Evidence for Supersymmetry in the Random-Field Ising Model at $D = 5$

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We provide a nontrivial test of supersymmetry in the random-field Ising model at five spatial dimensions, by means of extensive zero-temperature numerical simulations. Indeed, supersymmetry relates correlation functions in a D-dimensional disordered system with some other correlation functions in a $D-2$ clean system. We first show how to check these relationships in a finite-size scaling calculation and then perform a high-accuracy test. While the supersymmetric predictions are satisfied even to our high accuracy at $D = 5$, they fail to describe our results at $D = 4$.

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Introduction.—The suggestion [1] that the random-field Ising model (RFIM) at the critical point [2–4] obeys supersymmetry came as a major surprise in theoretical physics. One of the implications of supersymmetry is dimensional reduction [5,6]: The critical exponents of a disordered system at space dimension D and those of a pure (i.e., nondisordered) system at dimension $D-2$ coincide. Let us remark that dimensional reduction is a consequence of [1,7], but not necessarily equivalent to, supersymmetry.

However, in spite of its power and elegance, it was soon clear that the applicability of supersymmetry is problematic. The original argument [1] was based on the study of the solutions of the stochastic Landau-Ginsburg equations in the presence of a random magnetic field. Unfortunately, the crucial assumption of uniqueness of the solution of these equations [1] (which holds at all orders in the perturbation theory) fails beyond the perturbation theory. In fact, it was immediately clear that in the RFIM the predicted dimensional reduction is absent at low dimensions (but not for branched polymers [8], where dimensional reduction has been mathematically proven [9–11]): The RFIM has a ferromagnetic phase at $D = 3$ [12,13], while the $D = 1$ pure Ising model has no transition. Nonperturbative effects (e.g., bound states in replica space [14–17]) are obviously important in $D = 3$. Yet, their relevance for $D > 3$ (especially upon approaching the presumed upper critical dimension $D_u = 6$) is unclear. If we consider the case of $D = 6 - \epsilon$, different scenarios are possible, as listed below.

(1) Nonperturbative effects could destroy supersymmetry at a finite order in the ϵ expansion or, even worse, at $D = 6$.

(2) Violations of supersymmetry might be exponentially small $\sim \exp(-A/\epsilon)$ (see, e.g., Refs. [18,19]; the computation of A is still an unsolved problem).

(3) Supersymmetry has been suggested to be exact but only for $D > D_{\text{int}} \approx 5.1$ [20–22]. For $D < D_{\text{int}}$, the supersymmetric fixed point becomes unstable with respect to nonsupersymmetric perturbations.

In order to discriminate among these three scenarios, we need accurate simulations aimed to test some of the many predictions of supersymmetry. In the past few years, the development of a powerful panoply of simulation and statistical analysis methods [23–25] set the basis for a fresh revision of the problem. Great emphasis was made on the anomalous dimensions η and $\bar{\eta}$ related to the decay of the connected and disconnected correlations functions, respectively [see Eq. (2)]. Supersymmetry predicts $\eta = \bar{\eta}$ (moreover, the D-dimensional RFIM $\eta = \bar{\eta}$ are predicted to be equal to the anomalous dimension of the pure Ising model in dimension $D - 2$). Extensive numerical simulations at a zero temperature showed that these relations fail at $D = 3$ [23] and $D = 4$ [25], but they are valid with good accuracy at $D = 5$ [26]. These numerical results suggest that supersymmetry may be really at play at $D = 5$. We should mention as well a recent work using the conformal boostrap [27], where it was found that dimensional reduction holds in the RFIM for $D \geq 5$.

The predictions of supersymmetry go further beyond those regarding the critical exponents: They involve both finite volume effects and high-order correlations functions. Here, we will show that several nontrivial supersymmetry predictions hold at $D=5$ to a very high numerical accuracy. This is the first direct confirmation that supersymmetry holds in the RFIM at high dimensions. As a consistency check, we show that the same relations are definitively not satisfied at $D=4$.

Simulation setup.—The Hamiltonian of the RFIM is

$$\mathcal{H} = -J\sum_{\langle xy\rangle} S_x S_y - \sum_x h_x S_x, \tag{1}$$

with the spins $S_x = \pm 1$ on a hypercubic lattice in D dimensions with nearest-neighbor ferromagnetic interactions and h_x independent random magnetic fields with zero mean and variance σ^2. Given our previous universality confirmations [28], we have restricted ourselves to normal-distributed h_x. We work directly at a zero temperature [29–33], because the relevant fixed point of the model lies there [34–36]. The system has a ferromagnetic phase at small σ, that, upon increasing the disorder, becomes paramagnetic at the critical point σ_c. Here, we work directly at σ_c, namely, at $6.02395 \approx \sigma_c(D=5)$ [26] and at $4.17749 \approx \sigma_c(D=4)$ [25].

We consider *two* correlation functions, namely, the connected and disconnected propagators, $C^{(\mathrm{con})}_{xy}$ and $C^{(\mathrm{dis})}_{xy}$, respectively:

$$C^{(\mathrm{con})}_{xy} \equiv \frac{\partial\overline{\langle S_x\rangle}}{\partial h_y}, \qquad C^{(\mathrm{dis})}_{xy} \equiv \overline{\langle S_x\rangle\langle S_y\rangle}, \tag{2}$$

where the $\langle\cdots\rangle$ are thermal mean values as computed for a given realization, a *sample*, of the random fields $\{h_x\}$. An overline refers to the average over the samples.

For each of these two propagators, we scrutinize the second-moment correlation lengths [37], as adapted to our geometrical setting. In particular, our chosen geometry is an elongated hypercube with periodic boundary conditions and linear dimensions $L_x = L_y = L_z = L$ and $L_t = L_u = RL$ (at $D=4$, we chose $L_x = L_y = L$ and $L_z = L_t = RL$) with aspect ratio $R \geq 1$. In fact, the supersymmetric identities that we will check in the critical region hold in the limit $R \to \infty$, which should be taken *before* the standard thermodynamic limit.

We simulated lattice sizes in the range $L = 4$–14 at $D=5$ ($L = 4$–28 at $D=4$) and aspect ratios $1 \leq R \leq 5$. Additional simulations for $R=10$ and $L \leq 10$ were performed at both 5D and 4D for consistency reasons. For each pair of (L, R) values, we computed ground states for 10^5 disorder samples. Our simulations and analysis closely follows the methodology outlined in our previous works at $D=3$ and 4 [23,25] (for full technical details, see Ref. [24]).

Supersymmetric predictions.—Let us consider a point in the 5D lattice, $\mathbf{r} = (\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = (x, y, z)$ refers to the first three Cartesian coordinates, while $\mathbf{u} = (t, u)$. In a similar vein, for the 4D case, we split $\mathbf{r} = (x, y, z, t) = (\mathbf{x}, \mathbf{u})$ as $\mathbf{x} = (x, y)$ and $\mathbf{u} = (z, t)$. The supersymmetric predictions (see [7,38–41] for a more paused exposition) are particularly simple for disconnected correlation functions:

$$C^{(\mathrm{dis}),D}_{\mathbf{x}_1,\mathbf{u};\mathbf{x}_2,\mathbf{u}} = \mathcal{Z} G^{\mathrm{Ising},D-2}_{\mathbf{x}_1;\mathbf{x}_2}, \tag{3}$$

where G is the pure Ising model correlator and $\mathcal{Z}$ is a position-independent normalization constant that will play no role (see below). Note that the left-hand side depends on both linear dimensions L and RL, while the right-hand side depends only on L. Therefore, we must carefully consider under which conditions Eq. (3) is expected to hold. In a more conventional study, one would require an hierarchy of length scales $LR \gg L \gg \xi \gg 1$ (recall that ξ is the correlation length), while we demand for the $D-2$ Euclidean distance $\|\mathbf{x}_1 - \mathbf{x}_2\|/\xi \sim 1$. We shall put under stress Eq. (3) by demanding it to hold as well in the finite-size scaling regime

$$LR \gg L \sim \xi \gg 1, \qquad \|\mathbf{x}_1 - \mathbf{x}_2\|/\xi \sim 1. \tag{4}$$

These preliminaries lead us to consider a $D-2$ Fourier transform in the D-dimensional RFIM

$$\hat{C}^{(\mathrm{dis}),D}_{\mathbf{k}} = \frac{1}{L^{D-2}} \sum_{\mathbf{x}_1,\mathbf{x}_2} e^{i(\mathbf{x}_1-\mathbf{x}_2)\cdot\mathbf{k}} \overline{\langle S_{\mathbf{x}_1,\mathbf{u}}\rangle\langle S_{\mathbf{x}_2,\mathbf{u}}\rangle}. \tag{5}$$

Note that the $\mathbf{u}$ dependence vanishes due to the disorder average (hence we average over $\mathbf{u}$ in order to gain statistics). We then compute the second-moment correlation length from the ratio of $\hat{C}^{(\mathrm{dis}),D}_{\mathbf{k}}$ at $\mathbf{k} = \mathbf{0}$ and $\mathbf{k}_{\min} = (2\pi/L, 0, 0)$ [37] [$\mathbf{k}_{\min} = (2\pi/L, 0)$ for $D=4$]. The important observation is that, because the constant $\mathcal{Z}$ in the rhs of Eq. (3) cancels when computing the ratio, the dimensionless ratio $\xi^{(\mathrm{dis})}/L$ as computed in the D-dimensional RFIM coincides with ξ/L as computed in the $D-2$ Ising model. This equality holds if $\xi^{(\mathrm{dis})}/L$ is computed precisely at the critical point σ_c and if the thermodynamic limit is taken under conditions (4).

If we now consider the four-body disconnected correlation function, supersymmetry predicts a relation analogous to Eq. (3) (the normalization in the rhs changes to $\mathcal{Z}^2$), so we may compute as well a $(D-2)$-dimensional U_4 parameter,

$$M_{\mathbf{u}} = \sum_{\mathbf{x}} S_{\mathbf{x},\mathbf{u}}, \qquad U_4 = \overline{\langle M_{\mathbf{u}}^4\rangle}/\overline{\langle M_{\mathbf{u}}^2\rangle}^2, \tag{6}$$

TABLE I. Universal quantities as computed in the pure Ising model at two and three spatial dimensions. The somewhat controversial situation with the corrections to scaling exponent ω in two dimensions is discussed in Ref. [41].

$D-2$	ξ/L	U_4	ω
2	0.905 048 8… [47]	1.167 93… [47]	1.75
3	0.6431(1) [48]	1.6036(1) [48]	0.829 66(9) [49]

that is predicted to coincide with that of the critical $D-2$ Ising model (under the same condition discussed above for $\xi^{(\mathrm{dis})}/L$). Again, we improve our statistics by averaging both $\overline{\langle M_{\mathbf{u}}^4\rangle}$ and $\overline{\langle M_{\mathbf{u}}^2\rangle}$ over $\mathbf{u}$.

We finally address the supersymmetric predictions for the connected correlation function. It is convenient to consider the correlation functions K defined as

$$K_{\mathbf{x}_1;\mathbf{x}_2} = \sum_{\mathbf{u}} C^{(\mathrm{con})}_{\mathbf{x}_1,\mathbf{0};\mathbf{x}_2,\mathbf{u}}. \tag{7}$$

The Ward identity for supersymmetry [38] implies (see [41]) that the second-moment correlation length $\xi^{(\mathrm{con})}_{\sigma-\eta}$ computed from K [46] is equal to the disconnected correlation length. This prediction $\xi^{(\mathrm{con})}_{\sigma-\eta} = \xi^{(\mathrm{dis})}$ does not make direct reference to dimensional reduction.

Results.—Let us start by recalling in Table I the $(D-2) = 2$, 3 universal quantities from the pure Ising model that we aim to recover from the D-dimensional RFIM. We shall need as well the value of the leading corrections to scaling exponent ω; the analysis we present is done using the exponent ω given by dimensional reduction, which is not far from the one computed in the large-scale simulations at $D = 5$ [26].

First, we consider the dimensionless ratio $\xi^{(\mathrm{dis})}(L,R)/L$ in Fig. 1. Our first task [recall Eq. (4)] is to extract the large-R limit. The good news is that we expect this limit to be reached exponentially in R and uniformly in L [50]. In fact, the comparison of our numerical results for $R = 5$ and 10 suggests that (within our statistical accuracy) $R = 5$ is large enough. Therefore, we focus the analysis on $R = 5$, where we reach our largest L value, namely, $L = 14$. As is clear from Fig. 1, our data are accurate enough to resolve corrections to scaling. Furthermore, the nonmonotonic L evolution of $\xi^{(\mathrm{dis})}(L, R = 5)/L$ implies that subleading corrections cannot be neglected. Hence, we have attempted to represent these subleading corrections in an effective way by means of a fit to a polynomial in $L^{-\omega}$. We have included in the fit only data with $L \geq L_{\min}$. We have attempted to keep both $L_{\min}$ and the order of the polynomial as low as possible. We find a fair fit ($\chi^2/\mathrm{dof} = 3.24/2$, p value $= 20\%$) with a cubic polynomial and $L_{\min} = 6$. The corresponding extrapolation to $L = \infty$ is

$$\lim_{L\to\infty}\left(\lim_{R\to\infty}\frac{\xi^{(\mathrm{dis})}(L,R)}{L}\right) = 0.654(13), \tag{8}$$

which is statistically compatible to the three-dimensional result in Table I. Hence, our first check of supersymmetry

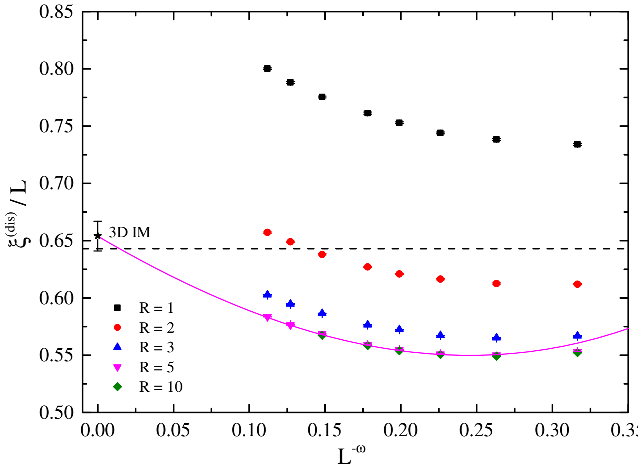

FIG. 1. $\xi^{(\mathrm{dis})}(L,R)/L$ vs $L^{-\omega}$ for various R values, as computed in the $D = 5$ RFIM. The value of the corrections to scaling exponent ω corresponds to the pure Ising model in three spatial dimensions; see Table I (the value from Ref. [49] is so accurate that we took their central value as numerically exact). The dashed horizontal line corresponds to the value for ξ/L, also shown in Table I. The continuous line is a fit to our $R = 5$ data (see the text for details). The extrapolation to $L = \infty$ obtained from the fit is compatible with the pure Ising model value, as predicted by supersymmetry.

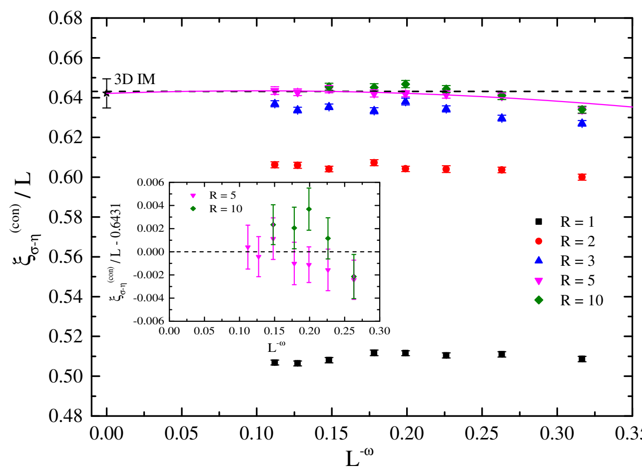

FIG. 2. As in Fig. 1, but for the $\xi^{(\mathrm{con})}_{\sigma-\eta}(L,R)/L$ data, as computed in the $D = 5$ RFIM. The agreement of the $L = \infty$ extrapolation with the value of ξ/L from the pure Ising model is a direct confirmation of the supersymmetric Ward identity [41]. Inset: Enlargement of the main panel data corresponding to $R = 5$, 10 and $L > 4$. For the sake of clarity, in the vertical axis, we have subtracted the value of the pure Ising model (see also Table I).

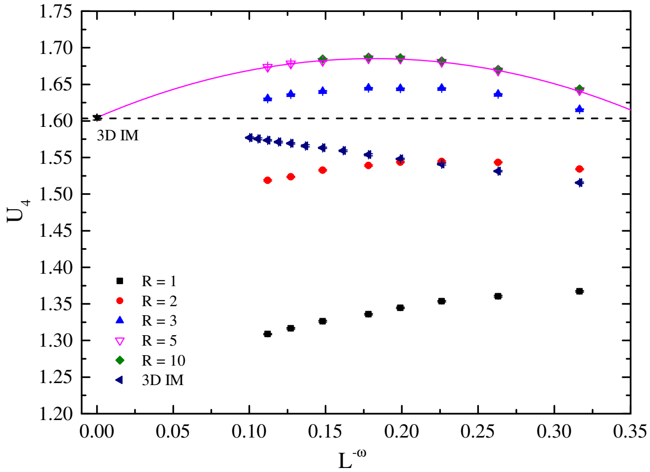

FIG. 3. As in Fig. 1, but for the $U_4(L, R)$ data, as computed in the $D = 5$ RFIM. For comparison, we also show data for the pure Ising model in three spatial dimensions. Corrections to scaling in the pure model are of a similar size (but opposite sign) to those of the large R limit for the RFIM at $D = 5$.

has been passed. The strength of this check is quantified by our 2% accuracy.

The analysis of $\xi^{(\mathrm{con})}_{\sigma-\eta}(L, R)/L$ (see Fig. 2) is carried out along the same lines. We find a good fit ($\chi^2/\mathrm{dof} = 0.63/3$, p value = 89%) with a quadratic polynomial in $L^{-\omega}$ and $L_{\min} = 6$. The corresponding extrapolation to $L = \infty$ is

$$\lim_{L\to\infty}\left(\lim_{R\to\infty}\frac{\xi^{(\mathrm{con})}_{\sigma-\eta}(L, R)}{L}\right) = 0.642(7). \tag{9}$$

It follows that we have checked supersymmetry to a 1% accuracy.

Our $U_4(L, R)$ data (see Fig. 3) can be analyzed in a similar vein. We find a fair fit ($\chi^2/\mathrm{dof} = 6.85/4$, p value = 14%) with a quadratic polynomial in $L^{-\omega}$ and $L_{\min} = 5$. The corresponding extrapolation to $L = \infty$ is

$$\lim_{L\to\infty}[\lim_{R\to\infty} U_4(L, R)] = 1.604(3), \tag{10}$$

again compatible with the three-dimensional pure Ising model value (Table I). Supersymmetry is checked to the 0.2% level this time.

Finally, as a comparison, we show our data for the 4D RFIM Ising model in Fig. 4. Even after carrying out the double limit $L \to \infty$ and $R \to \infty$, all three dimensionless quantities differ from their values in the 2D pure Ising ferromagnet. Although this is hardly a surprise (recall, for instance, exponents η and $\bar{\eta}$ [25]), the discrepancy is at least at the 10% level.

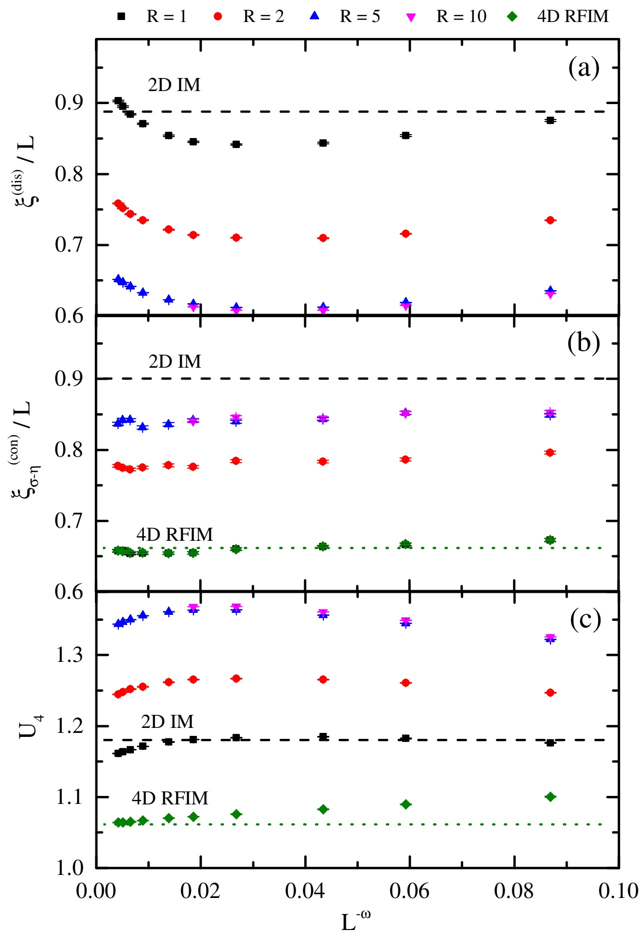

FIG. 4. Dimensionless quantities $\xi^{(\mathrm{dis})}(L, R)/L$ (a), $\xi^{(\mathrm{con})}_{\sigma-\eta}(L, R)/L$ (b), and $U_4(L, R)$ (c) vs $L^{-\omega}$ as computed in the $D = 4$ RFIM. We set $\omega = 1.75$ from Table I. We show the corresponding universal values for the 2D pure Ising model (black dashed lines). Note that for $R = 1$ there are two natural ways of computing U_4. One way (black squares) is averaging over a codimension-two manifold [this is the natural way for a supersymmetry check; recall Eq. (6)]. The other way, which is the natural one when studying the $D = 4$ RFIM *per se*, is averaging over the full four-dimensional lattice (green diamonds). Clearly, the two choices differ, both at finite L and in the large-L limit. Instead, for $\xi^{(\mathrm{con})}_{\sigma-\eta}(L, R)/L$ these two kinds of spatial averaging coincide by construction. The horizontal green dotted lines are the large-L limit, as obtained for the $D = 4$ RFIM [25].

Conclusions.—The finding of supersymmetry and dimensional reduction in the RFIM is, arguably, one of the most surprising results in theoretical physics. Here, thanks to state-of-the-art numerical techniques, we have carried out a precision test of supersymmetry. Although supersymmetry is clearly broken at $D = 4$, the $D = 5$ RFIM is supersymmetric with good accuracy. Hence, scenario 1 in the introduction is plainly discarded.

The only remaining contenders are scenarios 2 and 3. Exponent ω might help to settle the question. In the ϵ expansion ($\epsilon = 6 - D$), we find at least two exponents:

$\omega_{\rm DR} = \epsilon + \mathcal{O}(\epsilon^2)$ (obtained through dimensional reduction) and $\omega_{\rm NS} = 2 + \mathcal{O}(\epsilon^2)$ (due to irrelevant nonsupersymmetric operators). The large value of ω found here and in Ref. [26] [the values for $\omega(D)$ are in Ref. [41]], agrees with dimensional reduction and favors scenario 2. Indeed, in scenario 3, supersymmetry is broken only for space dimension $D < D_{\rm int}$, suggesting a much smaller value $\omega(D=5) \sim D_{\rm int} - D \approx 0.1$. However, further studies are needed to resolve this delicate issue.

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