Interplay between $SU(N_f)$ chiral symmetry, $U(1)_A$ axial anomaly, and massless bosons

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The standard wisdom on the origin of massless bosons in the spectrum of a quantum field theory (QFT) describing the interaction of gauge fields coupled to matter fields is based on two well-known features: gauge symmetry and spontaneous symmetry breaking of continuous global symmetries. However, we will show in this article how the topological properties, which originate the $U(1)_A$ axial anomaly in a QFT that describes the interaction of fermion matter fields and gauge bosons, are the basis of an alternative mechanism to generate massless bosons in the chiral limit, if the non-Abelian $SU(N_f)_A$ chiral symmetry is fulfilled in the vacuum. We will also test our predictions with the results of a well-known two-dimensional model, the two-flavor Schwinger model, which was analyzed by Coleman long ago, and will give a reliable answer to some of the questions he asked himself on the spectrum of the model in the strong-coupling (chiral) limit. We will also analyze what the expectations for the $U(N)$ gauge-fermion model in two dimensions are and will discuss the impact of our results in the chirally symmetric high-temperature phase of QCD, which was present in the early Universe and is expected to be created in heavy-ion collision experiments. To keep mathematical rigor, we perform our calculations using a lattice regularization and Ginsparg-Wilson fermions.

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I. INTRODUCTION

There are two well-known mechanisms in quantum field theory that allow us to understand the existence of massless bosons in the spectrum of a given model of gauge fields coupled to matter fields: gauge symmetry and spontaneous symmetry breaking of continuous global symmetries. The gauge symmetry is, for instance, responsible for the photon not having mass. On the other hand, the spontaneous breaking of the $SU(2)_A$ chiral symmetry in QCD allows us to understand, via the Nambu-Goldstone theorem, why pions are so light; indeed, they would be massless if the up and down quark masses vanished.

However, there are also some well-known examples, for instance, two-flavor quantum electrodynamics in $(1+1)$ dimensions, in which chiral quasimassless bosons appear in the spectrum of the model near the chiral limit [1] and in which the explanation of this phenomenon escapes the two aforementioned mechanisms to generate massless bosons. Hence, it is worth wondering if this happens because of some uninteresting peculiarities of two-dimensional models or if there is a deeper and general explanation for this phenomenon.

We want to show here how the topological properties of quantum field theories, which describe the interaction of fermion matter fields and gauge bosons and which exhibit $U(1)_A$ axial anomaly, can be the basis of an alternative mechanism to generate massless bosons in the chiral limit. More precisely, we will show, with the help of three distinct argumentation lines, that a gauge-fermion quantum field theory, with $U(1)_A$ axial anomaly and in which the chiral condensate vanishes in the chiral limit, typically because of an exact non-Abelian chiral symmetry, should exhibit a divergent correlation length in the correlation function of the scalar condensate, in the chiral limit. The nonanomalous Ward-Takahashi identities will tell us then that, in such a case, also some pseudoscalar correlation functions should exhibit a divergent correlation length, associated to what would be the Nambu-Goldstone bosons if the non-Abelian chiral symmetry were spontaneously broken.

We will also test our predictions with the results of a well-known two-dimensional model, the aforementioned two-flavor Schwinger model [1], and will discuss what expectations are for the $U(N)$ gauge-fermion model in two dimensions, the spectrum of which was analyzed some time ago in the large N limit [2] and later on by means of bosonization techniques [3,4].

Some of the basic ideas here developed can be found in Refs. [5] and [6]. However, and in order to make this article self-contained, we will expose all of them in a detailed way.

Since the $Q = 0$ topological sector will play a main role in our physical discussions, we will devote Sec. II to reviewing some results concerning the relation between vacuum expectation values of local and nonlocal operators computed in the $Q = 0$ topological sector, with their corresponding values in the full theory, taking into account the contribution of all topological sectors, and will see how, notwithstanding that the $Q = 0$ sector breaks spontaneously the $U(1)_A$ axial symmetry and shows a divergent pseudoscalar susceptibility in the chiral limit, the associated pseudoscalar correlation length remains finite in this sector, and the Nambu-Goldstone theorem is not fulfilled. To this end, we will analyze in this section the one-flavor model as well as the $N_f > 1$-flavor theory, the latter in the case in which the non-Abelian axial symmetry is spontaneously broken, as happens in the low-temperature phase of QCD. In Sec. III, we will analyze what the physical expectations are for the $N_f > 1$ model when the non-Abelian $SU(N_f)_A$ chiral symmetry is fulfilled in the vacuum and will show, with the help of three distinct argumentation lines, how a theory that verifies the aforementioned properties should exhibit, in the chiral limit, a divergent correlation length and a rich spectrum of massless chiral bosons. Section IV is devoted to testing the main prediction of this paper with well-known results of the two-flavor Schwinger model. In Sec. V, we analyze our expectations for the $U(N)$ gauge-fermion model in two dimensions, and in the last section, we report our conclusions and discuss the possible implications of our results in the high-temperature chiral symmetry restored phase of QCD.

II. SOME RELEVANT FEATURES OF THE $Q=0$ TOPOLOGICAL SECTOR

In this article, we are interested in the analysis of some physical phenomena induced by the topological properties of a fermion-gauge theory with $U(1)_A$ axial anomaly. In this analysis, the $Q = 0$ topological sector will play an essential role, and this is the reason why we devote this section to reviewing some results concerning the relation between vacuum expectation values of local and nonlocal operators computed in the $Q = 0$ sector, with their corresponding values in the full theory, in which we take into account the contribution of all topological sectors. In particular, we will recall that the vacuum expectation value of local or intensive operators computed in the $Q = 0$ topological sector is equal, in the infinite volume limit, to their corresponding value in the full theory. While this property is, in general, not true for nonlocal operators, we will see later that there are exceptions. We will also show how, even if the aforementioned property implies that the $U(1)_A$ symmetry is spontaneously broken in the $Q = 0$ topological sector, the Goldstone theorem is not realized because the divergence of the pseudoscalar susceptibility does not come from a divergent correlation length [5].

To begin, let us write the continuum Euclidean Lagrangian for the most popular gauge-fermion system with $U(1)_A$ axial anomaly, QCD in four space-time dimensions. We want to remark, however, that all the results reported in this paper apply to any gauge-fermion system with $U(1)_A$ anomaly, and, indeed, in Secs. IV and V, we will analyze the Schwinger model (two-dimensional QED) and the $U(N)$ model in two dimensions. The one-flavor QCD Euclidean action in the presence of a θ-vacuum term reads as

$$S = \int \mathrm{d}^4x \bigg\{ \bar{\psi}(x)(\gamma_\mu D_\mu(x) + m)\psi(x) + \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) + i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \bigg\}, \tag{1}$$

where $D_\mu(x)$ is the covariant derivative and

$$Q = \frac{g^2}{64\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \tag{2}$$

is the topological charge of the gauge configuration, which is an integer number.

To give mathematical rigor to all developments throughout this paper, we will avoid ultraviolet divergences with the help of a lattice regularization. We will also assume Ginsparg-Wilson (G-W) fermions [7], the overlap fermions [8,9] being an explicit realization of them. G-W fermions share with the continuum formulation all essential ingredients. Indeed, G-W fermions show an explicit $U(1)_A$ anomalous symmetry [10], good chiral properties, and a quantized topological charge and allow us to establish and exact index theorem on the lattice [11]. We recall here a few essential features of Ginsparg-Wilson fermions that will be useful for understanding the rest of the paper.

The lattice fermionic action for a massless G-W fermion can be written in a compact form as

$$S_F = \bar{\psi} D \psi, \tag{3}$$

where D, the Dirac-Ginsparg-Wilson operator, obeys the essential anticommutation equation

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D, \tag{4}$$

where a is the lattice spacing, and thus the right-hand side of (4) vanishes in the naive continuum limit, $a \to 0$.

It can be easily shown that action (3) is invariant under the following lattice $U(1)_A$ chiral rotation:

$$\psi \to e^{i\alpha\gamma_5(I-aD)}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}. \tag{5}$$

However, the integration measure of Grassmann variables is not invariant, and the change of variables (5) induces a Jacobian,

$$e^{-i2\alpha\frac{a}{2}\mathrm{tr}(\gamma_5 D)}, \tag{6}$$

where

$$\frac{a}{2}\mathrm{tr}(\gamma_5 D) = n_- - n_+ = Q \tag{7}$$

is an integer number, the difference between left-handed and right-handed zero modes, which can be identified with the topological charge Q of the gauge configuration. Thus, Eqs. (6) and (7) show us how Ginsparg-Wilson fermions reproduce the $U(1)_A$ axial anomaly.

We can also add a symmetry-breaking mass term, $m\bar{\psi}(1-\frac{a}{2}D)\psi$, to action (3), so G-W fermions with mass are described by the fermion action

$$S_F = \bar{\psi}D\psi + m\bar{\psi}\left(1-\frac{a}{2}D\right)\psi, \tag{8}$$

and it can also be shown that the scalar and pseudoscalar condensates

$$S = \bar{\psi}\left(1-\frac{a}{2}D\right)\psi \qquad P = i\bar{\psi}\gamma_5\left(1-\frac{a}{2}D\right)\psi \tag{9}$$

transform, under the chiral $U(1)_A$ rotations (5), as a vector, just in the same way as $\bar{\psi}\psi$ and $i\bar{\psi}\gamma_5\psi$ do in the continuum formulation.

The partition function of the N_f-flavor model in a finite lattice is the sum over all topological sectors, Q, of the partition function in each topological sector times a θ-phase factor,

$$Z = \sum_Q Z_Q e^{i\theta Q}, \tag{10}$$

where Q, which takes integer values, is bounded at finite volume by the number of degrees of freedom. At large lattice volume, the partition function should behave as

$$Z(\beta, m_f, \theta) = e^{-VE(\beta, m_f, \theta)}, \tag{11}$$

where $E(\beta, m_f, \theta)$ is the free energy density, β is the inverse gauge coupling, m_f is the f-flavor mass, and $V = V_s \times L_t$ is the lattice volume in units of the lattice spacing. Moreover, the partition function and the mean value of any local or intensive operator O, as, for instance, the scalar and pseudoscalar condensates, or any correlation function, in the $Q = 0$ topological sector, can be computed, respectively, as

$$Z_{Q=0} = \frac{1}{2\pi}\int \mathrm{d}\theta Z(\beta, m_f, \theta) \tag{12}$$

$$\langle O\rangle^{Q=0} = \frac{\int \mathrm{d}\theta \langle O\rangle_\theta Z(\beta, m_f, \theta)}{\int \mathrm{d}\theta Z(\beta, m_f, \theta)}, \tag{13}$$

where $\langle O\rangle_\theta$, which is the mean value of O computed with the lattice regularized integration measure (1), is a function of the inverse gauge coupling β, flavor masses m_f, and θ, and it takes a finite value in the infinite lattice volume limit. Then, since the free energy density, as a function of θ, has its absolute minimum at $\theta = 0$ for nonvanishing fermion masses, the following relations hold in the infinite volume limit,

$$E_{Q=0}(\beta, m_f) = E(\beta, m_f, \theta)_{\theta=0} \tag{14}$$

$$\langle O\rangle^{Q=0} = \langle O\rangle_{\theta=0}, \tag{15}$$

where $E_{Q=0}(\beta, m_f)$ is the vacuum energy density of the $Q = 0$ topological sector.[1] As we will show below, Eq. (15) is in general not true if O is a nonlocal operator, while there are exceptions to this rule.

We will devote the rest of this section to showing that Eq. (15) is consistent with the $U(1)_A$ axial anomaly. To this end, let us start with the analysis of the one-flavor model at zero temperature.

In the one-flavor model, the only axial symmetry is an anomalous $U(1)_A$ symmetry. The standard wisdom on the vacuum structure of this model in the chiral limit is that it is unique at each given value of θ, the θ vacuum. Indeed, the only plausible reason to have a degenerate vacuum in the chiral limit would be the spontaneous breakdown of chiral symmetry, but since it is anomalous, actually there is no symmetry. Furthermore, due to the chiral anomaly, the model shows a mass gap in the chiral limit, and therefore all correlation lengths are finite in physical units. Since the model is free from infrared divergences, the vacuum energy density can be expanded in powers of the fermion mass m_u, treating the quark mass term as a perturbation [12]. This expansion will be then an ordinary Taylor series,

$$E(\beta, m_u, \theta) = E_0(\beta) - \Sigma(\beta)m_u\cos\theta + O(m_u^2), \tag{16}$$

giving rise to the expansions for the scalar and pseudoscalar condensates

$$\langle S_u\rangle = -\Sigma(\beta)\cos\theta + O(m_u) \tag{17}$$

$$\langle P_u\rangle = -\Sigma(\beta)\sin\theta + O(m_u), \tag{18}$$

where S_u and P_u are the scalar and pseudoscalar condensates (9) normalized by the lattice volume. The topological

[1] We want to notice that, as we will see later, Eq. (15) is, in general, wrong if some fermion mass vanishes.

susceptibility χ_T is given, on the other hand, by the following expansion:

$$\chi_T = \Sigma(\beta) m_u \cos\theta + O(m_u^2). \tag{19}$$

The resolution of the $U(1)_A$ problem is obvious if we set down the Ward-Takahashi identity, which relates the pseudoscalar susceptibility $\chi_\eta = \sum_x \langle P_u(x) P_u(0)\rangle$, the scalar condensate $\langle S_u\rangle$, and the topological susceptibility χ_T,

$$\chi_\eta = -\frac{\langle S_u\rangle}{m_u} - \frac{\chi_T}{m_u^2}. \tag{20}$$

Indeed, the divergence in the chiral limit of the first term in the right-hand side of (20) is canceled by the divergence of the second term in this equation, giving rise to a finite pseudoscalar susceptibility and a finite nonvanishing mass for the pseudoscalar η boson.

Now, we can apply Eq. (13) to the computation of vacuum expectation values of local operators, as the two-point pseudoscalar correlation function, but before that, we want to notice two relevant features of the $Q = 0$ topological sector:

(1) In the $Q = 0$ sector, the integration measure is invariant under global $U(1)_A$ chiral transformations because the full topological charge vanishes for any gauge configuration. This means that the global $U(1)_A$ axial symmetry is not anomalous in this sector.

(2) If we apply Eq. (13) to the computation of the vacuum expectation value of the scalar condensate, which is an intensive operator, we get that the $U(1)_A$ symmetry is spontaneously broken in the $Q = 0$ sector because the chiral limit of the infinite volume limit of the scalar condensate, the limits taken in this order, does not vanish.

The two-point pseudoscalar correlation function $\langle P_u(x) P_u(0)\rangle$ is also an intensive operator, and Eq. (13) tell us that, in the infinite volume limit and for $m_u \neq 0$, we can write

$$\langle P_u(x) P_u(0)\rangle^{Q=0} = \langle P_u(x) P_u(0)\rangle_{\theta=0}. \tag{21}$$

This equation implies that the mass of the pseudoscalar boson, m_η, which can be extracted from the long-distance behavior of the two-point correlation function, computed in the $Q = 0$ sector, is equal to the value we should get in the full theory, taking into account the contribution of all topological sectors. On the other hand, the topological susceptibility, χ_T, vanishes in the $Q = 0$ sector, and hence the Ward-Takahashi identity (20)) in this sector reads as follows:

$$\chi_\eta^{Q=0} = -\frac{\langle S_u\rangle^{Q=0}}{m_u}. \tag{22}$$

This identity gives us an expected result: the pseudoscalar susceptibility in the $Q = 0$ sector diverges in the chiral limit $m_u \to 0$ because the $U(1)_A$ symmetry is spontaneously broken in this sector. Even if expected, this is, however, a very surprising result because it suggests that the pseudoscalar boson would be a Goldstone boson, and therefore its mass, m_η, would vanish in the limit $m_u \to 0$.

The loophole to this paradoxical result is that in systems with a global constraint the divergence of the susceptibility does not necessarily imply a divergent correlation length. The susceptibility is the infinite volume limit of the integral of the correlation function over all distances, in this order, and in systems with a global constraint, the infinite volume limit and the space integral of the correlation function do not necessarily commute. A very simple and illustrative example is the Ising model at infinite temperature with an even number of spins and vanishing full magnetization as a global constraint [5]. In such a case, one has for the spin-spin correlation function

$$\langle s_i^2\rangle = 1$$
$$\langle s_i s_j\rangle = -\frac{1}{V-1} \quad i \neq j.$$

The integral of the infinite volume limit of the correlation function is equal to 1, whereas the infinite volume limit of the integrated correlation function vanishes. The correlation function has a contribution of order $1/V$, which violates cluster at finite volume, and vanishes in the infinite volume limit, but that gives a finite contribution to the integrated correlation function. We will see in what follows how this is qualitatively what happens when computing the pseudoscalar correlation function in the $Q = 0$ sector.

The $\langle P_u(x) P_u(0)\rangle^{Q=0}$ correlation function at any finite space-time volume V verifies the equation

$$\langle P_u(x) P_u(0)\rangle^{Q=0} = \frac{\int d\theta \langle P_u(x) P_u(0)\rangle_\theta e^{-VE(\beta,m,\theta)}}{\int d\theta e^{-VE(\beta,m,\theta)}}, \tag{23}$$

and we are interested not only in the infinite volume limit of this expression but also in the $O(\frac{1}{V})$ corrections.

On the other hand, it is standard wisdom that QCD has no phase transition at $\theta = 0$, and hence we can expand the pseudoscalar correlation function in powers of the θ angle as

$$\langle P_u(x) P_u(0)\rangle_\theta = \langle P_u(x) P_u(0)\rangle_{\theta=0} + h(x, m_u)\theta^2 + O(\theta^4), \tag{24}$$

where

$$h(x, m_u) = \langle S_u(x) S_u(0)\rangle_{\theta=0} - \langle P_u(x) P_u(0)\rangle_{\theta=0} + O(m_u). \tag{25}$$

$O(m_u)$ in (25) stays to indicate terms that vanish at least linearly with m_u as $m_u \to 0$, in contrast with the first two terms in the right-hand side of (25) that take a nonvanishing value in the chiral limit.

The vacuum energy density can also be expanded in powers of θ as

$$E(\beta, m_u, \theta) = E_0(\beta, m_u) - \frac{1}{2}\chi_T(\beta, m_u)\theta^2 + O(\theta^4). \quad (26)$$

Taking into account Eqs. (23), (24), and(26) and making an expansion around the saddle-point solution, we can write the expansion in powers of $\frac{1}{V}$ of the pseudoscalar correlation function in the zero-charge topological sector,

$$\begin{aligned} &\langle P_u(x)P_u(0)\rangle^{Q=0} \\ &\quad = \langle P_u(x)P_u(0)\rangle_{\theta=0} \\ &\qquad + \frac{1}{V}\frac{\langle S_u(x)S_u(0)\rangle_{\theta=0} - \langle P_u(x)P_u(0)\rangle_{\theta=0} + O(m_u)}{\chi_T} \\ &\qquad + O\left(\frac{1}{V^2}\right), \end{aligned} \quad (27)$$

which shows, as in the simple Ising model case, a violation of cluster at finite volume for the pseudoscalar correlation function in the zero-charge topological sector, as follows from the fact that

$$\lim_{|x|\to\infty} \langle S_u(x)S_u(0)\rangle_{\theta=0} = \Sigma^2. \quad (28)$$

The cluster-violating term is of the order of $\frac{1}{V}$, and because the topological susceptibility $\chi_T = m_u\Sigma$ is linear in m_u, it is singular at $m_u = 0$. It is just this term that is responsible for the divergence of the pseudoscalar susceptibility in the $Q = 0$ sector in the chiral limit. However, in the infinite volume limit, the pseudoscalar correlation function in the zero-charge topological sector and in the full theory at $\theta = 0$ agree, as expected.

In what concerns the pseudoscalar susceptibility, Eqs. (23) and (24) allow us to relate this quantity in the $Q = 0$ sector and in the full theory as

$$\chi_\eta^{Q=0} = \chi_{\eta_{\theta=0}} + \frac{(\langle S_u\rangle_{\theta=0} - m_u\chi_{\eta_{\theta=0}})^2}{\chi_T}, \quad (29)$$

which shows explicitly how $\chi_\eta^{Q=0}$ diverges as $\frac{\Sigma}{m_u}$ when $m_u \to 0$.

Summarizing, we have shown that, even if the $Q = 0$ topological sector breaks spontaneously the $U(1)_A$ axial symmetry to give account of the anomaly, the Goldstone theorem is not fulfilled because the divergence of the pseudoscalar susceptibility does not come from a divergent correlation length but from some peculiar features of the pseudoscalar correlation function which can emerge in systems with global constraints.

The inclusion of more flavors does not change the qualitative results reported in this section when the $SU(N_f)$ chiral symmetry is spontaneously broken, as happens in the low-temperature phase of QCD. The quantitative changes are essentially reduced to replace the one-flavor scalar and pseudoscalar condensates by the flavor-singlet scalar and pseudoscalar condensates, respectively, and the topological susceptibility χ_T by $N_f^2\chi_T$ in Eqs. (20), (23)–(25), and (25)–(29).

The case in which the $SU(N_f)$ chiral symmetry is fulfilled in the vacuum will be discussed in detail in the next section.

III. TWO FLAVORS AND EXACT $SU(2)$ CHIRAL SYMMETRY

We will discuss in this section the physical expectations in a fermion-gauge theory with two (or more flavors), exact $SU(2)$ chiral symmetry, and a $U(1)_A$ axial anomaly. In this discussion, the main ideas developed in the previous section will play an essential role. We will see how a theory that verifies the aforementioned properties should show, in the chiral limit, a divergent correlation length and a rich spectrum of massless chiral bosons. In Sec. III A, we will give, under very general assumptions, a short demonstration of this result. Section III B contains a qualitative but powerful argument supporting the results of Sec. III A, and in Sec. III C, we will show how we can get the same qualitative result using general properties of the spectral density of the Lee-Yang zeros of the partition function of the zero charge topological sector.

A. Vacuum energy density of the $Q = 0$ topological sector

As previously stated, we consider a fermion-gauge model with two flavors, up and down, with masses m_u and m_d, exact $SU(2)_A$ chiral symmetry, and a $U(1)_A$ axial anomaly, for instance, the two-flavor Schwinger model or the high-temperature phase of QCD. We will assume that the flavor-singlet scalar susceptibility $\chi_\sigma(m_u, m_d)$ and hence also the flavor-singlet pseudoscalar susceptibility $\chi_\eta(m_u, m_d)$ take a finite value in the chiral limit and will show that, in such a case, we get a quite surprising result: the scalar $\chi_\sigma(m_u, m_d)$ and pseudoscalar $\chi_\eta(m_u, m_d)$ susceptibilities are equal in the chiral limit, in contrast with what we would expect in a theory with two flavors and a $U(1)_A$ anomaly.

To start the proof, let us write the Euclidean fermion-gauge action (8) for the two-flavor model,

$$\begin{aligned} S_F = &\, m_u\bar{\psi}_u\left(1 - \frac{a}{2}D\right)\psi_u + m_d\bar{\psi}_d\left(1 - \frac{a}{2}D\right)\psi_d \\ &+ \bar{\psi}_u D\psi_u + \bar{\psi}_d D\psi_d, \end{aligned} \quad (30)$$

where D is the Dirac-Ginsparg-Wilson operator. This action can be written in a compact form as

$$S_F = m_+\bar{\psi}\left(1 - \frac{a}{2}D\right)\psi - m_-\bar{\psi}\left(1 - \frac{a}{2}D\right)\tau_3\psi + \bar{\psi}D\psi, \tag{31}$$

where $m_+ = \frac{m_u+m_d}{2}$ and $m_- = \frac{m_d-m_u}{2}$. ψ is a Grassmann field carrying site, Dirac, color, and flavor indices and τ_3 is the third Pauli matrix acting in flavor space.

The vacuum energy density $E(m_+, m_-, \beta)$ of our model is a function of the quark masses, m_+, m_-, and the inverse gauge coupling β. Since we are assuming that the flavor-singlet scalar susceptibility χ_σ and hence χ_η are finite in the chiral limit, and because the pseudoscalar susceptibility χ_η is equal to the δ-meson susceptibility χ_δ in this limit due to the exact $SU(2)_A$ axial symmetry, we can write a second-order Taylor expansion for the free energy density as

$$E(m_+, m_-, \beta) = \frac{1}{2}m_+^2\chi_\sigma(\beta) + \frac{1}{2}m_-^2\chi_\eta(\beta) + E_2(m_+, m_-, \beta), \tag{32}$$

where $E_2(m_+, m_-, \beta)$ verifies that

$$\lim_{m_+,m_-\to 0} \frac{E_2(m_+, m_-, \beta)}{m_+^2 + m_-^2} = 0.$$

We have shown in Sec. II, Eq. (14), that the vacuum energy density of the $Q = 0$ topological sector is equal, in the thermodynamic limit, to the vacuum energy density in the full theory at $\theta = 0$. Hence, we can write

$$E_{Q=0}(m_+, m_-, \beta) = E(m_+, m_-, \beta). \tag{33}$$

We can perform, in the $Q = 0$ topological sector, an Abelian axial rotation of the up quark in the path integral, with angle $\theta = \pi$, while leaving the down quark unchanged. This variable change, the Jacobian of which is trivial in this sector, is equivalent to interchange m_+ and m_-, and so we get the following symmetry relation:

$$E_{Q=0}(m_+, m_-, \beta) = E_{Q=0}(m_-, m_+, \beta). \tag{34}$$

Equations (32), (33), and (34) can only be verified if $\chi_\sigma(\beta) = \chi_\eta(\beta)$, and this concludes the proof.

This result tells us that a finite value of the flavor-singlet scalar susceptibility in the chiral limit seems to be incompatible with the presence of the $U(1)_A$ axial anomaly in the two-flavor model. In the next subsection, we will give an argument pointing also to the divergence of the flavor-singlet scalar susceptibility in the chiral limit for any $N_f \geq 2$.

B. Phase diagram and the Landau approach

We have shown in Sec. III A that a fermion-gauge theory with $U(1)_A$ anomaly and exact $SU(2)$ chiral symmetry should exhibit a divergent correlation length in the scalar sector in the chiral limit. In this section, we want to give what is perhaps the strongest indication supporting this result, which comes from a qualitative but powerful argument. To this end, we will explore the expected phase diagram of the model in the $Q = 0$ topological sector [6] and will apply the Landau theory of phase transitions to it.

Since the $SU(2)$ chiral symmetry is assumed to be fulfilled in the vacuum and the flavor singlet scalar condensate is an order parameter for this symmetry, its vacuum expectation value $\langle S \rangle = 0$ vanishes in the limit in which the fermion mass $m \to 0$. However, if we consider two nondegenerate fermion flavors, up and down, with masses m_u and m_d, respectively, and take the limit $m_u \to 0$ keeping $m_d \neq 0$ fixed, the up condensate S_u will reach a nonvanishing value

$$\lim_{m_u\to 0}\langle S_u \rangle = s_u(m_d) \neq 0 \tag{35}$$

because the $U(1)_u$ axial symmetry, which exhibits our model when $m_u = 0$, is anomalous, and the $SU(2)$ chiral symmetry, which would enforce the up condensate to be zero, is explicitly broken if $m_d \neq 0$.

Obviously, the same argument applies if we interchange m_u and m_d, and we can therefore write an equation symmetric to (35) for the down condensate,

$$\lim_{m_d\to 0}\langle S_d \rangle = s_d(m_u) \neq 0, \tag{36}$$

and since when m_u, $m_d \to 0$ the $SU(2)$ chiral symmetry is recovered and fulfilled in the vacuum, we get

$$\lim_{m_d\to 0} s_u(m_d) = \lim_{m_u\to 0} s_d(m_u) = 0. \tag{37}$$

Let us consider now our model, with two nondegenerate fermion flavors, restricted to the $Q = 0$ topological sector. As discussed in Sec. II, the mean value of any local or intensive operator in the $Q = 0$ topological sector will be equal, if we restrict ourselves to the region in which both $m_u > 0$ and $m_d > 0$, to its mean value in the full theory in the infinite lattice volume limit.[2] We can hence apply this result to $\langle S_u \rangle$ and $\langle S_d \rangle$ and write the following equations:

$$\begin{aligned} &\lim_{m_u\to 0}\langle S_u \rangle^{Q=0} = s_u(m_d) \neq 0 \\ &\lim_{m_d\to 0}\langle S_d \rangle^{Q=0} = s_d(m_u) \neq 0. \end{aligned} \tag{38}$$

[2]Since the two-flavor model with $m_u < 0$ and $m_d < 0$ at $\theta = 0$ is equivalent to the same model with $m_u > 0$ and $m_d > 0$, this result is also true if both $m_u < 0$ and $m_d < 0$.

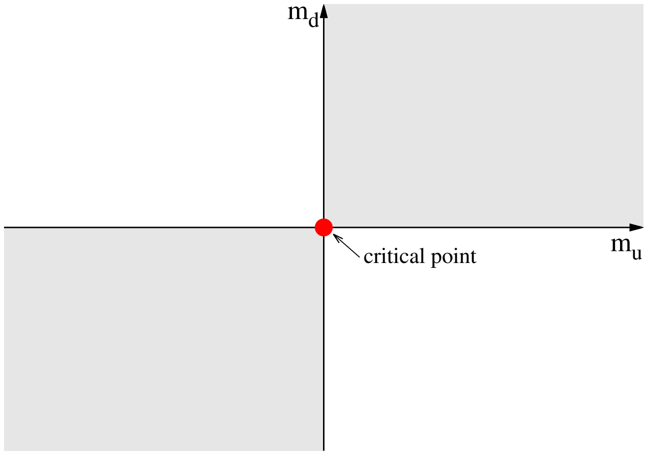

FIG. 1. Phase diagram of the two-flavor model in the $Q = 0$ topological sector. The coordinate axes in the (m_u, m_d) plane are first-order phase transition lines. The origin of coordinates is the end point of all first-order transition lines. The vacuum energy density, its derivatives, and expectation values of local operators of the two-flavor model at $\theta = 0$ only agree with those of the $Q = 0$ sector in the first $(m_u > 0, m_d > 0)$ and third $(m_u < 0, m_d < 0)$ quadrants (the darkened areas).

In the $Q = 0$ sector, the $U(1)_u$ axial symmetry of our model at $m_u = 0$ and the $U(1)_d$ symmetry at $m_d = 0$ are good symmetries of the action because the Jacobian associated to a chiral $U(1)_{u,d}$ transformation is the unit. Then, Eq. (38) tells us that both the $U(1)_u$ symmetry at $m_u = 0$, $m_d \neq 0$ and the $U(1)_d$ symmetry at $m_u \neq 0$, $m_d = 0$ are spontaneously broken. This is not surprising at all since the present situation is similar to what happens in the one-flavor model discussed in Sec. II, and, as in that case, the Goldstone theorem is not verified because the divergence of the pseudoscalar up or down susceptibilities does not come from a divergent correlation length.

Figure 1 is a schematic representation of the phase diagram for the two-flavor model in the $Q = 0$ topological sector and in the (m_u, m_d) plane, which emerges from the previous discussion. The two coordinate axes show first-order phase transition lines. If we perpendicularly cross the $m_d = 0$ axis, the mean value of the down condensate jumps from $s_d(m_u)$ to $-s_d(m_u)$, and the same is true if we interchange up and down. All first-order transition lines end, however, at a common point, the origin of coordinates $m_u = m_d = 0$, where all condensates vanish because at this point we recover the $SU(2)$ chiral symmetry, which is assumed to be also a symmetry of the vacuum. Notice that if the $SU(2)$ chiral symmetry is spontaneously broken, as happens, for instance, in the low-temperature phase of QCD, the phase diagram in the (m_u, m_d) plane would be the same as that of Fig. 1 with the only exception that the origin of coordinates is not an end point.

Landau's theory of phase transitions predicts that the end point placed at the origin of coordinates in the (m_u, m_d) plane is a critical point, the scalar condensate should show a nonanalytic dependence on the fermion masses m_u and m_d as we approach the critical point, and hence the scalar susceptibility should diverge. But since the vacuum energy density in the $Q = 0$ topological sector matches the vacuum energy density in the full theory, and therefore the same is true for the critical equation of state, Landau's theory of phase transitions predicts a nonanalytic dependence of the flavor-singlet scalar condensate on the fermion mass and a divergent correlation length in the chiral limit of our full theory, in which we take into account the contribution of all topological sectors.

More precisely, we can apply the Landau approach to analyze the critical behavior around the two first-order transition lines in Fig. 1 near the end point or critical point. In the analysis of the $m_d = 0$ transition line, we consider m_d as an external "magnetic field"and m_u as the "temperature," and vice versa for the analysis of the $m_u = 0$ line. Then, the standard Landau approach tells us that the up and down condensates verify the two equations of state

$$m_u \langle S_u \rangle^{-3} = -C_1 m_d \langle S_u \rangle^{-2} + C_2$$
$$m_d \langle S_d \rangle^{-3} = -C_1 m_u \langle S_d \rangle^{-2} + C_2, \tag{39}$$

where C_1 and C_2 are two positive constants. If we fix the ratio of the up and down masses $\frac{m_u}{m_d} = \lambda$, the equations of state (39) allow us to write the following expansions for the up and down condensates:

$$\langle S_u \rangle = m_u^{\frac{1}{3}} \left(\left(\frac{1}{4C_2} \right)^{\frac{1}{3}} + \frac{C_1}{3(2C_2^2)^{\frac{1}{3}} \lambda} m_u^{\frac{1}{3}} + \cdots \right)$$
$$\langle S_d \rangle = m_d^{\frac{1}{3}} \left(\left(\frac{1}{4C_2} \right)^{\frac{1}{3}} + \frac{C_1 \lambda}{3(2C_2^2)^{\frac{1}{3}}} m_d^{\frac{1}{3}} + \cdots \right). \tag{40}$$

Equation (40) explicitly shows the nonanalytical behavior of the up and down condensates. The flavor-singlet scalar condensate scales as $m_u^{\frac{1}{3}} + m_d^{\frac{1}{3}}$ near the critical point, and for the degenerate flavor case, $m_u = m_d = m$, we get

$$\langle S \rangle = \langle S_u \rangle + \langle S_d \rangle = \left(\frac{2}{C_2} \right)^{\frac{1}{3}} m^{\frac{1}{3}} + \cdots$$
$$\chi_\sigma(m) = \frac{1}{3} \left(\frac{2}{C_2} \right)^{\frac{1}{3}} m^{-\frac{2}{3}} + \cdots, \tag{41}$$

which explicitly shows the divergence of the flavor-singlet scalar susceptibility in the chiral limit.

The power law dependence of the scalar condensate (41) in the Landau approach reproduces the mean field critical exponents. In general, mean field exponents are expected to be correct in four or higher dimensions. In lower dimensions, the effect of fluctuations can change the critical exponents, and this means that in these cases the Landau approach give us a good qualitative description of the phase diagram but fails in its quantitative predictions of critical exponents.

In general, and beyond the Landau approach, we can parametrize the critical behavior of the flavor-singlet scalar condensate for degenerate flavors with a critical exponent $\delta > 1$,

$$\langle S\rangle_{m\to 0} \sim m^{\frac{1}{\delta}}, \tag{42}$$

which gives us a divergent scalar susceptibility $\chi_\sigma(m) \sim m^{\frac{1-\delta}{\delta}}$ and hence a massless scalar boson as $m \to 0$.

If on the other hand, we write the Ward-Takahashi identity for the isotriplet of "pions," which follows from the $SU(2)_A$ nonanomalous chiral symmetry

$$\chi_{\bar\pi}(m) = \frac{\langle S\rangle}{m}; \tag{43}$$

we get that also $\chi_{\bar\pi}(m)$ diverges when $m \to 0$ as $m^{\frac{1-\delta}{\delta}}$, and a rich spectrum of massless bosons $(\sigma, \bar\pi)$ emerges in the chiral limit.

To conclude this section, we would like to point out that the results reported here can be generalized in a straightforward way to a number of flavors $N_f > 2$.

C. Spectral density of the Lee-Yang zeros of the partition function

The results of Secs. III A and III B have been obtained with the help of some general properties of the vacuum energy density of the $Q = 0$ topological sector. In view of the relevance of these results, it is worth it to explore some alternative way to corroborate it. In this section, we will show how we can get, using general properties of the spectral density of the Lee-Yang zeros of the partition function of the zero charge topological sector, the same qualitative result in an independent way.

We consider here a generic gauge-fermion model with $U(1)_A$ axial anomaly and two fermion flavors of equal mass, in which the $SU(2)$ chiral symmetry is fulfilled in the ground state for massless fermions. Our starting assumption here, as in Sec. III A, is that the flavor-singlet scalar susceptibility, $\chi_\sigma(m)$, is a continuous function of the quark mass, m, at $m = 0$, in the full theory, taking into account the contribution of all topological sectors. Under this assumption, we will prove that the flavor-singlet scalar susceptibility, $\chi_\sigma^{Q=0}(m)$, in the $Q = 0$ topological sector, is also a continuous function of the quark mass, m, at $m = 0$, a result that, together with the identities

$$\chi_\sigma^{Q=0}(m=0) = \frac{1}{2}\chi_\sigma(m=0) + \frac{1}{2}\chi_\eta(m=0)$$
$$\chi_\sigma^{Q=0}(m) = \chi_\sigma(m) \quad \forall\ m \neq 0, \tag{44}$$

will lead us to the same paradoxical conclusion, $\chi_\sigma(m=0) = \chi_\eta(m=0)$, obtained in Sec. III A.[3]

The zeros in the complex quark mass plane of the partition function of the zero charge topological sector are distributed following several symmetry properties. If we denote with μ the absolute value of a given zero, and with α its phase, the density of zeros $\rho(\mu,\alpha)$ in the infinite lattice volume limit verifies the symmetry relations

$$\rho(\mu,\alpha) = \rho(\mu,-\alpha)$$
$$\rho(\mu,\alpha) = \rho(\mu,\pi+\alpha), \tag{45}$$

and then we can write the following expressions for the scalar condensate and the flavor-singlet scalar susceptibility, the last for massless fermions, in the $Q = 0$ topological sector

$$\langle S\rangle^{Q=0}(m) = -2m\int d\alpha \int d\mu \frac{m^2-\mu^2\cos(2\alpha)}{m^4-2m^2\mu^2\cos(2\alpha)+\mu^4}\rho(\mu,\alpha) \tag{46}$$

$$\chi_\sigma^{Q=0}(m=0) = 2\int d\alpha \int d\mu \frac{\cos(2\alpha)}{\mu^2}\rho(\mu,\alpha), \tag{47}$$

where α runs in the interval $(0,\pi)$, while it is true that, using the symmetry relations (45), the interval in α can be further reduced to $(0,\pi/2)$.

Since we are assuming that the flavor-singlet scalar susceptibility, $\chi_\sigma(m)$, takes a finite value when the quark mass goes to zero, the scalar condensate at small fermion mass will be linear in the fermion mass, plus higher-order corrections. Furthermore, as discussed in previous sections, the scalar condensate and the scalar susceptibility computed in the $Q = 0$ topological sector agree, in the infinite lattice volume limit, with the corresponding quantities computed in the full theory taking into account the contribution of all topological sectors. Hence, the chiral limit of $\chi_{\sigma^{Q=0}}(m)$ can be computed as

$$\lim_{m\to 0}\chi_\sigma^{Q=0}(m) = \lim_{m\to 0}\frac{\langle S\rangle^{Q=0}(m)}{m} = -\lim_{m\to 0} 2\int d\alpha \int d\mu \frac{m^2-\mu^2\cos(2\alpha)}{m^4-2m^2\mu^2\cos(2\alpha)+\mu^4}\rho(\mu,\alpha), \tag{48}$$

and the rest of this section will be devoted to showing that the chiral limit (48) is the massless flavor-singlet scalar susceptibility $\chi_\sigma^{Q=0}(m=0)$ (47).

[3] The first of these identities can be easily derived from the θ dependence of the massless scalar susceptibility in the full two-flavor theory, $\chi_\sigma(\theta) = \cos^2\frac{\theta}{2}\chi_\sigma(m=0) + \sin^2\frac{\theta}{2}\chi_\eta(m=0)$.

We should remark that the denominator in the right-hand side integral of (48) vanishes at $m^2 = \mu^2 e^{\pm i2\alpha}$, but since we assume that the model has no phase transitions in the fermion mass m near $m = 0$, except at most at $m = 0$, the zeros of the partition function should stay at a finite distance of the real positive axis in the complex mass plane. Hence, the only candidate to be a singular point in the integrand of (48) is $m = 0$, $\mu = 0$. This means that if we split the μ integral into two regions, $\mu < \epsilon$, and $\mu > \epsilon$, with $\epsilon \ll 1$, we can write

$$\lim_{m\to 0} 2 \int d\alpha \int_{\mu>\epsilon} d\mu \frac{m^2 - \mu^2 \cos(2\alpha)}{m^4 - 2m^2\mu^2 \cos(2\alpha) + \mu^4} \rho(\mu,\alpha) = -2 \int d\alpha \int_{\mu>\epsilon} d\mu \frac{\cos(2\alpha)}{\mu^2} \rho(\mu,\alpha), \tag{49}$$

and therefore we will concentrate on the chiral limit of the integral in the $\mu < \epsilon$ region.

Since we assume a finite massless scalar susceptibility $\chi_\sigma(m=0)$, $\chi_\sigma^{Q=0}(m=0) = \frac{1}{2}\chi_\sigma(m=0) + \frac{1}{2}\chi_\eta(m=0)$ will also be finite, and Eq. (47) tells us that the spectral density of zeros $\rho(\mu,\alpha)$ should vanish when $\mu \to 0$ fast enough in order to keep the $\mu = 0$ singularity integrable. Hence, we can parametrize the behavior of the spectral density of zeros near $\mu = 0$ as

$$\rho(\mu,\alpha)_{\mu\le\epsilon} \approx \mu^{p(\alpha)} f(\alpha) \tag{50}$$

with $p(\alpha) > 1$.[4]

To compute the chiral limit of

$$2 \int_0^\pi d\alpha \int_0^\epsilon d\mu \frac{m^2 - \mu^2 \cos(2\alpha)}{m^4 - 2m^2\mu^2 \cos(2\alpha) + \mu^4} \rho(\mu,\alpha), \tag{51}$$

we perform a change of variables and replace the spectral density of zeros in the previous expression by its small μ value (50), and so we get

$$2 \int_0^\pi d\alpha \int_0^\epsilon d\mu \frac{m^2 - \mu^2 \cos(2\alpha)}{m^4 - 2m^2\mu^2 \cos(2\alpha) + \mu^4} \rho(\mu,\alpha) = \frac{2}{m} \int_0^\pi d\alpha m^{p(\alpha)} f(\alpha) \int_0^{\frac{\epsilon}{m}} dt \frac{1 - t^2 \cos(2\alpha)}{1 - 2t^2 \cos(2\alpha) + t^4} t^{p(\alpha)}. \tag{52}$$

[4] A value of $p(\alpha) \le 1$ could also give a finite massless susceptibility (47) if large cancellations when performing the α integral happen in a fine-tuning way. For instance, if we assume $p(\alpha)$ constant and less than 1, Eq. (47) would be still finite if $\int d\alpha \cos(2\alpha) f(\alpha) = 0$. However, it can be shown that, in such an unlikely case, the qualitative results obtained in this section do not change.

It is easy to check that, for $p(\alpha) > 1$,

$$\lim_{m\to 0} \frac{2}{m} \int d\alpha m^{p(\alpha)} f(\alpha) \int_0^{\frac{\epsilon}{m}} dt \frac{1 - t^2 \cos(2\alpha)}{1 - 2t^2 \cos(2\alpha) + t^4} t^{p(\alpha)} = -2 \int d\alpha \frac{f(\alpha)\cos(2\alpha)}{p(\alpha) - 1} \epsilon^{p(\alpha)-1}, \tag{53}$$

and since the right-hand side of Eq. (53) vanishes when $\epsilon \to 0$, we get that

$$\lim_{\epsilon\to 0} \lim_{m\to 0} 2 \int d\alpha \int_0^\epsilon d\mu \frac{m^2 - \mu^2 \cos(2\alpha)}{m^4 - 2m^2\mu^2 \cos(2\alpha) + \mu^4} \rho(\mu,\alpha) = 0, \tag{54}$$

a result that, together Eqs. (48) and (49), allows us to write

$$\lim_{m\to 0} \chi_\sigma^{Q=0}(m) = \lim_{\epsilon\to 0} 2 \int_0^\pi d\alpha \int_{\mu>\epsilon} d\mu \frac{\cos(2\alpha)}{\mu^2} \rho(\mu,\alpha), \tag{55}$$

which tells us that the chiral limit of the flavor-singlet scalar susceptibility in the $Q = 0$ topological sector agrees with the massless scalar susceptibility in this sector, and therefore the scalar susceptibility is a continuous function of the fermion mass, m, at $m = 0$, in the $Q = 0$ sector. We should also notice that logarithmic violations to the power law behavior of the spectral density $\rho(\mu,\alpha)$ (50) do not change the previous qualitative result.

IV. SCHWINGER MODEL

The Schwinger model, or quantum electrodynamics in $(1+1)$ dimensions, is a good laboratory to test the results reported in the previous sections. The model is confining [13], exactly solvable at zero fermion mass, has nontrivial topology, and shows explicitly the $U(1)_A$ axial anomaly [14] through a nonvanishing value of the chiral condensate in the chiral limit in the one-flavor case. Furthermore, in the multiflavor Schwinger model, the $SU(N_f)_A$ nonanomalous axial symmetry in the chiral limit is fulfilled in the vacuum, and this property makes this model a perfect candidate to check the main conclusion of this article, namely, the existence of light scalar and pseudoscalar bosons in the spectrum of the model, the mass of which vanishes in the chiral limit.

The Euclidean continuum action is

$$S = \int d^2x \left\{ \sum_{f=1}^{N_f} \bar\psi_f(x) \gamma_\mu (\partial_\mu + iA_\mu(x)) \psi_f(x) + m \sum_{f=1}^{N_f} \bar\psi_f(x) \psi_f(x) + \frac{1}{4e^2} F_{\mu\nu}^2(x) \right\}, \tag{56}$$

where m is the fermion mass and e is the electric charge or gauge coupling, which has the same dimension as m.

$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and γ_μ are 2×2 matrices satisfying the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{57}$$

This action is apparently invariant in the chiral limit under $SU(N_f)_A$ and $U(1)_A$ chiral transformations. However, the $U(1)_A$ axial symmetry is broken at the quantum level because of the axial anomaly. The divergence of the axial current is

$$\partial_\mu J^A_\mu(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x), \tag{58}$$

where $\epsilon_{\mu\nu}$ is an antisymmetric tensor and hence does not vanish. The axial anomaly induces a topological θ term in the action of the form $i\theta Q$, where

$$Q = \frac{1}{4\pi}\int d^2x\epsilon_{\mu\nu}F_{\mu\nu}(x) \tag{59}$$

is the quantized topological charge.

The Schwinger model was analyzed years ago by Coleman [1], computing some quantitative properties of the theory in the continuum for both weak coupling, $\frac{e}{m} \ll 1$, and strong coupling $\frac{e}{m} \gg 1$.

For the one-flavor model, Coleman computed the particle spectrum of the model, which shows a mass gap in the chiral limit, and conjectured the existence of a phase transition at $\theta = \pi$ and some intermediate fermion mass m separating a weak-coupling phase ($\frac{e}{m} \ll 1$) in which the Z_2 symmetry of the model at $\theta = \pi$ is spontaneously broken from a strong-coupling phase ($\frac{e}{m} \gg 1$) in which the Z_2 symmetry is realized in the vacuum. A simple analysis of this model on the lattice also suggests that it should undergo a phase transition at some intermediate fermion mass m and $\theta = \pi$, even at finite lattice spacing. Indeed, the lattice model is analytically solvable in the infinite fermion mass limit (pure gauge two-dimensional electrodynamics with topological term) [15], and it is well known that the density of topological charge approaches a nonvanishing vacuum expectation value at $\theta = \pi$ for any value of the inverse square gauge coupling β, showing spontaneous symmetry breaking. On the other hand, by expanding the vacuum energy density in powers of m, treating the fermion mass as a perturbation, one gets for the vacuum expectation value of the density of topological charge the θ dependence

$$\langle -iq\rangle = m\Sigma\sin\theta + \frac{1}{2}m^2\sin(2\theta)(\chi_\sigma - \chi_\eta) + \ldots, \tag{60}$$

where Σ is the vacuum expectation value of the chiral condensate in the chiral limit and at $\theta = 0$ ($\Sigma = e^{\gamma_e}e/2\pi^{3/2}$ in the continuum limit) and χ_η and χ_σ are the pseudoscalar and scalar susceptibilities, respectively. Equation (60) shows how the Z_2 symmetry at $\theta = \pi$ is realized order by order in the perturbative expansion of the topological charge in powers of the fermion mass m. Therefore, a critical point separating the large and small fermion mass phases is expected, and this qualitative result has been recently confirmed by numerical simulations of the Euclidean-lattice version of the model [16].

What is, however, more interesting for the content of this article is the Coleman analysis of the two-flavor model. The theory has an internal $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$ symmetry in the chiral limit, and the $U(1)_A$ axial symmetry is anomalous. Since continuous internal symmetries cannot be spontaneously broken in a local field theory in two dimensions [17], the $SU(2)_A$ symmetry has to be fulfilled in the vacuum, and the scalar condensate, which is an order parameter for this symmetry, will therefore vanish in the chiral limit, notwithstanding the chiral $U(1)_A$ anomaly. Hence, the two-flavor Schwinger model verifies all the conditions we assumed in Sec. III.

We summarize here Coleman's main findings for the two-flavor model:

(1) For weak coupling, $\frac{e}{m} \ll 1$, the results on the particle spectrum are almost the same as for the massive Schwinger model.

(2) For strong coupling, $\frac{e}{m} \gg 1$, the low-energy effective theory depends only on one mass parameter, $m^{\frac{2}{3}}e^{\frac{1}{3}}\cos^{\frac{2}{3}}\frac{\theta}{2}$; the vacuum energy density will be then proportional to

$$E(m,e,\theta) \propto m^{\frac{4}{3}}e^{\frac{2}{3}}\cos^{\frac{4}{3}}\frac{\theta}{2}, \tag{61}$$

and the chiral condensate at $\theta = 0$ is therefore

$$\langle\bar{\psi}\psi\rangle \propto m^{\frac{1}{3}}e^{\frac{2}{3}}. \tag{62}$$

(3) The lightest particle in the theory is an isotriplet, and the next lightest is an isosinglet. The isosinglet/isotriplet mass ratio is $\sqrt{3}$. If there are other stable particles in the model, they must be $O([\frac{e}{m}]^{\frac{2}{3}})$ times heavier than these. The light boson mass, M, has a fractional power dependence on the fermion mass m:

$$M \propto m^{\frac{2}{3}}e^{\frac{1}{3}}\left(\cos\frac{\theta}{2}\right)^{\frac{2}{3}}. \tag{63}$$

Many of these results have been corroborated by several authors both in the continuum [18–22] and using the lattice approach [23,24]. Coleman concluded his paper [1] by asking some questions concerning things he did not understand, and we cite two of them here:

(1) Why are the lightest particles in the theory a degenerate isotriplet?

(2) Why does the next-lightest particle have $I^{PG} = 0^{++}$, rather than 0^{--}?

We think that the results of Sec. III allow us to give a reliable answer to these questions. The interplay between $U(1)_A$ anomaly and exact $SU(2)_A$ chiral symmetry enforces the divergence of both the flavor-singlet scalar susceptibility χ_σ, and the "pion" susceptibility $\chi_{\bar{\pi}}$ in the chiral limit. As discussed in Sec. III B, both susceptibilities have the same fractional power dependence on the fermion mass m, $\chi_\sigma, \chi_{\bar{\pi}} \propto m^{\frac{1-\delta}{\delta}} e^{\frac{\delta-1}{\delta}}$, and since

$$\chi_{\sigma m\to 0} \sim \frac{|\langle 0|\hat{O}_\sigma|\sigma\rangle|^2}{m_\sigma} \quad \chi_{\bar{\pi} m\to 0} \sim \frac{|\langle 0|\hat{O}_{\bar{\pi}}|\bar{\pi}\rangle|^2}{m_{\bar{\pi}}}, \tag{64}$$

we expect the σ and $\bar{\pi}$ masses have also the same dependence on the fermion mass m,

$$m_\sigma \propto |\langle 0|\hat{O}_\sigma|\sigma\rangle|^2 m^{\frac{\delta-1}{\delta}} e^{\frac{1-\delta}{\delta}}, \quad m_{\bar{\pi}} \propto |\langle 0|\hat{O}_{\bar{\pi}}|\bar{\pi}\rangle|^2 m^{\frac{\delta-1}{\delta}} e^{\frac{1-\delta}{\delta}}. \tag{65}$$

Coleman's analysis predicts $\delta = 3$, which is the mean field critical exponent, and a finite nonvanishing value for $\langle 0|\hat{O}_\sigma|\sigma\rangle$ and $\langle 0|\hat{O}_{\bar{\pi}}|\bar{\pi}\rangle$ in the chiral limit.

Concerning the σ meson pion mass ratio

$$\frac{m_\sigma}{m_{\bar{\pi}}} = \sqrt{3}$$

reported by Coleman in Ref. [1], we find a discrepancy. The critical behavior of the flavor-singlet scalar condensate $\langle S\rangle_{m\to 0} \sim m^{\frac{1}{\delta}}$ besides the nonanomalous Ward-Takahashi identity (43) tells us that the ratio of the pion and σ meson susceptibilities will reach the value δ in the chiral limit

$$\lim_{m\to 0} \frac{\chi_{\bar{\pi}}(m,e)}{\chi_\sigma(m,e)} = \delta, \tag{66}$$

and since the $SU(2)_A$ chiral symmetry is not spontaneously broken in the chiral limit, we expect from (64) that

$$\lim_{m\to 0} \frac{\chi_{\bar{\pi}}(m,e)}{\chi_\sigma(m,e)} = \lim_{m\to 0} \frac{m_\sigma}{m_{\bar{\pi}}}, \tag{67}$$

which, for $\delta = 3$, gives us the value 3 instead of $\sqrt{3}$ for the mass ratio. The origin of this discrepancy may reside in the strong-coupling limit approximation made by Coleman in Ref. [1]. Indeed, the bosonized two-flavor Schwinger model is a generalized sine-Gordon model that cannot be solved in closed form, but in the strong-coupling limit ($\frac{e}{m} \gg 1$) approximation, the flavor-singlet pseudoscalar field is treated as a static field, and the model is reduced to a special case of the standard sine-Gordon model for the isotriplet pseudoscalar field. It is inside the standard sine-Gordon model where Coleman found that the $\sigma - \bar{\pi}$ mass ratio is $\sqrt{3}$, but when going from the generalized sine-Gordon model to the standard sine-Gordon model, the structure of the mass term in the two-flavor Schwinger model is changed, and hence the nonanomalous Ward-Takahashi identity (43), which depends on the structure of the mass term, also changes. We want to notice, in this context, that the results for the $\sigma - \bar{\pi}$ mass ratio of a numerical simulation of the two-flavor Schwinger model with Kogut-Susskind fermions reported in Ref. [23] show a systematic deviation, at large inverse gauge coupling $\beta = \frac{1}{e^2 a^2}$ and small values of the fermion mass, from the $\sqrt{3}$ value, pointing to a larger value in the chiral limit. This is, however, a rather old calculation, and an improvement of the results of Ref. [23] could clarify this point.

We conclude this section by remarking that the results reported in Sec. III tell us that the existence of quasimassless chiral bosons in the spectrum of the two-flavor Schwinger model near the chiral limit does not originate in some uninteresting peculiarities of two-dimensional models but it should be a consequence of the interplay between exact non-Abelian chiral symmetry and a $U(1)_A$ axial anomaly, and this is a picture that also holds, for instance, in a much more interesting case, the high-temperature phase of four-dimensional QCD. What is a two-dimensional peculiarity is the fact that in the chiral limit, when all fermion masses vanish, these quasimassless bosons become unstable and the low-energy spectrum of the model reduces to a massless noninteracting boson, in accordance with Coleman's theorem [17], which forbids the existence of massless interacting bosons in two dimensions.

V. $U(N)$ MODEL IN TWO DIMENSIONS

The analysis of the previous section on the multiflavor Schwinger model applies also to the $U(N)$ model in $(1+1)$ dimensions. The Euclidean continuum action is

$$S = \int \mathrm{d}^2x \left\{ \sum_{f=1}^{N_f} \bar{\psi}_f(x)(\gamma_\mu D_\mu(x) + m_f)\psi_f(x) + \frac{1}{4e^2} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) \right\}, \tag{68}$$

where $D_\mu(x)$ is the covariant derivative, $\psi_f(x)$ is an N-multiplet fermion field, m_f is the mass of flavor f, and the index a runs from 1 to N^2. Since the $U(1)$ electromagnetic field is also gauged in the $U(N)$ model, the $U(1)_A$ axial symmetry is, like in the Schwinger model, also anomalous in the $U(N)$ model in $(1+1)$ dimensions. Furthermore, the dimensionful coupling constant e has mass dimensions, and the model is also superrenormalizable.

In the one-flavor model, we expect, as in the Schwinger model or in one-flavor four-dimensional QCD, a mass gap in the spectrum in the chiral limit because of the $U(1)_A$ axial anomaly. The spectrum of the $U(N)$ model in $(1+1)$ dimensions was analyzed time ago in the large N limit by 't Hooft [2], and he found, in the one-flavor case, a spectrum of masses of the order of the gauge coupling,

e, plus a single mass that vanishes with the fermion mass. This massless boson appears in the large N limit because the effects of the $U(1)_A$ anomaly disappear at leading order in this limit. Indeed, at finite N, the one-flavor model shows a mass gap in the chiral limit [3], as expected.

In what concerns the multiflavor $U(N)$ model, we can apply the main conclusions of this paper. In the multiflavor case, the model has a $SU(N_f)_A$ nonanomalous chiral symmetry and an anomalous $U(1)_A$ axial symmetry in the chiral limit. The $SU(N_f)_A$ chiral symmetry, as any continuous symmetry in two dimensions, is not spontaneously broken [17], and hence the scalar condensate $\langle S \rangle = 0$ vanishes in the chiral limit, notwithstanding the $U(1)_A$ anomaly. The results of Sec. III lead us to conclude that the model should exhibit a divergent correlation length in the chiral limit that, together with the Ward-Takahashi identities analogous to (20), tells us that the spectrum of the model should show N_f^2 quasimassless chiral bosons near the chiral limit, one of them scalar and the other of them $N_f^2 - 1$ pseudoscalar.

VI. CONCLUSIONS

The standard wisdom on the origin of massless bosons in the spectrum of a quantum field theory describing the interaction of gauge fields coupled to matter fields is based on two well-known features: gauge symmetry and spontaneous symmetry breaking of continuous symmetries. However, we have shown in this article that the topological properties, which originate the $U(1)_A$ axial anomaly in a QFT that describes the interaction of fermion matter fields and gauge bosons, are the basis of an alternative mechanism to generate massless bosons in the chiral limit, if the non-Abelian $SU(N_f)_A$ chiral symmetry is fulfilled in the vacuum. More precisely, we have shown, with the help of three distinct argumentation lines, that a gauge-fermion QFT, with the $U(1)_A$ axial anomaly, and in which the chiral condensate vanishes in the chiral limit, typically because of an exact non-Abelian chiral symmetry, should exhibit a divergent correlation length in the correlation function of the scalar condensate, in the chiral limit. The nonanomalous Ward-Takahashi identities tell us then that, in such a case, also some pseudoscalar correlation functions should exhibit a divergent correlation length, associated to what would be the Nambu-Goldstone bosons if the non-Abelian chiral symmetry were spontaneously broken.

The two-flavor Schwinger model, or quantum electrodynamics in two space-time dimensions, is a good test bed for our predictions. Indeed, the Schwinger model shows a nontrivial topology, which induces the $U(1)_A$ axial anomaly. Moreover, in the two-flavor case, the non-Abelian $SU(2)_A$ chiral symmetry is fulfilled in the vacuum, as required by Coleman's theorem [17] on the impossibility of spontaneously breaking continuous symmetries in two dimensions.

The two-flavor Schwinger model was analyzed by Coleman long ago in Ref. [1], in which he computed some quantitative properties of the theory in the continuum for both weak coupling, $\frac{e}{m} \ll 1$, and strong coupling $\frac{e}{m} \gg 1$. In what concerns the strong-coupling results, the Coleman's main findings are qualitatively in agreement with our predictions. The vacuum energy density (61) and the chiral condensate (62) show a singular dependence on the fermion mass, m, in the chiral limit, and the flavor-singlet scalar susceptibility diverges when $m \to 0$. Moreover, our results establish a reliable answer to some questions Coleman had himself [1] concerning the following two things he did not understand about the low-energy spectrum of the model:

(1) Why are the lightest particles in the theory a degenerate isotriplet?
(2) Why does the next-lightest particle have $I^{PG} = 0^{++}$, rather than 0^{--}?

Indeed, the interplay between the $U(1)_A$ anomaly and an exact $SU(2)_A$ chiral symmetry enforces the divergence of the flavor-singlet scalar susceptibility, $\chi_\sigma \sim m^{\frac{1-\delta}{\delta}}$, $\delta > 1$, in the $m \to 0$ limit, and the nonanomalous Ward-Takahashi identity tells us that also the pion susceptibility $\chi_{\bar{\pi}} \sim m^{\frac{1-\delta}{\delta}}$ diverges in the chiral limit. The ratio value of these susceptibilities

$$\lim_{m\to 0} \frac{\chi_{\bar{\pi}}}{\chi_{\bar{\sigma}}} = \delta$$

implies, on the other hand, that the pion is lighter than the σ meson.

The multiflavor $U(N)$ model in $1+1$ dimensions is another test bed for our predictions, and we have analyzed this model in Sec. V. The results of this analysis are qualitatively similar to those of the multiflavor Schwinger model; the model spectrum should show N_f^2 quasimassless chiral bosons near the chiral limit, one of them scalar and the other of them $N_f^2 - 1$ pseudoscalar.

It is worth wondering if the reason for the rich spectrum of light chiral bosons near the chiral limit found in the Schwinger and $U(N)$ models lies in some uninteresting peculiarities of two-dimensional models or if there is a deeper and general explanation for this phenomenon. We want to remark, concerning this, that our results reported in Sec. III tell us that the existence of quasimassless chiral bosons in the spectrum of these models near the chiral limit does not originate in some uninteresting peculiarities of two-dimensional models but it should be a consequence of the interplay between an exact non-Abelian chiral symmetry and the $U(1)_A$ axial anomaly. What is a two-dimensional peculiarity is the fact that in the chiral limit, when all fermion masses vanish, these quasimassless bosons become unstable and the low-energy spectrum of the model reduces to a massless noninteracting boson [3,4], in accordance with Coleman's theorem [17], which forbids the existence of massless interacting bosons in two dimensions.

In what concerns QCD, the analysis of the effects of the $U(1)_A$ axial anomaly in its high-temperature phase, in which the non-Abelian chiral symmetry is restored in the ground state, has aroused much interest in recent time because of its relevance in axion phenomenology. Moreover, the way in which the $U(1)_A$ anomaly manifests itself in the chiral symmetry restored phase of QCD at high temperature could be tested when probing the QCD phase transition in relativistic heavy-ion collisions.

The first investigations on this subject started a long time ago. The idea that the chiral symmetry restored phase of two-flavor QCD could be symmetric under $U(2) \times U(2)$ rather than $SU(2) \times SU(2)$ was raised by Shuryak in 1994 [25] based on an instanton liquid-model study. In 1996, Cohen [26] also got this result formally from the QCD functional integral under some assumptions. However, immediately after, several calculations questioning this result appeared [27–30]. On the other hand, a more recent analytic calculation of two-flavor QCD in the lattice, with overlap fermions, has shown [31] that the axial $U(1)_A$ anomaly becomes invisible in the scalar and pseudoscalar meson susceptibilities, suggesting again that the effects of the anomaly disappear in the high-temperature phase. However, as stated by the authors of Ref. [31], their result strongly relies on their assumption that the vacuum expectation values of quark-mass independent observables, as the topological susceptibility, are analytic functions of the square quark mass, m^2, if the non-Abelian chiral symmetry is restored. Conversely, Coleman's result for the topological susceptibility in the two-flavor Schwinger model, which follows from Eq. (61),

$$\chi_T \propto m^{\frac{4}{3}} e^{\frac{2}{3}},$$

explicitly shows a nonanalytic quark-mass dependence and casts serious doubts on the validity of this assumption.

The dilute instanton gas model [32–35] predicts, on the other hand, a topological susceptibility for three light flavors, $\chi_T \sim \frac{1}{T^8}$, which decays with a power law of the temperature at high T, and a recent lattice calculation [36] of the topological properties of full QCD with physical quark masses and temperatures around 500 MeV gives as a result a small but nonvanishing topological susceptibility, although with large error bars in the continuum limit extrapolations, suggesting that the effects of the $U(1)_A$ axial anomaly still persist at these temperatures.

We can therefore do the reasonable hypothesis that the effects of the anomaly, although diminished, still persist in the high-temperature phase of QCD, and under such an assumption, the main conclusions of this paper should also apply to this phase. Taking into account the recent lattice determination of the light quark masses [37] ($m_u \simeq 2$ MeV, $m_d \simeq 5$ MeV, $m_s \simeq 94$ MeV), we can consider QCD with two quasimassless quarks as a good approach. Hence, our results predict a large value for the σ and $\bar{\pi}$ meson susceptibilities and a spectrum of light σ and $\bar{\pi}$ mesons at $T \gtrapprox T_c$, and the presence of these light scalar and pseudoscalar mesons in the chirally symmetric high-temperature phase of QCD could, on the other hand, significantly influence the dilepton and photon production observed in the particle spectrum [38] at heavy-ion collision experiments.

There are, on the other hand, two recent lattice calculations of mesonic screening masses in two- [39] and three-flavor [40] QCD around and above the critical temperature. The reported results are not enough to allow a good check of our spectrum prediction. However, the results of Ref. [40] show a small change of the pion screening mass when crossing the critical temperature and a decreasing screening mass, at $T \gtrapprox T_c$, when going from the $\bar{u}s$ to the $\bar{u}d$ channel, compatible with a vanishing pion screening mass in the chiral limit.

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