/n)),$$

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and k^α is increasing (as a function of n) for $\alpha > 1$, decreasing for $0 < \alpha < 1$, and $k^1(n) = 1$ for $n \in \mathbb{N}_0$. Moreover, the Gautschi inequality implies

$$(2) \quad \frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} \leq k^\alpha(n) \leq \frac{n^{\alpha-1}}{\Gamma(\alpha)}, \quad n \in \mathbb{N}, \quad 0 < \alpha \leq 1.$$

For more details see [6, 16, 26].

Recall that a bounded linear operator T on X is called (C, α) -Cesàro bounded if the Cesàro mean of order α of T , $(M_T^\alpha(n))_{n \in \mathbb{N}_0}$, is uniformly bounded. The particular cases $\alpha = 0$ and $\alpha = 1$ are well-known, the power boundedness and Cesàro boundedness respectively. It is known that (C, α) -Cesàro boundedness implies (C, β) -Cesàro boundedness for $\beta > \alpha \geq 0$, however the converse is not true in general: the Assani matrix is Cesàro bounded but it is not power bounded. In the case that $(M_T^\alpha(n))_{n \in \mathbb{N}_0}$ converges in the strong topology of X , we say that the operator T is (C, α) -mean ergodic (for $\alpha = 1$ it said to be mean ergodic). Properties, characterization thorough functional calculus and ergodic results for (C, α) -Cesàro bounded operators can be found in [2, 3, 7, 8, 9, 10, 15] and references therein.

Also, there are connections between some boundedness of the Cesàro means and the resolvent of T . We recall that the condition

$$\|M_{\lambda T}^2(n)\| \leq C, \quad \text{for } |\lambda| = 1 \text{ and } n \in \mathbb{N}_0,$$

is equivalent to the *Kreiss bounded* condition

$$\|(\lambda - T)^{-1}\| \leq \frac{C}{(|\lambda| - 1)}, \quad \text{for all } |\lambda| > 1,$$

and

$$\|M_{\lambda T}(n)\| \leq C, \quad \text{for } |\lambda| = 1 \text{ and } n \in \mathbb{N}_0,$$

is equivalent to the *Uniformly Kreiss bounded* condition

$$\left\| \sum_{k=0}^n \lambda^{-k-1} T^k \right\| \leq \frac{C}{(|\lambda| - 1)}, \quad \text{for all } |\lambda| > 1, \quad n \in \mathbb{N}_0,$$

where C is a positive constant. There exist Kreiss bounded operators which are not Cesàro bounded, and reciprocally, Cesàro boundedness does not imply Kreiss boundedness, see [22]. Also, there are Kreiss bounded operators that they are not uniformly Kreiss bounded, see [17]. Moreover, in finite dimensional Hilbert spaces, Kreiss bounded and power bounded are equivalent classes of operators, see [17].

Growth orders for the Cesàro means of any order (especially for the natural powers) of (C, α) -Cesàro bounded or Kreiss bounded operators have been studied in a large amount of papers ([1, 9, 15, 19, 22] and references therein). In the following remark, we include some interesting particular cases.

Remark 1.1. Given $T \in X$, $\sigma(T) \cap \partial\mathbb{D}$ denotes the spectrum of T on the unit circle and $m(\sigma(T) \cap \partial\mathbb{D})$ its Lebesgue measure. From [22], [19] and [5], it follows that if one of next conditions is satisfied:

- i) T is Cesàro bounded and $\sigma(T) \cap \partial\mathbb{D} \subset \{1\}$.
- ii) T is Kreiss bounded and $m(\sigma(T) \cap \partial\mathbb{D}) = 0$.

iii) T is uniformly Kreiss bounded and X is a Hilbert space.

Then $\|T^n\| = o(n)$.

Furthermore, a similar result to the part i) is true for $\alpha \geq 1$, that is, if T is (C, α) -Cesàro bounded with $\alpha \geq 1$ and $\sigma(T) \cap \partial\mathbb{D} \subset \{1\}$, then $\|T^n\| = o(n^\alpha)$ ([1, Theorem 4.3]). Also, if we only consider that T is Cesàro bounded or Kreiss bounded, then $\|T^n\| = O(n)$ ([20, 21]).

Recently in [12], the authors introduced a new concept about the growth of the Cesàro means. An operator $T \in \mathcal{B}(X)$ is called *absolutely Cesàro bounded*, if there exists $C > 0$ such that

$$\sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^n \|T^j x\| \leq C \|x\|, \quad x \in X.$$

The starting point of the above concept is related to the distributional chaos theory. In [5], some examples of absolutely Cesàro bounded are given, as well as its connection with power bounded and Kreiss bounded properties. Also they prove that if T is an absolutely Cesàro bounded operator, then $\|T^n\| = o(n)$. In this paper we introduce an extension of the absolutely Cesàro boundedness, and our main aim is to construct simple examples, to study the asymptotic behaviour of the orbits, to compare with other mentioned concepts, and to show ergodic results.

Definition 1.1. Let $\alpha > 0$. We say that a bounded linear operator T on X is *absolutely (C, α) -Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}_0} \frac{1}{k^{\alpha+1}(n)} \sum_{j=0}^n k^\alpha(n-j) \|T^j x\| \leq C \|x\|,$$

for all $x \in X$.

For $\alpha = 1$, the above definition is the absolutely Cesàro boundedness. In order to clarify to the reader, we show the following sketch:

$$\begin{aligned} \text{Power bounded} &\Rightarrow \text{Absolutely } (C, \alpha)\text{-Cesàro bounded} \\ &\Rightarrow (C, \alpha)\text{-Cesàro bounded} \Rightarrow \|T^n\| = O(n^\alpha) \end{aligned}$$

Particularly, it follows that all absolutely Cesàro bounded operators are uniformly Kreiss bounded, and that every absolutely $(C, 2)$ -Cesàro bounded operator is Kreiss bounded.

The outline of this paper is as follows: In Section 2, for $0 < \alpha \leq 1$, we construct a class of weighted backward shift operators $(T_\beta)_{\beta > 0}$ on $\ell^p(\mathbb{N})$, for $1 \leq p < \infty$, which are mixing absolutely (C, α) -Cesàro bounded with $\|T^n\| = (n+1)^\beta$ for $0 < \beta < \alpha/p$ (Theorem 2.1 and Corollary 2.1), and not (C, α) -Cesàro bounded if $\beta \geq 1/p$ (Remark 2.1). Moreover, we prove that all absolutely (C, α) -Cesàro bounded operators in a Banach space satisfy that $\|T^n\| = o(n^\alpha)$ with $0 < \alpha \leq 1$ (Corollary 2.2), generalizing the result obtained in [5] for $\alpha = 1$, and $\|T^n\| = O(n)$ for $\alpha > 1$ (Corollary 2.3). In Hilbert spaces, for $0 < \alpha \leq 1$, if T is an absolutely (C, α) -Cesàro bounded operator then $\|T^n\| = o(n^{\min\{\alpha, 1/2\}})$ (Corollary 2.5). In Section 3, we study some ergodic applications. We prove that the absolutely (C, α) -Cesàro boundedness in a

reflexive Banach space implies (C, α) -mean ergodicity (Corollary 3.1). Also, we study stability results for the Cesàro means of fractional order of a bounded linear operator T , assuming spectral and growth conditions on T (Theorem 3.1 and Corollary 3.3).

2. EXAMPLES AND GROWTH ORDERS OF ABSOLUTELY CESÀRO BOUNDED OPERATORS

We denote by $\{e_n\}_{n \in \mathbb{N}}$ the standard canonical basis on $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$, that is, $e_n = (\delta_{n,j})_{j \in \mathbb{N}} := (0, \dots, 0, \underbrace{1}_n, 0, \dots)$, where $\ell^p(\mathbb{N})$ is the Lebesgue space of

complex sequences $x = \sum_{j=1}^{\infty} \alpha_j e_j$, with $\|x\|_p := \left(\sum_{j=1}^{\infty} |\alpha_j|^p \right)^{\frac{1}{p}} < \infty$.

In the literature, there are only simple examples of (C, α) -Cesàro bounded operators (no power bounded) for $\alpha \in \mathbb{N}$. In the following, we construct a class of (C, α) -Cesàro bounded backward shift operators on $\ell^p(\mathbb{N})$, with $0 < \alpha < 1$, which are not power bounded.

Theorem 2.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $0 < \beta < \frac{\alpha}{p}$. The unilateral weighted backward shift operator T , defined by $Te_1 := 0$ and $Te_j := w_j e_{j-1}$ for $j > 1$, with $w_j := \left(\frac{j}{j-1} \right)^\beta$, is absolutely (C, α) -Cesàro bounded on $\ell^p(\mathbb{N})$.*

Proof. Denote $\varepsilon := \alpha - \beta p > 0$. Let $x \in \ell^p(\mathbb{N})$ whose Fourier representation is $x := \sum_{j=1}^{\infty} \alpha_j e_j$. We assume without loss of generality that $\|x\|_p = 1$. For $N \in \mathbb{N}$ we have

$$\begin{aligned}
 \sum_{n=0}^N k^\alpha(N-n) \|T^n x\|_p^p &= \sum_{n=0}^N k^\alpha(N-n) \sum_{j=n+1}^{\infty} |\alpha_j|^p \left(\frac{j}{j-n} \right)^{\alpha-\varepsilon} \\
 &= \sum_{j=1}^{\infty} |\alpha_j|^p \sum_{n=0}^{\min\{N, j-1\}} k^\alpha(N-n) \left(\frac{j}{j-n} \right)^{\alpha-\varepsilon} \\
 (3) \quad &= \sum_{j=1}^N |\alpha_j|^p j^{\alpha-\varepsilon} \sum_{n=0}^{j-1} k^\alpha(N-n) (j-n)^{\varepsilon-\alpha} \\
 (4) \quad &+ \sum_{j=N+1}^{2N} |\alpha_j|^p j^{\alpha-\varepsilon} \sum_{n=0}^N k^\alpha(N-n) (j-n)^{\varepsilon-\alpha} \\
 (5) \quad &+ \sum_{j=2N+1}^{\infty} |\alpha_j|^p \sum_{n=0}^N k^\alpha(N-n) \left(\frac{j}{j-n} \right)^{\alpha-\varepsilon}.
 \end{aligned}$$

In what follows, we estimate each of the above summands.

First, notice that for $j \geq 2N+1$ and $n \leq N$, one gets

$$\left(\frac{j}{j-n} \right)^{\alpha-\varepsilon} \leq 2^{\alpha-\varepsilon} < 2^\alpha.$$

Hence

$$(5) < 2^\alpha k^{\alpha+1}(N) \sum_{j=2N+1}^{\infty} |\alpha_j|^p \leq 2^\alpha k^{\alpha+1}(N).$$

Secondly, taking $j \leq N$ to estimate the summand (3), we obtain

$$\begin{aligned} \sum_{n=0}^{j-1} k^\alpha(N-n)(j-n)^{\varepsilon-\alpha} &= \sum_{n=1}^j k^\alpha(N-j+n)n^{\varepsilon-\alpha} \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{n=1}^j (N-j+n)^{\alpha-1} n^{\varepsilon-\alpha} \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{n=1}^j n^{\varepsilon-1} < \frac{1}{\Gamma(\alpha)} \left(1 + \int_1^j x^{\varepsilon-1} dx\right) \\ &= \frac{1}{\Gamma(\alpha)} \frac{j^\varepsilon}{\varepsilon}, \end{aligned}$$

where we have used (2).

Now, in order to estimate (4) we take $j > N$, and we have

$$\begin{aligned} \sum_{n=0}^N k^\alpha(N-n)(j-n)^{\varepsilon-\alpha} &= \sum_{n=0}^N k^\alpha(n)(j-N+n)^{\varepsilon-\alpha} \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{n=1}^N n^{\alpha-1} (j-N+n)^{\varepsilon-\alpha} + 1 \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{j-1} n^{\varepsilon-1} + 1 < \frac{1}{\Gamma(\alpha)} \left(1 + \int_1^{j-1} x^{\varepsilon-1} dx\right) + 1 \\ &= \frac{1}{\Gamma(\alpha)} \frac{(j-1)^\varepsilon}{\varepsilon} + 1 < \frac{1}{\Gamma(\alpha)} \frac{j^\varepsilon}{\varepsilon} + 1. \end{aligned}$$

Using the previous estimates we get

$$\begin{aligned} \sum_{n=0}^N k^\alpha(N-n) \|T^n x\|_p^p &\leq \sum_{j=1}^N |\alpha_j|^p j^{\alpha-\varepsilon} \frac{1}{\Gamma(\alpha)} \frac{j^\varepsilon}{\varepsilon} \\ &\quad + \sum_{j=N}^{2N} |\alpha_j|^p j^{\alpha-\varepsilon} \left(1 + \frac{j^\varepsilon}{\Gamma(\alpha)\varepsilon}\right) + 2^\alpha k^{\alpha+1}(N) \\ &= \sum_{j=1}^{2N} |\alpha_j|^p \left(1 + \frac{1}{\Gamma(\alpha)\varepsilon}\right) j^\alpha + 2^\alpha k^{\alpha+1}(N) \\ &\leq \left(1 + \frac{1}{\Gamma(\alpha)\varepsilon}\right) (2N)^\alpha \sum_{j=1}^{2N} |\alpha_j|^p + 2^\alpha k^{\alpha+1}(N) \\ &\leq C(\alpha, \varepsilon) k^{\alpha+1}(N), \end{aligned}$$

and by Jensen's inequality

$$\left(\frac{1}{k^{\alpha+1}(N)} \sum_{n=0}^N k^{\alpha}(N-n) \|T^n x\|_p \right)^p \leq \frac{1}{k^{\alpha+1}(N)} \sum_{n=0}^N k^{\alpha}(N-n) \|T^n x\|_p^p \leq C(\alpha, \varepsilon).$$

□

Recall that a bounded linear operator T on X is *topologically mixing* if for any pair U, V of non-empty open subsets of X , there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. Therefore, we deduce the following result.

Corollary 2.1. *Let $1 \leq p < \infty$ and $0 < \varepsilon < \alpha \leq 1$. There exists an absolutely (C, α) -Cesàro bounded operator T on $\ell^p(\mathbb{N})$ such that it is mixing and $\|T^n\| = (n+1)^{\frac{\alpha-\varepsilon}{p}}$.*

Proof. By Theorem 2.1 we have that the unilateral weighted backward shift operator on $\ell^p(\mathbb{N})$ is absolutely (C, α) -Cesàro bounded and

$$(6) \quad \|T^n\| = (n+1)^{\frac{\alpha-\varepsilon}{p}}.$$

Moreover, by [11, Theorem 4.8] we have that T is mixing since $(\prod_{k=1}^n w_k)^{-1} \rightarrow 0$ as $n \rightarrow \infty$. □

The unilateral weighted backward shift operator on $\ell^p(\mathbb{N})$, with $\beta = \frac{1}{p}$, given in Theorem 2.1, is not Cesàro bounded, and so it is not (C, α) -Cesàro bounded for $0 < \alpha < 1$, see [5]. It is natural to ask if this operator is (C, α) -Cesàro bounded for some $\alpha > 1$. The answer is negative as the following proposition shows.

Proposition 2.1. *The unilateral weighted backward shift operator T on $\ell^p(\mathbb{N})$ with $1 \leq p < \infty$, given by $Te_1 := 0$ and $Te_j := w_j e_{j-1}$ for $j > 1$, with $w_j := \left(\frac{j}{j-1}\right)^{1/p}$, is not (C, α) -Cesàro bounded for any α .*

Proof. By the above comment it is enough to prove the result for $\alpha > 1$. Let N an

even natural number. We define $y_{N+1} := \frac{1}{(N+1)^{1/p}} \sum_{l=0}^{N+1} e_l$. Then

$$\begin{aligned}
& \left\| \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^{\alpha}(N-j) T^j y_{N+1} \right\|_p^p \\
&= \left\| \frac{1}{k^{\alpha+1}(N)(N+1)^{1/p}} \sum_{l=1}^{N+1} \left(\sum_{j=l}^{N+1} k^{\alpha}(N+l-j)(j/l)^{1/p} \right) e_l \right\|_p^p \\
&= \frac{1}{(k^{\alpha+1}(N))^p (N+1)} \sum_{l=1}^{N+1} \frac{1}{l} \left(\sum_{j=l}^{N+1} k^{\alpha}(N+l-j) j^{1/p} \right)^p \\
&\geq \frac{1}{(k^{\alpha+1}(N))^p (N+1)} \sum_{l=1}^{N/2+1} \frac{1}{l} \left(\sum_{j=N/2+1}^{N+1} k^{\alpha}(N+l-j) j^{1/p} \right)^p \\
&\geq \frac{1}{(k^{\alpha+1}(N))^p (N+1)} \sum_{l=1}^{N/2+1} \frac{1}{l} \left(\sum_{j=N/2+1}^{N+1} k^{\alpha}(N+1-j) j^{1/p} \right)^p \\
&= \frac{1}{(k^{\alpha+1}(N))^p (N+1)} \sum_{l=1}^{N/2+1} \frac{1}{l} \left(\sum_{j=0}^{N/2} k^{\alpha}(N/2-j)(j+N/2+1)^{1/p} \right)^p \\
&\geq \frac{1}{(k^{\alpha+1}(N))^p (N+1)} \sum_{l=1}^{N/2+1} \frac{1}{l} \left(\sum_{j=0}^{N/2} k^{\alpha}(N/2-j)(N/2)^{1/p} \right)^p \\
&\geq C \sum_{l=1}^{N/2+1} \frac{1}{l} \geq C \ln(N/2+1),
\end{aligned}$$

with C a positive constant, where we have used that k^α is increasing as function of n for $\alpha > 1$, identity (1) and $k^\alpha * k^1 = k^{\alpha+1}$. So, we conclude that the Cesàro mean of order α of T is not uniformly bounded. \square

Remark 2.1. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\beta > 0$ and T be the unilateral weighted backward shift operator on $\ell^p(\mathbb{N})$ defined in Theorem 2.1. Then the operator T is absolutely (C, α) -Cesàro bounded if $\beta < \alpha/p$ (Theorem 2.1) and not (C, α) -Cesàro bounded if $\beta \geq 1/p$ (Proposition 2.1).

From now to the end of the section we focus on studying the growth of the natural powers of absolutely (C, α) -Cesàro bounded operators.

Theorem 2.2. Let $0 < \alpha \leq 1$, X be a Banach space and $T \in B(X)$ satisfying $\|T^n\| = O(n^\alpha)$. Then either $\|T^n\| = o(n^\alpha)$ or the set

$$\left\{ x \in X : \sup_N \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^{\alpha}(N-j) \|T^j x\| = \infty \right\}$$

is residual in X .

Proof. Suppose that $\frac{\|T^n\|}{n^\alpha} \not\rightarrow 0$ as $n \rightarrow \infty$. So, there exists a positive constant c such that

$$\limsup_{n \rightarrow \infty} n^{-\alpha} \|T^n\| > c.$$

Also, by hypothesis, there is $C > 0$ such that $\|T^n\| \leq Cn^\alpha$ for all $n \in \mathbb{N}$.

For $s \in \mathbb{N}$ we denote

$$M_s = \left\{ x \in X : \sup_N \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j x\| > s \right\}.$$

It is clear that M_s is an open set.

First, we show that M_s contains a unit vector for $s \in \mathbb{N}$. Note that there exists $N \in \mathbb{N}$ enough large such that $\frac{cN^\alpha}{k^{\alpha+1}(N)} \left(\frac{1}{C\Gamma(\alpha)2^{1-\alpha}} \ln N + 1 \right) > s$ and $\|T^N\| > cN^\alpha$. Therefore we can take a unit vector $x \in X$ such that $\|T^N x\| > cN^\alpha$. Using that $\|T^N x\| \leq \|T^j\| \cdot \|T^{N-j} x\|$ for $j = 1, \dots, N-1$, one gets

$$\|T^{N-j} x\| \geq \frac{\|T^N x\|}{\|T^j\|} \geq \frac{cN^\alpha}{Cj^\alpha}.$$

Thus,

$$\begin{aligned} \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j x\| &\geq \frac{1}{k^{\alpha+1}(N)} \left(\sum_{j=0}^{N-1} k^\alpha(N-j) \frac{cN^\alpha}{C(N-j)^\alpha} + cN^\alpha \right) \\ &\geq \frac{cN^\alpha}{k^{\alpha+1}(N)} \left(\frac{1}{C} \sum_{j=1}^N k^\alpha(j) \frac{1}{j^\alpha} + 1 \right) \\ &\geq \frac{cN^\alpha}{k^{\alpha+1}(N)} \left(\frac{1}{C\Gamma(\alpha)2^{1-\alpha}} \sum_{j=1}^N \frac{1}{j} + 1 \right) \\ &\geq \frac{cN^\alpha}{k^{\alpha+1}(N)} \left(\frac{1}{C\Gamma(\alpha)2^{1-\alpha}} \ln N + 1 \right) > s, \end{aligned}$$

and so $x \in M_s$.

Now, we prove that M_s is dense for each $s \in \mathbb{N}$. Indeed, let $y \in X$ and $\varepsilon > 0$. Let $s' > \frac{s}{\varepsilon}$, and we take $x \in M_{s'}$ with $\|x\| = 1$. For each $j \in \mathbb{N}$ we have

$$\|T^j(y + \varepsilon x)\| + \|T^j(y - \varepsilon x)\| \geq 2\varepsilon \|T^j x\|,$$

and then

$$\begin{aligned} \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j(y + \varepsilon x)\| &+ \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j(y - \varepsilon x)\| \\ &\geq \frac{2\varepsilon}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j x\| \\ &> 2\varepsilon s' > 2s. \end{aligned}$$

Hence either $y + \varepsilon x \in M_s$ or $y - \varepsilon x \in M_s$. Since $\varepsilon > 0$ was arbitrary, we conclude that M_s is dense.

The Baire category theorem implies that

$$\bigcap_{s=1}^{\infty} M_s = \left\{ x \in X : \sup_N \frac{1}{k^{\alpha+1}(N)} \sum_{j=0}^N k^\alpha(N-j) \|T^j x\| = \infty \right\}$$

is a residual set. □

Corollary 2.2. *Let $0 < \alpha \leq 1$ and T be an absolutely (C, α) -Cesàro bounded operator. Then $\|T^n\| = o(n^\alpha)$.*

Proof. Note that by the absolutely (C, α) -Cesàro boundedness of T , we have

$$\|T^n x\| \leq \sum_{j=0}^n k^\alpha(n-j) \|T^j x\| \leq C k^{\alpha+1}(n) \|x\| \leq C n^\alpha \|x\|, \quad C > 0.$$

By Theorem 2.2, we conclude the result, since the second possibility in Theorem 2.2 contradicts the absolutely (C, α) -Cesàro boundedness of T . \square

Theorem 2.3. *Let T be an absolutely (C, α) -Cesàro bounded operator with $\alpha > 1$. Then T is Kreiss bounded.*

Proof. For each $\beta > \alpha$ the operator T is an absolutely (C, β) -Cesàro bounded operator. Then, by [23, Corollary 2.3] we get the result. \square

The following corollary is a straightforward consequence of Theorem 2.3 and [21, p.344] (or [20, Proposition 2]).

Corollary 2.3. *Let T be an absolutely (C, α) -Cesàro bounded operator with $\alpha > 1$. Then $\|T^n\| = O(n)$*

Also, by Theorem 2.1 we have the next result.

Corollary 2.4. *Let T be an absolutely (C, α) -Cesàro bounded operator with $\alpha > 1$. Then $\|T^n\| = o(n)$ or T is not absolutely Cesàro bounded.*

Remark 2.2. For any $0 < \beta < 1$, there exists an operator on $\ell^1(\mathbb{N})$ which is absolutely (C, α) -Cesàro bounded for $\alpha > \beta$ (Theorem 2.1) and no absolutely (C, γ) -Cesàro bounded for $0 < \gamma \leq \beta$ (Corollary 2.1, Corollary 2.2). It would be interesting to find an example of an absolutely (C, α) -Cesàro bounded operator with $\alpha > 1$ such that it is not absolutely Cesàro bounded.

In Hilbert spaces, it is known that if there exists $\varepsilon > 0$ such that $T \in \mathcal{B}(X)$ satisfies $\|T^n\| \geq C n^{\frac{1}{2}+\varepsilon}$ for all n , then there exists $x \in X$ such that $\|T^n x\| \rightarrow \infty$, see by [18, Theorem 3]. Therefore T is not absolutely (C, α) -Cesàro bounded for any $\alpha > 0$.

As consequence of Corollary 2.2 and [5, Theorem 2.4], we have

Corollary 2.5. *Let H be a Hilbert space and T be an absolutely (C, α) -Cesàro bounded operator with $0 < \alpha \leq 1$. Then $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n^{\min\{\alpha, 1/2\}}} = 0$*

3. ERGODIC APPLICATIONS FOR THE CESÀRO MEANS

In this section we study the ergodic behaviour of the Cesàro means supposing several assumptions, both the geometry of the Banach space X , and spectral and growth conditions on the operator T .

In the first results, the reflexive property plays role. As consequence of Corollary 2.2, Corollary 2.3 and [10, Proposition 3.2], we obtain the following result.

Corollary 3.1. *Let $\alpha > 0$. Every absolutely (C, α) -Cesàro bounded operator in a reflexive Banach space is (C, α) -mean ergodic.*

Hence, by Corollary 2.1, we have the next corollary.

Corollary 3.2. *Let $\alpha > 0$. There exists a mixing (C, α) -mean ergodic operator on $\ell^p(\mathbb{N})$ for $1 < p < \infty$.*

Results according to previous ones are shown in [3, 4, 5] for $\alpha = 1$.

From now on, we focus on stability results for the Cesàro mean differences of size n and $n + 1$ for bounded operators. In 1986, Y. Katznelson and L. Tzafriri proved that if $T \in \mathcal{B}(X)$ is power bounded, then $\lim_{n \rightarrow \infty} \|T^n - T^{n+1}\| = 0$ if and only if $\sigma(T) \cap \partial\mathbb{D} \subset \{1\}$, see [13, Theorem 1]. If T is $(C, 1)$ -Cesàro bounded and $\sigma(T) \cap \partial\mathbb{D} = \{1\}$, but T is not power bounded, then $\|T^n - T^{n+1}\|$ need not converge to zero. See examples in [14, 24]. Also, the result is not true if the condition $(C, 1)$ -Cesàro bounded is changed by Kreiss bounded, see [19, Example 4]. Recently, in [1], the author proved that if T is (C, α) -Cesàro bounded, with $\alpha > 0$, and $\sigma(T) \cap \partial\mathbb{D} \subset \{1\}$, then $\lim_{n \rightarrow \infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0$.

We recall the following lemma which will be used in the proof of the main theorem of this section.

Lemma 3.1. ([25, Lemma 1]) *Let $\alpha > 0$ and $T \in \mathcal{B}(X)$ such that $\|T^n\| = o(n^\omega)$, where $w = \min(1, \alpha)$. Then $\lim_{n \rightarrow \infty} \|M_T^\alpha(n)(T - I)\| = 0$.*

Theorem 3.1. *If any of the following conditions is true:*

- a) *T is an absolutely (C, α) -Cesàro bounded operator and $0 < \alpha \leq 1$,*
- b) *T is Kreiss bounded, $1 \leq \alpha < 2$ and $m(\sigma(T) \cap \partial\mathbb{D}) = 0$,*
- c) *T is Kreiss bounded and $\alpha \geq 2$,*

then

$$\lim_{n \rightarrow \infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0.$$

Proof. a) If $\alpha = 1$, from the known identity

$$\frac{n+2}{n+1}M_T(n+1) - M_T(n) = \frac{1}{n+1}T^{n+1},$$

one gets

$$M_T(n+1) - M_T(n) = \frac{1}{n+1}T^{n+1} - \frac{1}{n+1}M_T(n+1).$$

The result follows from [5, Corollary 2.6] and the $(C, 1)$ -Cesàro boundedness of T .

If $0 < \alpha < 1$, note that

$$M_T^\alpha(n+1) - M_T^\alpha(n) = \frac{\alpha}{n+\alpha+1}I + \frac{n+1}{n+\alpha+1}M_T^\alpha(n)(T-I) + \frac{\alpha}{n+\alpha+1}M_T^\alpha(n),$$

see [1, p.79]. By Corollary 2.2 and Lemma 3.1 we have

$$\lim_{n \rightarrow \infty} \|M_T^\alpha(n)(T-I)\| = 0,$$

and therefore $\lim_{n \rightarrow \infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0$.

b) Note that by Remark 1.1 ii) we have that $\|T^n\| = o(n)$. Then by Lemma 3.1, one gets $\lim_{n \rightarrow \infty} \|M_T^\alpha(n)(T-I)\| = 0$ for $1 \leq \alpha < 2$. Using that the Kreiss condition

implies $\|M_T(n)\| = O(\ln(n))$ ([21, Theorem 6.2]), we have that $\lim_{n \rightarrow \infty} \|\frac{1}{n}M_T^\alpha(n)\| \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{0 \leq j \leq n} \|M_T(j)\| = 0$. By the above comments and the following identity ([1, p.78])

$$M_T^\alpha(n+1) - M_T^\alpha(n) = M_T^\alpha(n)(T - I) + \frac{\alpha}{n+1}(I - M_T^\alpha(n+1)),$$

we conclude the result.

c) By [21, Theorem 6.2], since $\alpha \geq 2$ we have

$$\lim_{n \rightarrow \infty} \|\frac{1}{n}M_T^{\alpha-1}(n)\| \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{0 \leq j \leq n} \|M_T(j)\| = 0.$$

Doing use of the identity ([1, p.78])

$$M_T^\alpha(n+1) - M_T^\alpha(n) = \frac{\alpha}{n+1}M_T^{\alpha-1}(n+1) - \frac{\alpha}{n+1}M_T^\alpha(n+1),$$

we get

$$\lim_{n \rightarrow \infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0.$$

□

Since all absolutely (C, α) -Cesàro bounded operators are Kreiss bounded, we get the following corollary.

Corollary 3.3. *Let T be an absolutely (C, α) -Cesàro bounded operator. If any of the following conditions is true:*

- a) $1 < \alpha < 2$ and $m(\sigma(T) \cap \partial\mathbb{D}) = 0$,
- b) $\alpha \geq 2$,

then

$$\lim_{n \rightarrow \infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0.$$

We deduce from Theorem 3.1 a) that if $0 < \alpha \leq 1$ and T is an absolutely (C, α) -Cesàro bounded operator, then $\|M_T^\alpha(n)(T - I)\|$ converges to zero as $n \rightarrow \infty$. So, a natural question is if the following extended version of the Katznelson-Tzafriri theorem is true:

Question 3.1. Is there α with $0 < \alpha \leq 1$ such that if T is an absolutely (C, α) -Cesàro bounded operator with $\sigma(T) \cap \partial\mathbb{D} = \{1\}$, $\|T^n(T - I)\|$ converges to zero as $n \rightarrow \infty$?

On the other hand, assuming a less restrictive condition ($1 < \alpha < 2$), we obtain the next ergodic result.

Theorem 3.2. *Let T be an absolutely (C, α) -Cesàro bounded operator with $1 < \alpha < 2$. Then $\lim_{n \rightarrow \infty} \|M_T^\alpha(n)(T - I)^2\| = 0$.*

Proof. If $1 < \alpha < 2$, we have by Corollary 2.3 that $\|T^n\| = o(n^\alpha)$, then following the ideas in [9, p.9 and 10] one gets

$$\lim_{n \rightarrow \infty} \|M_T^\alpha(n)(T - I)^2\| = 0.$$

□

To finish, we comment the relevance of the conditions for Theorem 3.1 in the Kreiss bounded case.

Remark 3.1. The condition $m(\sigma(T) \cap \partial\mathbb{D}) = 0$ in Theorem 3.1 b) is not redundant. Indeed, in the proof of [19, Theorem 6], the author constructs a bounded linear operator A on a Banach space of analytic functions in the unit disc such that A is Kreiss bounded and $\sigma(A) \cap \partial\mathbb{D}$ has positive measure. Moreover, it satisfies that

$$(7) \quad \|A^n\| \geq 1 + \frac{n}{2}m(\sigma(A) \cap \partial\mathbb{D}).$$

Then the identity

$$M_A(n+1) - M_A(n) = \frac{1}{n+1}A^{n+1} - \frac{1}{n+1}M_A(n+1),$$

identity (7) and [21, Theorem 6.2] imply

$$\lim_{n \rightarrow \infty} \|M_A(n+1) - M_A(n)\| \neq 0.$$

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