# Diagonalizable algebraic groups and gradings on algebras

**Eduardo de Lorenzo Poza**
Trabajo de fin de grado en Matemáticas
Universidad de Zaragoza

Director del trabajo: Alberto Carlos Elduque Palomo
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# Resumen

Los grupos algebraicos son el análogo algebraico a los grupos de Lie de la geometría diferencial. No cabe duda pues, de que el estudio de estos objetos, entre los que se encuentran grupos tan conocidos como los grupos clásicos de matrices, es del mayor interés. Siguiendo la línea marcada por Grothendieck, los matemáticos se dieron cuenta hace algún tiempo de que era posible comprender estos objetos fijándose en la estructura de los morfismos entre ellos. Tanto es así que en este trabajo presentamos una descripción totalmente funtorial de los mismos, sin apelar directamente a la geometría en ningún momento. Esto nos permite comenzar a trabajar con grupos algebraicos sin tener que tratar primero con nociones de geometría algebraica. A cambio, es necesario estar familiarizado con nociones básicas de teoría de categorías. La fuente principal de esta parte del trabajo es [8], y allí se puede encontrar también la relación entre esta versión funtorial de los grupos algebraicos y su naturaleza geométrica.

Empezaremos asumiendo conocidos los primeros conceptos de la teoría de categorías, tales como la propia definición de categoría, la definición de funtor o la definición de transformación natural. No será necesario mucho más que eso, puesto que el primer capítulo comienza dando la definición de funtor representable y estableciendo uno de los resultados fundamentales sobre los que descansa el resto de la teoría: el lema de Yoneda. A grandes rasgos y en la versión que utilizaremos aquí, este importante lema dice lo siguiente.

**Lema de Yoneda.** *Sea $\mathcal{C}$ una categoría y sean $E, F : \mathcal{C} \to \mathbf{Set}$ dos funtores representables. Las transformaciones naturales de $E$ a $F$ están en correspondencia con los morfismos entre los objetos representantes, en sentido inverso.*

Comprender este lema es primordial, puesto que es el diccionario que nos permite traducir lo que sucede en la categoría de esquemas-grupo afines (una versión ligeramente más general de los grupos algebraicos) a la categoría de álgebras de Hopf, un nuevo objeto algebraico que introducimos y estudiamos en el capítulo 1. Más adelante, en el capítulo 2, el lema de Yoneda volverá a entrar en juego y nos servirá de nuevo como puente, esta vez entre las representaciones lineales de grupos algebraicos y los comódulos, otra nueva estructura algebraica.

Con esta herramienta en mano, el resto del capítulo 1 transcurre estudiando las propiedades algebraicas típicas de los esquemas-grupo afines y las álgebras de Hopf, tales como los morfismos entre objetos de la misma categoría; los subobjetos y monomorfismos; y los cocientes y epimorfismos. Seguidamente se introducen dos tipos de grupos algebraicos afines que resultarán útiles en los capítulos siguientes: los grupos algebraicos diagonalizables y los esquemas-grupo constantes. El capítulo finaliza con una versión restringida de la dualidad de Cartier, que nos permite poner en relación estos dos últimos tipos de esquemas-grupo.

El capítulo 2 comienza con un cambio súbito de dirección. Abandonamos momentáneamente el terreno funtorial para introducir la noción de álgebra (no necesariamente asociativa) graduada y algunos conceptos relacionados, como el de realización de una graduación. La definición básica es la que sigue.

**Definición.** Dados un grupo $G$ y un álgebra (no necesariamente asociativa) $\mathcal{A}$, una *graduación por $G$ de $\mathcal{A}$* o una *$G$-graduación de $\mathcal{A}$*, es una descomposición de $\mathcal{A}$ como espacio vectorial, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, que satisface $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh}$ para todo $g, h \in G$.

A continuación aparecen distintas nociones de isomorfismo para álgebras graduadas, según si consideramos el grupo por el que se gradúa como parte de la definición o no. Los teoremas principales de este capítulo 2, presentados en la sección 2.4, lidian con los diferentes tipos de isomorfismo por separado. Antes de poder formular dichos resultados es necesario desarrollar el lenguaje apropiado. Este es el propósito de la sección 2.3 que, empleando el lema de Yoneda y con tan solo una conexión adicional, pone en relación las graduaciones de un álgebra no asociativa $\mathcal{A}$ por un grupo abeliano finito $G$ con las representaciones lineales del grupo algebraico diagonalizable $G^D$ en el esquema-grupo de automorfismos de $\mathcal{A}$, $\mathbf{Aut}(\mathcal{A})$. Todo esto y mucho más se puede encontrar en [1], fuente de este capítulo.

Por último, el capítulo 3 pretende sacar a relucir una aplicación directa de lo presentado en el capítulo 1, mostrando que los conceptos allí descritos siguen siendo relevantes hoy en día. Para ello nos servimos de temas y artículos de actualidad. Calcularemos aquí los esquemas-grupo de automorfismos de algunas álgebras de evolución de dimensión 2. Estas álgebras fueron introducidas en 2006 por Tian, y él mismo ha indicado sus múltiples aplicaciones (aliciente del intenso trabajo que ha habido en este campo en los últimos años) en [6]. Para el cómputo de estos esquemas-grupo nos valdremos de las técnicas presentadas en dos artículos de este año 2019, a saber [2] y [3].

Estos cálculos, aparentemente inocuos, pueden volverse realmente complicados. Las interesantes ideas presentadas en [2] simplificarán las cuentas a cambio de introducir nueva maquinaria. El capítulo comenzará con la introducción de estos nuevos métodos, que incluyen grafos y secuencias exactas cortas de esquemas-grupo afines, y finalizará con el cálculo de algunos de los esquemas-grupo de automorfismos antes nombrados. En este capítulo nos volveremos a encontrar con los dos tipos de esquemas-grupo afines que habían aparecido anteriormente en el capítulo 1, diagonalizables y constantes.

# Summary


Algebraic groups are the algebraic counterpart of Lie groups. It goes without saying that the study of such objects, among which we may find well-known groups including the classical matrix groups, is of great interest. Following the path outlined by Grothendieck, mathematicians from the last century have come to the realization that it is possible to understand these algebraic groups by focusing on the structure of the morphisms between them. This is so much so that in this work we are able to present a completely functorial description of the former, with no direct reference to the geometry behind the scenes. It is this way that we are capable of working with algebraic groups without having to first learn some notions from Algebraic Geometry. In return, one should be familiar with the basic concepts from Category Theory to properly understand this description. The main source for this first part of the work ahead is [8], where one may also find explained the relationship between the functorial presentation of algebraic groups and their geometric nature.

We will start off assuming that the reader is acquainted with the first principles of Category Theory — the definitions of category, functor and natural transformation. This is all that will be needed, as the first chapter sets off introducing the concept of representable functor and establishing one of the fundamental results: the Yoneda Lemma. In the form that we will use the most, this result says the following.

**Yoneda Lemma.** *Let $\mathcal{C}$ be a category and let $E, F : \mathcal{C} \to$ **Set** be two representable functors. Natural transformations from $E$ to $F$ are in correspondence with morphisms between the representing objects, in reverse order.*

A thorough understanding of this lemma is of the utmost importance, for it is a lexicon that allows us to translate from the category of affine group schemes (a slighlty generalized version of an algebraic group) to the category of Hopf algebras, a new object which we define and study throughout Chapter 1. Later on, in Chapter 2, the Yoneda Lemma will come into play again, this time around serving as a bridge between linear representations of algebraic groups and comodules, another new algebraic structure.

With this result in our toolbox, the remainder of Chapter 1 continues by studying the usual algebraic properties of affine group schemes and Hopf algebras, such as morphisms between the objects of the same category; subobjects and monomorphisms; and quotients and epimorphisms. Immediately after that, two families of affine algebraic groups enter the scene. Their names are diagonalizable algebraic groups and constant group schemes, and they will make an appearance in both subsequent chapters. The relation between the two is explicited via a restricted version of Cartier duality, with which we close the chapter.

Chapter 2 begins with a sudden change of direction. We temporarily leave the functorial realm behind to introduce the notion of a (not necessarily associative) graded algebra and some related topics, such as that of the realization of a grading. The basic definition goes as follows.

**Definition.** Let $G$ be a group and $\mathcal{A}$ be a (not necessarily associative) algebra. A *grading by $G$ on $\mathcal{A}$*, or a *$G$-grading on $\mathcal{A}$*, is a vector space decomposition $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ satisfying $\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh}$ for all $g, h \in G$.

Following this definition we encounter several different notions of isomorphism for graded algebras, depending on whether we consider the grading group as part of the definition or not. The main theorems of Chapter 2, presented in Section 2.4, deal with the different types of isomorphism separately. Before we are able to even formulate said theorems, we must develop the necessary language. Such is the goal of Section 2.3, in which, by means of the Yoneda Lemma and just one other connection, we are able to relate the gradings of a nonassociative algebra $\mathcal{A}$ by a finite abelian group $G$ with the linear representations of the diagonalizable algebraic group $G^D$ on the automorphism group scheme of $\mathcal{A}$, $\mathbf{Aut}(\mathcal{A})$. All of this and much more can be found in [1], the main source of this chapter.

Last but not least, Chapter 3 aims to present a direct application of the machinery of Chapter 1. In doing so we pretend to show that the concepts we have covered are still relevant nowadays. To achieve that, we will make use of topics and papers which only very recently have appeared in journals. We will proceed to compute the automorphism group schemes of some evolution algebras of dimension 2. These algebras were introduced on 2006 by Tian, who has pointed out their manifold applications in [6]. To do the computations we will employ the techniques from two papers that have come out this year 2019, namely [2] and [3].

These computations, although inocuous at first sight, can quickly become really complicated. The interesting ideas presented in [2] will simplify the work. The chapter starts by defining some new necessary concepts, and along the way we will encounter some agents from Graph Theory and short exact sequences of affine group schemes. The final section will consist in the promised computation of the automorphism group schemes mentioned above. In this chapter we will meet once again the two special types of affine group schemes that were introduced in Chapter 1.

# Contents

# Chapter 1

# Affine group schemes

## 1.1 Definitions

We begin with some background from category theory, which will be the language in which we express most of the results. Any elementary text in category theory will cover these topics among other basic concepts, in this case we used [4] and [1, App. A].

**Definition 1.1.** Let $\mathcal{C}$ be a category. A functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be *representable* if there is an object $A \in \mathcal{C}$ such that $F$ is naturally isomorphic to the hom functor $\mathrm{Hom}(A, -)$. If $F$ is instead a functor $\mathcal{C} \to \mathbf{Grp}$ then we say $F$ is representable if its composition with the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is representable.

Representing objects are unique (up to isomorphism), which is a consequence of the following more general result.

**Lemma 1.2** (Yoneda). *Let $E, F : \mathcal{C} \to \mathbf{Set}$ be functors and assume that $E$ is representable with representing object $A \in \mathcal{C}$. Then there is a one-to-one correspondence between natural transformations from $E$ to $F$ and elements of $F(A)$.*

*In particular, if $F$ is also representable with representing object $B$, then natural transformations from $E$ to $F$ correspond to morphisms $B \to A$ and composition of natural transformations corresponds to composition of these morphisms in reverse order.*

*Proof.* We proceed in the "naïve" way. Let $\Phi : E \to F$ be a natural transformation. Since we want an element of $F(A)$ it makes sense to take the image of some distinguished element of $E(A)$ under $\Phi^A : E(A) \to F(A)$. The only element we know for sure exists is $id_A \in \mathrm{Hom}(A, A) = E(A)$. Therefore to $\Phi$ we assign $\Phi^A(id_A) \in F(A)$.

Conversely, given an element $x \in F(A)$ we are going to construct a natural transformation $\Phi : E \to F$. To have a bijection we must impose $\Phi^A(id_A) = x$. Luckily this already fixes the whole natural transformation because the following diagram should commute for any $C \in \mathcal{C}$ by the naturality of $\Phi$:

$$\begin{array}{ccc} E(A) & \xrightarrow{E(f)} & E(C) \\ \Phi^A \downarrow & & \downarrow \Phi^C \\ F(A) & \xrightarrow{F(f)} & F(C) \end{array} .$$

Indeed, for any element $f \in E(C) = Hom(A, C)$ we have $f = f \circ id_A = E(f)(id_A)$ because of how the representable functor $E$ is defined. On the other hand we have fixed $\Phi^A(id_A) = x$. Finally $F(f)$ is an already defined map. Hence

$$\Phi^C(f) = \Phi^C(E(f)(id_A)) = F(f)(\Phi^A(id_A)) = F(f)(x).$$

Now one can easily check that this is indeed a natural transformation.

In the particular case that $F$ is represented by an object $B$ the function $F(f)$ is left composition with $f$, and the element $x$ is a function $x \in F(A) = \mathrm{Hom}(B, A)$. Therefore we can explicitly give $F(f)(x) = f \circ x \in \mathrm{Hom}(C, C) = F(C)$. In other words: in the case that both functors are representable, the natural transformation is given by right composition with the function $x : B \to A$. □

We are ready to define our main object of study. We denote by $\mathbf{Alg}_k$ the category of unital, associative, commutative algebras over a unital, associative, commutative ring $k$. Later on we will fix our attention in the case that $k$ is a field.

**Definition 1.3.** Let $k$ be a ring. An *affine group scheme* over $k$ is a representable functor $\mathbf{G} : \mathbf{Alg}_k \to \mathbf{Grp}$. The representing object will be denoted by $k[\mathbf{G}]$. If $k[\mathbf{G}]$ is finitely generated as a $k$-algebra we say that $\mathbf{G}$ is an *(affine) algebraic group*.

This definition seems admittedly dry and unmotivated. Let us step back a little bit and see where it can come from. All of the rings considered here will unital, associative, commutative rings (i.e. $\mathbb{Z}$-algebras). We follow the lines of [8, Chap. 1].

One of the most fruitful problems in the history of mathematics has been studying systems of polynomial equations with some restrictions on the coefficients and the allowed solution set. It is the case that for some equations the solution set admits a natural group structure. Perhaps the clearest example of this is the general linear group $\mathbf{GL}_n(R)$. If we fix a ring $R$, we may regard elements of $R^{n^2}$ as matrices, and in this case we know that the determinant is a polynomial expression in the coefficients of the matrix. Requiring the matrix $M \in R^{n^2}$ to be invertible is equivalent to requesting its determinant to be invertible in $R$, which in turn is the same as asking that the entries of $M$ be the solution of the polynomial equation $\det(M)Y = 1$, where $Y$ is some dummy indeterminate. Notice that this construction can be made for any ring $R$, and since the condition to be in the solution set is given by a polynomial with integer coefficients, a ring homomorphism $\varphi : R \to S$ takes solutions to solutions and thus induces a map from one solution set to another, i.e. a group homomorphism $\mathbf{GL}_n(R) \to \mathbf{GL}_n(S)$.

To get some other (albeit trivial) examples of this kind of construction notice that we may regard the underlying additive group $(R, +)$ of any ring $R$ as the solution set of an empty system of equations, and that a ring homomorphism $\varphi : R \to S$ induces a group homomorphism $(R, +) \to (S, +)$. Similarly taking the multiplicative group of units $(R^\times, \cdot)$ also comes from an equation, $XY = 1$, and respects homomorphisms (this is just the particular case $\mathbf{GL}_1(R)$ from the previous discussion). Since these examples will be important later on, we give them a notation similar to the one of the general linear group: define $\mathbf{G}_a(R) = (R, +)$ and $\mathbf{G}_m(R) = (R^\times, \cdot)$.

Generalizing slightly we stop requesting the coefficients of our equations to be integers and instead we fix a "field of coefficients" $k$ (it should be noted that some of the results ahead hold in more generality, namely for algebras over rings, but to avoid confusion we will be working over a field $k$). Hence it will only make sense to look for solutions of this polynomials over $k$-algebras instead of general rings, and the maps between them will need to be homomorphisms of $k$-algebras. There is however one complication that we have overlooked: seemingly different equations might give rise to the same groups. To solve this we move to the functorial picture.

It is a well known fact and easy to check that if we have a family of equations $(f_i)_{i \in I}$ with indeterminates $(X_j)_{j \in J}$ and coefficients in $k$, the solutions for these over a $k$-algebra $R$ correspond to $k$-algebra homomorphisms $A \to R$, where $A$ is the quotient algebra $k[(X_j)_{j \in J}]/(f_i)_{i \in I}$. For example elements of $\mathbf{G}_a(R)$ correspond to maps $k[X] \to R$, and elements of $\mathbf{G}_m(R)$ correspond to maps $k[X, Y]/(XY - 1) \to R$. This leads us to identify $\mathrm{Hom}(A, R)$ with the solution set. If as before we request that our equations have a solution set with a natural group structure (in the sense that algebra homomorphisms induce group homomorphisms between the solution sets) then we find out that we have been talking about the functor $\mathbf{Alg}_k \to \mathbf{Grp}$ represented by $A$. Since any algebra may be expressed as the quotient of some polynomial ring (maybe in infinitely many variables) by taking generators, we conclude that studying solution sets of polynomial equations with a natural group structure corresponds exactly to studying representable functors. Hence our definition of affine group scheme is motivated.

## 1.2 Hopf algebras

The Yoneda Lemma tells us that a representable functor is completely determined by its representing object and vice versa. It makes sense then to expect all of the information of an affine group scheme $\mathbf{G}$ to be encoded in $k[\mathbf{G}]$ in some way. This is indeed the case, and to see that we need the following proposition, the proof of which can be found in [5, 1.2.4, 1.2.5].

**Proposition 1.4.** *Let $\mathbf{G}, \mathbf{H}$ be affine group schemes. The functor $\mathbf{G}\times\mathbf{H}$ sending each $k$-algebra $R$ to the direct product $\mathbf{G}(R)\times\mathbf{H}(R)$ is represented by $k[\mathbf{G}\times\mathbf{H}] = k[\mathbf{G}]\otimes_k k[\mathbf{H}]$. The functor that sends all $k$-algebras to a trivial group $\{e\}$ is represented by $k$.*

Now comes a key observation. The fact that $\mathbf{G}$ is an affine group scheme means among other things that for any $k$-algebra homomorphism $\varphi : R \to S$ the induced map $\mathbf{G}(\varphi) : \mathbf{G}(R) \to \mathbf{G}(S)$ is a group homomorphism, so it commutes with multiplication. This is reflected in the following diagram

$$\begin{array}{ccc} \mathbf{G}(R)\times\mathbf{G}(R) & \xrightarrow{\text{mult}^R} & \mathbf{G}(R) \\ {\scriptstyle(\mathbf{G}\times\mathbf{G})(\varphi)}\downarrow & & \downarrow{\scriptstyle\mathbf{G}(\varphi)} \\ \mathbf{G}(S)\times\mathbf{G}(S) & \xrightarrow{\text{mult}^S} & \mathbf{G}(S). \end{array}$$

Since this works for *any* ring homomorphism, we deduce multiplication is a natural transformation from $\mathbf{G}\times\mathbf{G}$ to $\mathbf{G}$. Similarly the map "pointing" at the identity element of each group unit : $\{e\} \to \mathbf{G}$ and the inverse map inv : $\mathbf{G} \to \mathbf{G}$ are natural maps. The definition of group tells us that these maps with some relations between them are all we need to have a group structure. Hence a group functor is just a set functor together with these three natural transformations mult : $\mathbf{G}\times\mathbf{G} \to \mathbf{G}$, unit : $\{e\} \to \mathbf{G}$ and inv : $\mathbf{G} \to \mathbf{G}$ satisfying some relations given by the group axioms.

By the Yoneda Lemma, these natural transformations correspond to morphisms between the representing objects. If we denote $A := k[\mathbf{G}]$ these are

- $\Delta : A \to A\otimes_k A$, which is called *comultiplication* and corresponds to mult.
- $\epsilon : A \to k$, which is called *counit* or *augmentation* and corresponds to unit.
- $S : A \to A$, which is called *coinverse* or *antipode* and corresponds to inv.

The Lemma also says that composition behaves well when passing from natural transformation to algebra homomorphisms and vice versa. Hence the commutative diagrams expressing the group axioms turn into these ones for algebras

$$\begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{id_A\otimes\Delta} & A\otimes A \\ {\scriptstyle\Delta\otimes id_A}\uparrow & & \uparrow{\scriptstyle\Delta} \\ A\otimes A & \xleftarrow{\Delta} & A \end{array} \tag{1.1}$$

$$\begin{array}{ccccc} k\otimes A & \xleftarrow{\epsilon\otimes id_A} & A\otimes A & \xrightarrow{id_A\otimes\epsilon} & A\otimes k \\ {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\Delta} & & \uparrow{\scriptstyle\cong} \\ A & \xleftarrow{id_A} & A & \xrightarrow{id_A} & A \end{array} \tag{1.2}$$

$$\begin{array}{ccccc} A & \xleftarrow{(S,id_A)} & A \otimes A & \xrightarrow{(id_A,S)} & A \\ \uparrow & & \uparrow{\scriptstyle \Delta} & & \uparrow \\ k & \xleftarrow{\epsilon} & A & \xrightarrow{\epsilon} & k \end{array} \tag{1.3}$$

**Definition 1.5.** A *commutative Hopf algebra over* $k$ is a unital, associative, commutative $k$-algebra together with homomorphisms $\Delta : A \to A \otimes_k A$, $\epsilon : A \to k$ and $S : A \to A$ satisfying equations (1.1), (1.2) and (1.3).

What we have seen is that the representing object of an affine group scheme carries a commutative Hopf algebra structure. Conversely, one can prove (see [5, Th. 1.3.4] or [1, App. A]) that a Hopf algebra structure on the representing object of a set functor induces a group structure on each Hom set via

$$(fg)(a) := \sum f(a_{(1)})g(a_{(2)}) \quad \text{for all } a \in A;\ f, g \in \operatorname{Hom}(A, R).$$

Here we have used Sweedler's notation, which denotes $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. Written as composition this would mean that we define

$$fg := (f, g) \circ \Delta.$$

Notice that what we are doing is just recovering the natural transformation mult from the element $\Delta \in \mathbf{G}(A \otimes A)$ as in the proof of the Yoneda Lemma.

In this same line a morphism of affine group schemes $\Phi : \mathbf{G} \to \mathbf{H}$ induces a homomorphism of algebras $k[\mathbf{H}] \to k[\mathbf{G}]$ by the Yoneda Lemma. Summarizing our discussion, we have proved the following theorem.

**Theorem 1.6.** *There is an anti-equivalence of categories between the category of affine group schemes over $k$ and the category of commutative Hopf algebras over $k$. We call it an* anti-*equivalence because the arrows get reversed (see [7, 1.3.11]).*

As an example (from [1, Ex. A.10]) the Hopf algebra structure on $k[\mathbf{G}_m] = k[X, Y]/(XY - 1) \cong k[X, X^{-1}]$ is given by

$$\Delta(X) = X \otimes X, \quad \epsilon(X) = 1, \quad S(X) = X^{-1}.$$

Indeed, if $f, g \in \operatorname{Hom}(k[X, X^{-1}], R), y = f(X), z = g(X)$, then

$$(fg)(X) = (f, g)(\Delta(X)) = (f, g)(X \otimes X) = f(X)g(X) = yz.$$

Which proves that the product we have defined on $\operatorname{Hom}(k[X, X^{-1}], R)$ corresponds to the product from $\mathbf{G}_m(R) = R^\times$. Furthermore, since the multiplication completely determines the structure of any group (i.e. it also fixes the neutral element and the inverses) it follows that comultiplication completely determines the Hopf algebra structure:

$$\Delta \rightsquigarrow \text{mult} \rightsquigarrow \text{unit}, \text{inv} \rightsquigarrow \epsilon, S. \tag{1.4}$$

This is a particular instance of the more general observation that any statement that is true for all groups, i.e. can be deduced from the group axioms, has an analog true statement for Hopf algebras. More examples of this are explored in [8, Sect. 1.5].

Theorem 1.6 makes it clear that a good understanding of Hopf algebras provides a lot of insight into affine group schemes. Therefore we proceed to study the structure of Hopf algebras and affine group schemes as one would do with any other algebraic structure. The simultaneous study of both allows us to see the connections. For example, as affine group schemes are just functors the obvious definition of a morphism $\Phi : \mathbf{G} \to \mathbf{H}$ is that it is a natural transformation, and since the components of $\Phi$ are morphisms in **Grp** they must commute with mult. If we denote $A = k[\mathbf{G}], B = k[\mathbf{H}]$ and write the corresponding diagram for Hopf algebras right next to the diagram for affine group schemes, we have

$$\begin{array}{ccc} \mathbf{G}\times\mathbf{G} & \xrightarrow{\Phi\times\Phi} & \mathbf{H}\times\mathbf{H} \\ {\scriptstyle \mathrm{mult}_{\mathbf{G}}}\downarrow & & \downarrow{\scriptstyle \mathrm{mult}_{\mathbf{H}}} \\ \mathbf{G} & \xrightarrow{\Phi} & \mathbf{H} \end{array} \qquad \begin{array}{ccc} A\otimes A & \xleftarrow{f\otimes f} & B\otimes B \\ {\scriptstyle \Delta_A}\uparrow & & \uparrow{\scriptstyle \Delta_B} \\ A & \xleftarrow{f} & B \end{array}$$

where $f : B \to A$ is the comorphism corresponding to $\Phi$ by the Yoneda Lemma. This motivates the following definition.

**Definition 1.7.** Let $A, B$ be commutative Hopf algebras with comultiplication given by $\Delta_A, \Delta_B$ respectively. A *homomorphism of Hopf algebras* is a $k$-algebra homomorphism $f : B \to A$ satisfying the additional condition that

$$\Delta_A \circ f = (f \otimes f) \circ \Delta_B.$$

Once again, following (1.4) reveals that the condition to be a Hopf algebra homomorphism implies that $f$ also commutes with $\epsilon$ and $S$.

Since the components of $\Phi$ are given by precomposition with $f$, if $f$ is surjective then all of the components of $\Phi$ are injective. In this case we say that $\Phi$ is a *closed embedding*, and these are precisely the monomorphisms in the category of affine group schemes.

**Definition 1.8.** Let $\mathbf{G}, \mathbf{H}$ be affine group schemes. We will say that $\mathbf{H}$ is a *subgroupscheme* of $\mathbf{G}$ if, for any $k$-algebra $R$, the group $\mathbf{H}(R)$ is a subgroup of $\mathbf{G}(R)$, and the inclusions $\mathbf{H}(R) \hookrightarrow \mathbf{G}(R)$ form a natural map $\mathbf{H} \to \mathbf{G}$.

Notice that in this case the inclusion $\mathbf{H} \hookrightarrow \mathbf{G}$ is a closed embedding and therefore the corresponding $B \leftarrow A$ is a surjection, so $B$ is isomorphic to a quotient of $A$. The idea is that $\mathbf{H}$ is defined by the equations of $\mathbf{G}$ and some additional ones. However, we have already said that choosing equations at random does not result in an affine group scheme, and similarly quotienting by any ideal $I$ of $A$ will not work. Following a reasoning analogous to the one for the homomorphisms we find the following definition.

**Definition 1.9.** Let $A$ be a commutative Hopf algebra. A *Hopf ideal* is an ideal $I$ of $A$ as a ring, satisfying the conditions

$$\Delta(I) \subset I \otimes A + A \otimes I, \quad \epsilon(I) = 0, \quad S(I) \subset (I).$$

If $\mathbf{G}$ is the affine group scheme represented by $A$, the subgroupschemes of $\mathbf{G}$ are in correspondence with the Hopf ideals of $A$.

The largest Hopf ideal of any Hopf algebra $A$ is $\ker(\epsilon)$, known as the *augmentation ideal*. Clearly $A/\ker(\epsilon) \cong k$, so the corresponding subgroupscheme is the trivial one. Dually to the notion of closed embedding we may say that a morphism $\Phi : \mathbf{G} \to \mathbf{H}$ is a *quotient map* if the corresponding comorphism $A \leftarrow B$ is injective. Although these are the epimorphisms in the category of affine group schemes, it is *not* true that the components of a quotient map are surjective, as the following example shows.

**Example 1.10.** For any $n \in \mathbb{Z}$ and any $k$-algebra $R$ we may define a group homomorphism $R^\times \to R^\times : x \mapsto x^n$. This is natural by the definition of group homomorphism, so we get a morphism of affine group schemes $[n] : \mathbf{G}_m \to \mathbf{G}_m$. The corresponding comorphism is $k[X, X^{-1}] \to k[X, X^{-1}] : X \to X^n$, which is clearly injective. Hence $[n]$ is a quotient map. However already in $k[X, X^{-1}]$ the map $x \mapsto x^n$ is not surjective, as not every Laurent polynomial has an $n$-th root.

Any homomorphism of Hopf algebras $f : B \to A$ may be decomposed as $B \to B/I \to A$ where $I = \ker f$, where the first map is surjective and the second map is injective. Consequently any morphism of affine group schemes $\Phi : \mathbf{G} \to \mathbf{H}$ can be decomposed as $\mathbf{G} \to \mathbf{H}_0 \to \mathbf{H}$, where the first morphism is a quotient map and the second one is a closed embedding. In this case $\mathbf{H}_0$ is the subgroupscheme of $\mathbf{H}$ determined by the Hopf ideal $I = \ker f$. We call $\mathbf{H}_0$ the *image* of $\Phi$, and notice that in general $\Phi^R(\mathbf{G}(R)) \subsetneq \mathbf{H}_0(R)$.

On the other hand it is not hard to see that the functor defined by $\mathbf{N}(R) := \ker[\mathbf{G}(R) \xrightarrow{\Phi^R} \mathbf{H}(R)]$ is representable, and its representing object is $A \otimes_B k \cong A/(AI_B)$, where $I_B$ is the augmentation ideal. This shows that $\mathbf{N}$ is a subgroupscheme of $\mathbf{G}$, called the *kernel* of $\Phi$. This concepts of kernel and image allow us to talk about exact sequences of affine group schemes, which we will do in Chapter 3.

## 1.3 Diagonalizable group schemes

Now we turn to a concept that will be important in the following chapters: the notion of *diagonalizable* affine group scheme. We need a few definitions and results for that, which we state without proof. For a proof of those we refer to [8].

**Definition 1.11.** Let $A$ be a commutative Hopf algebra. An invertible element $a \in A$ is said to be *group-like* if $\Delta(a) = a \otimes a$.

**Proposition 1.12.** *A group-like element $a \in A$ always has $\epsilon(a) = 1$, $S(a) = a^{-1}$. The group-like elements of $A$ form a subgroup of $A^\times$, called the* group of characters. *Group-like elements are linearly independent.*

**Proposition 1.13.** *Let $G$ be a group and $kG$ be the corresponding group algebra. We may define a Hopf algebra structure on $kG$ by setting $\Delta(g) = g \otimes g$ for all $g \in G$. The group of characters of $kG$ is then precisely $G$. If $G$ is abelian then $kG$ is a commutative Hopf algebra, and therefore it corresponds to an affine group scheme, which we denote by $G^D$.*

The notation $G^D$ will be explained shortly. At this point we are ready to state the main definition of this section.

**Definition 1.14.** An affine group scheme is said to be *diagonalizable* if it is isomorphic to $G^D$ for some abelian group $G$. Its representing algebra is then isomorphic to $kG$, so $G^D$ is an algebraic group if and only if $G$ is finitely generated as a group.

Diagonalizable algebraic groups have a particularly nice structure, which follows from the structure theorem for finitely generated abelian groups. The proof comes from [8].

**Theorem 1.15.** *Let $\mathbf{G}$ be a diagonalizable algebraic group. Then $\mathbf{G}$ is isomorphic to a finite product of copies of $\mathbf{G}_m$ and various $\boldsymbol{\mu}_n$, where $\boldsymbol{\mu}_n$ is the affine group scheme represented by $k[X]/(X^n - 1)$, i.e. $\boldsymbol{\mu}_n(R) = \{x \in R \mid x^n = 1\}$.*

*Proof.* Since $\mathbf{G}$ is an algebraic group, its representing algebra is $kM$ for some finitely generated abelian group $M$. Recall that $k(M_1 \times M_2) \cong kM_1 \otimes kM_2$, and that the tensor product of representatives represents the cartesian product of affine group schemes. Hence we may assume without loss of generality (by the structure theorem for finitely generated abelian groups) that $M$ is either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$.

If $M = \mathbb{Z}$ then we have $k\mathbb{Z} \cong k[X, X^{-1}]$, where the isomorphism sends the 1 from $\mathbb{Z}$ to $X$ and $\Delta(1) = 1 \otimes 1$ implies $\Delta(X) = X \otimes X$. Therefore $\mathbf{G} \cong \mathbf{G}_m$. Similarly, if $M = \mathbb{Z}/n\mathbb{Z}$ then $k(\mathbb{Z}/n\mathbb{Z}) \cong k[X]/(X^n - 1)$ and thus $\mathbf{G} \cong \boldsymbol{\mu}_n$. □

To explain the notation $G^D$ we introduce the notions of constant group scheme and Cartier duality. It should be noted that Cartier duality holds in more generality than we present here. We follow [8, Sect. 2.3, 2.4].

Let us begin with constant group schemes. Apart from the trivial group scheme, we would like to know which other "simple" affine group schemes there are. A first idea could be to fix a finite group $G$ and assign it to every $k$-algebra. However it turns out that such a functor is only representable if $G$ has one element because of cardinality reasons (see [8, Exercise 1.1]). We can achieve something similar, namely assigning a group of our choice to all algebras with no idempotents except 0 and 1. Let $A = k^G$ be the set of functions from $G$ to $k$ endowed with pointwise addition and multiplication, and let $e_\sigma$ be the indicator function of the element $\sigma \in G$. Clearly $\{e_\sigma\}_{\sigma\in G}$ is a basis of $A$, and hence $A$ as a ring is just the direct product of $|G|$ copies of $k$. The elements of the basis satisfy

$$e_\sigma^2 = e_\sigma, \quad e_\sigma e_\tau = 0 \text{ if } \sigma \neq \tau, \quad \sum_{\sigma\in G} e_\sigma = 1.$$

If $R$ is a $k$-algebra with no idempotents other than 0 or 1 this means that any homomorphism $\varphi : A \to R$ must send one $e_\sigma$ to 1 and the others to 0, and this characterizes the homomorphism completely. Hence these homomorphisms are in one-to-one correspondence with the elements of $G$.

Now we want to define a Hopf algebra structure on $A$ that induces on $\mathrm{Hom}(A, R)$ the same group structure that $G$ has. To do this define

$$\Delta(e_\rho) = \sum_{\rho=\sigma\tau} (e_\sigma \otimes e_\tau).$$

Letting $f, g \in \mathrm{Hom}(A, R)$ correspond to $\nu_1, \nu_2 \in G$ respectively, we have

$$(fg)(e_\rho) = (f, g)(\Delta(e_\rho)) = \sum_{\rho=\sigma\tau} f(e_\sigma)g(e_\tau).$$

This expression is 1 if $\rho = \nu_1\nu_2$ and 0 otherwise, so indeed the product homomorphism $fg$ corresponds to the product in the group $\nu_1\nu_2$. Setting $S(e_\sigma) = e_{(\sigma^{-1})}$ and $\epsilon(e_\sigma) = 1$ if $\sigma$ is the neutral element of $G$ and 0 otherwise, we get a Hopf algebra structure on A. The affine group scheme it represents is called the *constant group scheme* for $G$ and will be denoted by $\mathsf{G}$.

Now we turn to Cartier duality. One can see there is a symmetry in the definition of a commutative Hopf algebra. This leads to the following theorem, the proof of which are only computations. To see some of them one can check [8]. Denote a Hopf algebra with its operations by $(A, \Delta, \epsilon, S, m, u)$, where $m : A \otimes A \to A$ is the multiplication of $A$ as a ring and $u : k \to A$ gives the $k$-algebra structure map.

**Theorem 1.16** (Cartier duality)**.** *Let* $\mathbf{G}$ *be a* finite *abelian group scheme, meaning that* $k[\mathbf{G}]$ *is a finite-dimensional* $k$*-vector space and* $\mathbf{G}(R)$ *is an abelian group for all* $k$*-algebras* $R$*. Denote by* $k[\mathbf{G}]^*$ *the dual vector space. Then* $k[\mathbf{G}]$ *is a finite-dimensional, commutative, cocommutative Hopf algebra. Furthermore* $(k[\mathbf{G}]^*, m^*, u^*, S^*, \Delta^*, \epsilon^*)$ *is also a finite-dimensional, commutative, cocommutative Hopf algebra, and therefore it represents a finite abelian group scheme, which we call its* Cartier dual *and denote by* $\mathbf{G}^D$*.*

This explains the notation $G^D$ for the finite diagonalizable group schemes. Indeed, if $G$ is a finitely generated abelian group, then we see that the dual algebra of $k^G$ is $kG$. Therefore the Cartier dual of the constant group scheme $G$ is just the diagonalizable group scheme $G^D$.

# Chapter 2

# Gradings on algebras

## 2.1 Definitions

In this chapter we will be working over a fixed field $k$. All vector spaces, linear maps, tensor products, algebras, etc. will be assumed over $k$ unless indicated otherwise. Almost all of the definitions and propositions here have been extracted from [1, Chap. 1].

The concept of grading is a generalization of the decomposition of polynomials into monomials of different degrees. Given a $k$-vector space $V$ we decompose it into a direct sum of "labeled" subspaces. As in any direct sum, not all elements get a label, but the labeled elements are enough to construct any other element as a linear combination of labeled elements. In the case of polynomials we know that we may decompose any polynomial as a sum of monomials and to each monomial we may attach a label, namely its degree. Here is the formal definition.

**Definition 2.1.** Let $V$ be a $k$-vector space and $S$ be a set. An *$S$-grading* $\Gamma$ on $V$ is any decomposition of $V$ into a direct sum of subspaces indexed by $S$,

$$\Gamma : \ V = \bigoplus_{s \in S} V_s.$$

The *support* of $\Gamma$ is the set $\operatorname{Supp} \Gamma \coloneqq \{s \in S \mid V_s \neq 0\}$. The subspace $V_s$ will be called the *homogeneous component* of degree $s$. If a grading $\Gamma$ is fixed, then $V$ will be referred to as a *graded vector space.*

If our vector space $\mathcal{A}$ is an algebra over $k$ (not necessarily associative) it seems appropiate that the labels should have some relation with the multiplication of the algebra. What we require is that the product of any two homogeneous components is contained in a homogeneous component, just like multiplying a monomial of degree 2 and a monomial of degree 3 always gives a monomial of degree 5.

**Definition 2.2.** Let $\mathcal{A}$ be a $k$-algebra and $S$ be a set. An *$S$-grading* $\Gamma$ on $\mathcal{A}$ is an $S$-grading $\Gamma$ of $\mathcal{A}$ as a vector space with the additional condition that for any $s_1, s_2 \in S$ there exists an $s_3 \in S$ such that

$$\mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}.$$

For the following discussion it will be useful to assume that $\operatorname{Supp} \Gamma = S$, i.e. that none of the homogeneous components is trivial. Notice that if $\mathcal{A}_{s_1}\mathcal{A}_{s_2} \neq 0$ then the element $s_3 \in S$ is uniquely determined, and therefore the algebra structure on $\mathcal{A}$ defines a partial operation on $S$ via

$$s_1 \cdot s_2 \coloneqq s_3 \qquad \text{whenever } 0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}. \tag{2.1}$$

A natural question to ask now is whether $S$ with its operation forms some kind of algebraic structure, or can be embedded in one. Going back to the polynomials, we know that the degree of the product of two monomials is the sum of the degrees of the monomials. The support of this grading are the natural numbers with 0. Therefore we may embed the support into the abelian group $\mathbb{Z}$. For our purpose we are only interested in the case where $S$ can be embedded in some (often abelian) group $G$. The precise definition is the following.

**Definition 2.3.** Let $G$ be a group. We will say that $\Gamma$ is *realized* as a $G$-grading if $G$ is a group containing $S$, the subspaces

$$\mathcal{A}_g := \begin{cases} \mathcal{A}_s & \text{if } g = s \in S, \\ 0 & \text{if } g \in G \setminus S; \end{cases}$$

satisfy $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh}$ for all $g, h \in G$, and $S$ generates $G$. If such a group exists we will say that $\Gamma$ is a *group grading*.

It has been shown that not all gradings can be realized as a group grading. In fact [1, Ex 1.9, 1.10] shows that there are cases where it isn't even possible to embed $S$ into a semigroup. On the other end, [1, Prop 1.12] shows that in some cases focusing on abelian groups is not that much of a restriction. When the grading $\Gamma$ can be realized as a $G$-grading for some group $G$, the group is in general not unique, as the following example shows (extracted from [1, Ex. 1.13]).

**Example 2.4.** Let $\mathcal{J}_i$ $(i = 1, 2)$ be Lie algebras with basis $\{x_i, y_i, h_i\}$ and multiplication given by

$$[h_i, x_i] = x_i, \quad [h_i, y_i] = -y_i, \quad [x_i, y_i] = h_i.$$

Consider the algebra $\mathcal{L} = \mathcal{J}_1 \oplus \mathcal{J}_2$ with grading

$$\Gamma : \mathcal{L} = \mathcal{L}_{s_1} \oplus \mathcal{L}_{s_2} \oplus \mathcal{L}_{s_3} \oplus \mathcal{L}_{s_4}.$$

where $\mathcal{L}_{s_1} = \text{span}\{h_1, h_2\}, \mathcal{L}_{s_2} = \text{span}\{x_2, y_2\}, \mathcal{L}_{s_3} = \text{span}\{x_1\}, \mathcal{L}_{s_4} = \text{span}\{y_1\}$. Then $\Gamma$ can be realized as a grading by the cyclic group $\langle g \rangle_6$ with $s_1 = e, s_2 = g^3, s_3 = g^2, s_4 = g^4$ and also as a grading by the symmetric group $S_3$ with $s_1 = e, s_2 = (12), s_3 = (123), s_4 = (132)$.

## 2.2 Isomorphisms, equivalences and weak isomorphisms

We would like to be able to have some notion of isomorphism of gradings to be able to tell when two gradings are the same. To get this we begin with not one but in fact two concepts of morphism, which will give rise to two different versions of isomorphism. We do this because one has to distinguish between *graded vector spaces* and *G-graded vector spaces*, that is, it is important to say whether the grading group is part of the definition or not.

**Definition 2.5.** Let $V$ be an $S$-graded vector space and let $W$ be a $T$-graded vector space. A linear map $f : V \to W$ is said to be *graded* if for any $s \in S$ there exists $t \in T$ such that $f(V_s) \subset W_t$. Clearly, if $f(V_s) \neq 0$, then $t$ is uniquely determined.

An *equivalence of graded vector spaces* is a linear isomorphism $f : V \to W$ such that both $f$ and $f^{-1}$ are graded maps. If $\Gamma : \mathcal{A} = \bigoplus_{s \in S} \mathcal{A}_s$ and $\Gamma' : \mathcal{B} = \bigoplus_{t \in T} \mathcal{B}_t$ are gradings on algebras (with supports $S$ and $T$) we say $\Gamma$ and $\Gamma'$ are *equivalent* if there exists an *equivalence of graded algebras* $\varphi : \mathcal{A} \to \mathcal{B}$, i.e. an isomorphism of algebras that is also an equivalence of graded vector spaces. It determines a bijection $\alpha : S \to T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$.

**Definition 2.6.** Let $V$ and $W$ be $S$-graded vector spaces. A linear map $f : V \to W$ is said to be a *homomorphism of S-graded vector spaces* if for all $s \in S$, we have $f(V_s) \subset W_s$.

Let $G$ be a group. Two $G$-graded algebras $\mathcal{A}, \mathcal{B}$ are said to be *isomorphic* if there exists an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

Another important concept will be that of the universal group of a group grading. This will allow us to go from realizations of gradings to group homomorphisms between the universal groups, where we have another concept of isomorphism.

**Definition 2.7.** Let $\Gamma$ be a grading on an algebra $\mathcal{A}$. Suppose that $\Gamma$ admits a realization as a $G_0$-grading for some group $G_0$ (i.e. $\Gamma$ is a group grading). We say that $G_0$ is a *universal group of* $\Gamma$ if for any other realization of $\Gamma$ as a $G$-grading, there exists a unique group homomorphism $G_0 \to G$ which restricts to the identity on $\operatorname{Supp}\Gamma$.

Using the standard trick for objects defined via universal properties one proves that universal groups are unique up to unique isomorphism, provided they exist. Their existance is the first part of the next proposition. The proof amounts to considering the free group on $\operatorname{Supp}\Gamma$ subject to the relations imposed by Eq. (2.1) and doing the necessary comprobations. This proposition also highlights the importance of the universal group that we mentioned previously. The complete proof is in [1, Prop. 1.18].

**Proposition 2.8.** *Let $\Gamma$ be a group grading on an algebra $\mathcal{A}$. Then there exists a universal group for $\Gamma$, which we denote by $U(\Gamma)$. Two group gradings, $\Gamma$ on $\mathcal{A}$ and $\Gamma'$ on $\mathcal{B}$, are equivalent if and only if there exist an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and a group isomorphism $\alpha : U(\Gamma) \to U(\Gamma')$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_{\alpha(g)}$ for all $g \in U(\Gamma)$.*

As a brief remark, the construction of $U(\Gamma)$ as the free group with some relations is always possible, regardless of whether $\Gamma$ is a group grading or not. The proof of the proposition shows that $\Gamma$ is as group grading if and only if the canonical map $\operatorname{Supp}\Gamma \to U(\Gamma)$ is injective.

**Corollary 2.9.** *For a given group grading $\Gamma$ and a group $G$, the realizations of $\Gamma$ as a $G$-grading are in one-to-one correspondence with the epimorphisms $U(\Gamma) \to G$ that are injective on* $\operatorname{Supp}\Gamma$.

There is another notion of isomorphism that will appear in our main theorems. To define it we need to introduce the concept of *change of group*.

Let $\alpha : G \to H$ be a group homomorphism and let $\Gamma : V = \bigoplus_{g\in G} V_g$ be a $G$-grading on the vector space $V$. The map $\alpha$ allows us to transform the $G$-grading into an $H$-grading in a natural way as follows. Define for each $h \in H$

$$V'_h := \bigoplus_{g\in G \,:\, \alpha(g)=h} V_g.$$

These subspaces clearly form an $H$-grading of $V$ which we denote by ${}^{\alpha}\Gamma : V = \bigoplus_{h\in H} V'_h$. If $V = \mathcal{A}$ is an algebra and $\Gamma$ is an algebra grading, then ${}^{\alpha}\Gamma$ is also an algebra grading. Indeed, if $\mathcal{A}'_{h_1} \neq 0 \neq \mathcal{A}'_{h_2}$, then $h_1$ and $h_2$ are in the image of $\alpha$ and

$$\mathcal{A}'_{h_1}\mathcal{A}'_{h_2} \subseteq \sum_{\substack{\alpha(g_1)=h_1\\ \alpha(g_2)=h_2}} \mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \sum_{\substack{\alpha(g_1)=h_1\\ \alpha(g_2)=h_2}} \mathcal{A}_{g_1g_2} \subseteq \sum_{\alpha(g)=h_1h_2} \mathcal{A}_g = \mathcal{A}'_{h_1h_2}.$$

**Definition 2.10.** Let $V$ be a $G$-graded vector space and $W$ be an $H$-graded vector space. We will say that $V$ and $W$ are *weakly isomorphic* if there exist a linear isomorphism $\varphi : V \to W$ and a group isomorphism $\alpha : G \to H$ such that $\varphi(V_g) = W_{\alpha(g)}$ for all $g \in G$. The same definition applies for algebras and algebra gradings by requiring the linear isomorphism $\varphi$ to be an algebra isomorphism.

Notice that this definition just says that two gradings $\Gamma$ and $\Gamma'$ are weakly isomorphic if and only if $\Gamma'$ is isomorphic to ${}^{\alpha}\Gamma$ for some group isomorphism $\alpha$. The definition also gives us a way to rephrase Proposition 2.8: two algebra gradings are equivalent if and only if they are weakly isomorphic when considered as gradings over their respective universal groups.

## 2.3 Gradings, comodules and linear representations

We want to bring back the concepts of Chapter 1 to get some powerful classification results. In order to do this we need to relate gradings with some affine group schemes. To do this we introduce an new object that will serve as a bridge between the two. Once again there is a more general definition, but we restrict to the case which is of interest to us right now.

**Definition 2.11.** Let $A$ be a commutative Hopf algebra. A *comodule over* $A$ (or $A$-*comodule*) is a vector space $V$ with a linear map $\rho : V \to V \otimes A$, called *coaction*, making the following diagrams commute

$$\begin{array}{ccc} V & \xrightarrow{\ \rho\ } & V \otimes A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho\otimes id_A} \\ V \otimes A & \xrightarrow{id_V\otimes\Delta} & V \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\ \rho\ } & V \otimes A \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle id_V\otimes\epsilon} \\ & & V \otimes k. \end{array}$$

Now we show how each vector space grading gives rise to a coaction. Let $G$ be an abelian group and let $\Gamma : V = \bigoplus_{g\in G} V_g$ be a $G$-grading on the vector space $V$. We have already seen that we can make $kG$ into a commutative Hopf algebra by setting $\Delta(g) = g \otimes g$ for all $g \in G$. We define a $kG$-comodule structure on $V$ via the coaction $\rho_\Gamma : V \to V \otimes kG$ defined by

$$\rho_\Gamma(v) \coloneqq v \otimes g \quad \text{for all} \quad g \in G,\ v \in V_g.$$

One easily checks that this map satisfies the conditions to be a coaction. Conversely, given a coaction $\rho : V \to V \otimes kG$ we define the following subspaces of $V$

$$V_g \coloneqq \{v \in V \mid \rho(v) = v \otimes g\} \quad \text{for all} \quad g \in G.$$

and now the comodule axioms imply that $V$ is the direct sum of the subspaces $V_g$. From the definition it is clear that the constructions are inverse to each other.

If the vector space is actually an algebra $\mathcal{A}$ with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, we can ask which condition should the coaction satisfy in order to have the vector space grading be an algebra grading. The condition $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh}$ translates to

$$\rho(ab) = ab \otimes gh \quad \text{for any} \quad a \in \mathcal{A}_g, b \in \mathcal{A}_h.$$

This is expressed in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\rho_{\mathcal{A}\otimes\mathcal{A}}} & \mathcal{A} \otimes \mathcal{A} \otimes kG \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu\otimes id_{kG}} \\ \mathcal{A} & \xrightarrow{\ \rho_{\mathcal{A}}\ } & \mathcal{A} \otimes kG. \end{array}$$

where $\rho_{\mathcal{A}}$ is the coaction of $\mathcal{A}$ and $\rho_{\mathcal{A}\otimes\mathcal{A}}$ is the natural coaction on $\mathcal{A}\otimes\mathcal{A}$ given by the composition

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes kG \otimes \mathcal{A} \otimes kG \cong \mathcal{A} \otimes \mathcal{A} \otimes kG \otimes kG \xrightarrow{id\otimes\text{mult}} \mathcal{A} \otimes \mathcal{A} \otimes kG.$$

We may read this diagram as saying that $\rho$ is a homomorphism of algebras or that $\mu$ is a morphism of $kG$-comodules (in the sense that it commutes with the respective coactions).

A remark that will be useful later on is the fact that a group homomorphism $\alpha : G \to H$ induces a Hopf algebra homomorphism $kG \to kH$, which we also denote by $\alpha$ by an abuse of notation. It is a straightforward computation to check that the coaction corresponding to the change of group is

$$\rho_{{}^\alpha\Gamma} = (id \otimes \alpha) \circ \rho_\Gamma. \tag{2.2}$$

On the other end of the correspondence we have the concept of a linear representation of an affine group scheme. If we fix a vector space $V$ the usual concept of a linear representation of a group $G$ on $V$ is just a linear action of $G$ on $V$, i.e. a group homomorphism $G \to GL_V$. In analogy with this we may define the following.

**Definition 2.12.** Fix a vector space $V$. We denote by $\mathbf{GL}_V$ the functor that assigns to each $k$-algebra $R$ the group of $R$-linear automorphisms of $V$ (by extension of scalars)

$$\mathbf{GL}_V(R) = \mathrm{Aut}_R(V \otimes R).$$

Notice that in principle this functor need not be representable, but if $V$ is finite-dimensional and we have a basis $\{v_1, \ldots, v_n\}$, then there is a natural isomorphism $\mathbf{GL}_V \cong \mathbf{GL}_n$, and we know this is an affine algebraic group.

**Definition 2.13.** Let $\mathbf{G}$ be an affine group scheme and $V$ be a vector space. A *linear representation* of $\mathbf{G}$ on $V$ is a morphism $\mathbf{G} \to \mathbf{GL}_V$.

The key observation is the following.

**Theorem 2.14.** *Linear representations of $\mathbf{G}$ on $V$ are in one-to-one correspondence with $k[\mathbf{G}]$-comodule structures on $V$.*

*Proof.* Let $\Phi : \mathbf{G} \to \mathbf{GL}_V$ be a linear representation. Denote $A = k[\mathbf{G}]$. Following a Yoneda-like approach consider the element $id_A \in \mathbf{G}(A)$ and send it through $\Phi$ to $\Phi^A(id_A) \in \mathrm{Aut}_A(V \otimes A)$. By $A$-linearity this map is determined by its restriction to $V \cong V \otimes k$, which we call $\rho : V \to V \otimes A$. Now the naturality of $\Phi$ implies that for any $k$-algebra $R$ and for any $g : A \to R$ in $G(R)$, the following diagram commutes:

$$\begin{array}{ccc} V \otimes A & \xrightarrow{\Phi^A(id_A)} & V \otimes A \\ {\scriptstyle id\otimes g}\downarrow & & \downarrow{\scriptstyle id\otimes g} \\ V \otimes R & \xrightarrow{\Phi^R(g)} & V \otimes R. \end{array}$$

Therefore in the restriction to $V \otimes k \subset V \otimes R$ the map $\Phi^R(g)$ acts via $(id \otimes g) \circ \rho$, and the rest is determined by $R$-linearity. Hence $\rho$ determines the whole natural transformation.

Conversely, given a linear map $\rho : V \to V \otimes A$ we get by linearity a linear endomorphism of $V \otimes A$ and thus a natural transformation $\Phi : \mathbf{G} \to \mathrm{End}(V \otimes -)$ by the Yoneda Lemma. The comodule axioms are seen to be equivalent to having this map be a representation. Indeed, the condition $(id \otimes \epsilon) \circ \rho = id$ is equivalent to having $\Phi$ send $id_R$ to $id_{V\otimes R}$ (the neutral element of $\mathrm{Aut}_R(V \otimes R)$), and the condition $(\rho \otimes id) \circ \rho = (id \otimes \Delta) \circ \rho$ is equivalent to $\Phi(g) \circ \Phi(h) = \Phi(gh)$ where $g, h$ are elements of $\mathbf{G}(R)$ and the multiplication is the one defined on $\mathbf{G}(R)$ (induced by $\Delta$). □

Therefore we may join the two correspondences and build the bridge

$$G\text{-gradings on } V \longleftrightarrow kG\text{-comodule structures on V} \longleftrightarrow \text{Linear representations of } G^D \text{ on } V.$$

We may express this relation directly and explicitly as follows. Given an abelian group $G$ and a finite-dimensional vector space $V$ with a $G$-grading $\Gamma$, we have already shown that we may define a $kG$-comodule structure on $V$ via the action $\rho_\Gamma$, sending each $v \in V_g$ to $v \otimes g$. Since $kG$ is the representing Hopf algebra for the diagonalizable group scheme $G^D$, $\rho_\Gamma$ induces a linear representation $\eta_\Gamma : G^D \to \mathbf{GL}_V$. Now we make use of the grading and the finite-dimensionality of $V$ by fixing a basis $\{v_1, \ldots, v_n\}$ of homogeneous elements, where $v_i \in V_{g_i}$ for each $i = 1, \ldots, n$. The basis gives a natural isomorphism $\mathbf{GL}_V \cong \mathbf{GL}_n$ identifying each $R$-linear automorphism of $V$ with its $R$-matrix in the basis $\{v_1, \ldots, v_n\}$. This exhibits $k[X_{ij}, \det^{-1}]$ as the representing algebra of $\mathbf{GL}_V$.

Since both $G^D$ and $\mathbf{GL}_V$ are affine group schemes, the Yoneda Lemma tells us that we may encode the linear representation $\eta_\Gamma$ in the corresponding comorphism of representing algebras $\eta_\Gamma^* : k[X_{ij}, \det^{-1}] \to kG$. To know how $\eta_\Gamma$ works, we take $f : kG \to R$ an element of $G^D(R)$ and see how its image acts on the basis elements.

$$\eta_\Gamma(f)(v_i \otimes 1) = (id \otimes f)(\rho(v_i)) = (id \otimes f)(v_i \otimes g_i) = v_i \otimes f(g_i). \tag{2.3}$$

The first equality comes from the correspondence between comodules and representations outlined in the proof of Theorem 2.14. This shows that the matrix of $\eta_\Gamma(f)$ in the basis $\{v_1, \ldots, v_n\}$ is the diagonal matrix

$$\begin{pmatrix} f(g_1) & & \\ & \ddots & \\ & & f(g_n) \end{pmatrix},$$

giving an idea of why the affine group scheme $G^D$ is called “diagonalizable”. Since the matrix of $\eta_\Gamma(f)$ is the one to which $f \circ \eta_\Gamma^*$ “points at” we must have (by the uniqueness of the Yoneda Lemma) the following expression for $\eta_\Gamma^*$:

$$\eta_\Gamma^*(X_{ij}) = \delta_{ij} g_i.$$

Conversely, given a linear representation $\eta : G^D \to \mathbf{GL}_V$ we get a $kG$-comodule structure on $V$, which defines a $G$-grading $\Gamma$. Taking an appropiate basis of $V$ we get back a diagonal representation of $\eta = \eta_\Gamma$.

Just like in the correspondence between comodules and gradings, assuming we are working with an algebra $\mathcal{A}$ we would like to know which conditions should $\eta_\Gamma$ satisfy so that $\Gamma$ is an algebra grading. It can be checked directly that $\Gamma$ is an algebra grading if and only if the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a morphism of $G^D$-representations, i.e. if and only if

$$\mu(\eta_\Gamma(f)(a) \otimes \eta_\Gamma(f)(b)) = \eta_\Gamma(f)(\mu(a \otimes b)) \quad \text{for all} \quad a, b \in \mathcal{A}, f \in G^D(R), R \in k\text{-}\mathbf{Alg}.$$

This means that the image by $\eta_\Gamma$ of any $f \in G^D(R)$ is not only an $R$-linear automorphism of $\mathcal{A}_R := \mathcal{A} \otimes R$ but is in fact an automorphism of $\mathcal{A}_R$ as an $R$-algebra. We denote the set of such automorphisms by $\mathrm{Aut}_R(\mathcal{A}_R)$ or $\mathrm{Aut}(\mathcal{A}_R)$ when it is clear from context what we mean. This leads to the following definition.

**Proposition 2.15.** *Let $\mathcal{A}$ be a (nonassociative) finite-dimensional algebra and define the following group for any $k$-algebra $R$*

$$\mathbf{Aut}(\mathcal{A})(R) := \mathrm{Aut}_R(\mathcal{A}_R).$$

*This defines a representable group functor, $\mathbf{Aut}(\mathcal{A})$, i.e. an affine group scheme. We call it the* automorphism group scheme *of $\mathcal{A}$.*

*Proof.* It is enough to find a representing object for $\mathbf{Aut}(\mathcal{A})$. Since algebra automorphisms are in particular linear automorphisms, we begin with the representing algebra for $\mathbf{GL}_n$ (where $n = \dim_k \mathcal{A}$) and quotient it by some equations which will guarantee that the resulting maps respect the multiplication.

Already in the identification with a subgroup of $\mathbf{GL}_n$ we are fixing a basis $\{a_1, \ldots, a_n\}$ of $\mathcal{A}$. Writing the product of the basis elements in terms of the basis gives us the so-called *structure constants*:

$$a_i a_j = \sum_{k=1}^{n} \lambda_{ij}^k a_k.$$

Let $\varphi$ be a linear automorphism of $\mathcal{A}_R$. If $(x_{ij})$ is the matrix of $\varphi$ in the basis $\{a_1, \ldots, a_n\}$, the condition $\varphi(a_i a_j) = \varphi(a_i)\varphi(a_j)$ may be written as

$$\sum_{s,t} \lambda_{st}^k x_{si} x_{tj} = \sum_l \lambda_{ij}^l x_{kl} \quad \text{for all} \quad i, j, k = 1, \ldots, n.$$

Define the polynomials $h_{ij}^k = \sum_{s,t} \lambda_{st}^k X_{si} X_{tj} - \sum_l \lambda_{ij}^l X_{kl}$. We have shown that the representing algebra of $\mathbf{Aut}(\mathcal{A})$ is

$$k[\mathbf{Aut}(\mathcal{A})] = k[X_{ij}, \det^{-1}]/(h_{ij}^k)_{i,j,k=1,\ldots,n}.$$

□

This allows us to summarize our previous discussion by saying that if $\Gamma$ is an algebra grading, then the image of $\eta_\Gamma$ is a subgroupscheme of $\mathbf{Aut}(\mathcal{A})$. And conversely, given a morphism $\eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{A})$ we obtain a $G$-grading on $\mathcal{A}$ that is actually an algebra grading.

## 2.4 Main theorems

Now that we have developed the tools and the language to express them, we are ready to present the main results of this chapter. In this section we continue with the same assumptions as before, namely $G, H$ are abelian groups and $\mathcal{A}, \mathcal{B}$ are finite-dimensional nonassociative algebras.

**Theorem 2.16.** *The $G$-gradings on $\mathcal{A}$ are in one-to-one correspondence with the morphisms of affine group schemes $G^D \to \mathbf{Aut}(\mathcal{A})$. Two $G$-gradings are isomorphic if and only if the corresponding morphisms are conjugate by an element of* $\mathrm{Aut}(\mathcal{A})$*. The weak isomorphism classes of gradings on $\mathcal{A}$ with the property that the support generates the grading group are in one-to-one correspondence with the* $\mathrm{Aut}(\mathcal{A})$*-orbits of diagonalizable subgroupschemes in* $\mathbf{Aut}(\mathcal{A})$*.*

*Proof.* We have described in the previous section the one-to-one correspondence between $G$-gradings of $\mathcal{A}$ and morphisms $G^D \to \mathbf{Aut}(\mathcal{A})$.

In the first place we should figure out what is the action by conjugation of $\mathrm{Aut}(\mathcal{A})$ on morphisms $G^D \to \mathbf{Aut}(\mathcal{A})$ that is mentioned in the statement of the proposition. Given a $k$-automorphism $\varphi \in \mathrm{Aut}(\mathcal{A})$ and an $R$-automorphism $f \in \mathrm{Aut}(\mathcal{A}_R)$ we can get another $R$-automorphism by conjugating $f$ with $\varphi$ on the original algebra and "leaving the $R$-part fixed". This results in a morphism $\mathrm{Ad}_\varphi : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{A})$:

$$\mathrm{Ad}^R_\varphi(f) := (\varphi \otimes id) \circ f \circ (\varphi^{-1} \otimes id) \text{ for all } f \in \mathrm{Aut}(\mathcal{A}_R). \tag{2.4}$$

To see how morphisms between the automorphism group schemes allow us to move gradings from one algebra to another, let $\mathcal{B}$ be another algebra and let $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ be a morphism. Any $G$-grading $\Gamma$ on $\mathcal{A}$ induces a $G$-grading on $\mathcal{B}$ via the morphism $\theta \circ \eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{B})$. We denote the induced grading by $\theta(\Gamma)$. In the particular case that $\theta = \mathrm{Ad}_\varphi$ and using (2.3) and (2.4) we see that the action on elements is

$$\begin{aligned}\eta^R_{\mathrm{Ad}_\varphi(\Gamma)}(f)(v_i \otimes 1) &= (\mathrm{Ad}^R_\varphi \circ \eta^R_\Gamma)(f)(v_i \otimes 1) = ((\varphi \otimes id) \circ \eta^R_\Gamma(f) \circ (\varphi^{-1} \otimes id))(v_i \otimes 1) = \\ &= v_i \otimes f(\alpha^{-1}(g_i)),\end{aligned}$$

where $f \in G^D(R)$, $v_i$ is a (homogeneous) basis element of degree $g_i$ and $\alpha : G \to G$ is the group isomorphism corresponding to $\varphi$. Looking at (2.3) this indicates that the $g$ component of the grading $\mathrm{Ad}_\varphi(\Gamma)$ is precisely the $\alpha(g)$ component of $\Gamma$, $\mathcal{A}_{\alpha(g)} = \varphi(\mathcal{A}_g)$. That is, $\mathrm{Ad}_\varphi(\Gamma) : \mathcal{A} = \bigoplus_{g\in G} \varphi(\mathcal{A}_g)$. By definition, this means precisely that the gradings $\Gamma$ and $\mathrm{Ad}_\varphi(\Gamma)$ are isomorphic. Conversely, given two isomorphic gradings $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' = \bigoplus_{g\in G} \mathcal{A}'_g$ there exists an autormorphism $\varphi \in \mathrm{Aut}(\mathcal{A})$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$. By the calculations above this means $\Gamma' = \mathrm{Ad}_\varphi(\Gamma)$. Therefore we can conclude that two $G$-gradings are isomorphic if and only if their morphisms are conjugate by an element of $\mathrm{Aut}(\mathcal{A})$.

For the second part of the statement recall that a group homomorphism $\alpha : G \to H$ gives rise to a Hopf algebra homomorphism $\alpha : kG \to kH$, which induces a morphism of affine group schemes $\alpha^D : H^D \to G^D$. Equation (2.2) implies directly that $\eta_{{}^\alpha\Gamma} = \eta_\Gamma \circ \alpha^D$. By associativity

$$(\theta \circ \eta_\Gamma) \circ \alpha^D = \theta \circ (\eta_\Gamma \circ \alpha^D) \implies {}^\alpha(\theta(\Gamma)) = \theta({}^\alpha\Gamma).$$

In the particular case that $\theta = \mathrm{Ad}_\varphi$, what this equation says is that gradings in the same weak isomorphism class are sent to the same weak isomorphism class via the action by conjugation of $\mathrm{Aut}(\mathcal{A})$. Therefore these classes are in bijection with the $\mathrm{Aut}(\mathcal{A})$-orbits. Furthermore, if the action is a closed embedding —and hence corresponds to a diagonalizable subgroupscheme of $\mathbf{Aut}(\mathcal{A})$— then the corresponding comorphism $k[X_{ij}, \det^{-1}]/(h_{ij}) \to kG$ is a surjection, meaning that the $g_i$ (which form the support of $\Gamma$) generate $G$. And conversely, if Supp $\Gamma$ generates $G$ then the comorphism is surjective (recall $\eta_\Gamma^*(X_{ij}) = \delta_{ij} g_i$) and the action is a closed embedding, whose image is a diagonalizable subgroupscheme of the automorphism group scheme. This concludes the proof. □

**Theorem 2.17.** *Assume we have a morphism $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$. Then, for any abelian group $G$, we have a mapping $\Gamma \to \theta(\Gamma)$ from $G$-gradings on $\mathcal{A}$ to $G$-gradings on $\mathcal{B}$. If $\Gamma$ and $\Gamma'$ are isomorphic (respectively weakly isomorphic), then $\theta(\Gamma)$ and $\theta(\Gamma')$ are isomorphic (respectively weakly isomorphic).*

*Proof.* We know from the previous theorem that isomorphism and weak isomorphism classes of gradings on $\mathcal{A}$ are related to the action by conjugation of $\mathrm{Aut}(\mathcal{A})$ on morphisms $G^D \to \mathbf{Aut}(\mathcal{A})$. Therefore we just have to study the relation between $\mathrm{Ad}_\varphi$ and $\theta$.

Let $\varphi \in \mathrm{Aut}(\mathcal{A})$ and define $\psi := \theta^k(\varphi) \in \mathrm{Aut}(\mathcal{B})$. For any $k$-algebra $R$ and any $R$-automorphism $f \in \mathrm{Aut}(\mathcal{A}_R)$ we have

$$\theta^R(\mathrm{Ad}_\varphi^R(f)) = \theta^R(\varphi \otimes id) \circ \theta^R(f) \circ \theta^R(\varphi^{-1} \otimes id) = (\psi \otimes id) \circ \theta^R(f) \circ (\psi^{-1} \otimes id) = \mathrm{Ad}_\psi^R(\theta^R(f)).$$

In the second equality we have used that $\theta^R(\varphi \otimes id) = \psi \otimes id$, which holds because $\theta$ is a natural transformation, so it commutes with the morphisms $- \otimes id_R$ (which are just the images under $\mathbf{Aut}(\mathcal{A}), \mathbf{Aut}(\mathcal{B})$ of the inclusion $k \to R$). This just shows that the diagram

$$\begin{array}{ccc} \mathbf{Aut}(\mathcal{A}) & \xrightarrow{\theta} & \mathbf{Aut}(\mathcal{B}) \\ {\scriptstyle \mathrm{Ad}_\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_\psi} \\ \mathbf{Aut}(\mathcal{A}) & \xrightarrow{\theta} & \mathbf{Aut}(\mathcal{B}) \end{array}$$

commutes. Hence if $\varphi$ sends $\Gamma$ (or ${}^\alpha\Gamma$ in the case of a weak isomorphism) to $\Gamma'$, then $\psi$ sends $\theta(\Gamma)$ (respectively $\theta({}^\alpha\Gamma) = {}^a(\theta(\Gamma))$) to $\theta(\Gamma')$. □

We could not end the chapter without pointing out the success of these last results —and others of their kind—, which have allowed algebrists to classify the gradings of the exceptional simple Lie algebras of types $G_2$ and $F_4$. These algebras and the octonion and Albert algebras have isomorphic automorphism group schemes, and our theorems allow us to move the better understood gradings of the latter algebras to the former.

# Chapter 3

# Computations

## 3.1 Evolution algebras

In this last chapter we are going to apply our knowledge from previous chapters to compute some automorphism group schemes (and hence the automorphism groups themselves) of a special kind of algebras called *evolution algebras*. They were introduced in 2006 by J.P. Tian and P. Vojtechovsky and have a wide range of connections to other fields such as graph theory, group theory or Markov chains. This is explored in [6] and we will not go into details in that regard.

Due to their applicability their properties have been thoroughly studied. We are particularly interested in papers [3] and [2]. In the former, a classification of two-dimensional evolution algebras is presented in Section 3 and afterwards the automorphism group scheme for each class of algebras is computed in Section 5 via direct computation. However, [2] introduces another method to compute the automorphism group scheme, which involves some nice properties of an associated graph. After briefly introducing all of these concepts, we will employ this alternative method to compute the automorphism group schemes of a couple of evolution algebras. Throughout this chapter, all algebras will be defined over a fixed field $k$ and their dimension will be finite. The definitions and theorems are extracted from [2].

**Definition 3.1.** An *evolution algebra* is an algebra $\mathcal{E}$ endowed with a basis $B = \{v_1, \dots, v_n\}$, called a *natural basis*, such that $v_i v_j = 0$ for any $1 \leq i \neq j \leq n$.

The main observation is that some properties of the algebra are determined by a graph that we may associate to it. Since the mixed products of the basis elements are zero by definition we just need to know the values of $v_i^2, i = 1, \dots, n$ to completely determine the multiplication. If we write these values in terms of that same natural basis and interpret the coefficients as some sort of adjacency matrix we get the definition of this graph.

**Definition 3.2.** Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{v_1, \dots, v_n\}$ and multiplication determined by $v_i^2 = \sum_{i=1}^n \alpha_{ij} v_j$. The *associated graph* of $\mathcal{E}$ is the graph $\Gamma = (V, E)$ whose set of vertices is $V = B$ and whose set of edges is

$$E \coloneqq \{(v_i, v_j) \in V \times V \mid \alpha_{ij} \neq 0\},$$

i.e. there is an arrow from $v_i$ to $v_j$ if the latter appears with nonzero coefficient in the expression of $v_i^2$.

The results we present here are true for a particular kind of evolution algebras. An evolution algebra $\mathcal{E}$ is said to be *perfect* if $\mathcal{E}^2 = \mathcal{E}$, or equivalently if the corresponding matrix $(\alpha_{ij})$ is regular. From this point on we assume that all evolution algebras that appear are perfect. Now we are going to define some graph-theoretical concepts that will appear in the theorems. We fix a directed graph $\Gamma$ with a finite set of vertices $V$ and a set of edges $E \subset V \times V$.

**Definition 3.3.**

- A *path* on $\Gamma$ is a sequence $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ where $n \geq 0$, $v_0, \ldots, v_n \in V$, $e_1, \ldots, e_n \in E$, and for each $i = 1, \ldots, n$, either $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$.
- The *balance* of the path $\gamma$ is the integer

$$\mathrm{b}(\gamma) = |\{i \mid 1 \leq i \leq n \text{ and } e_i = (v_{i-1}, v_i)\}| - |\{i \mid 1 \leq i \leq n \text{ and } e_i = (v_i, v_{i-1})\}|,$$

that is, $\mathrm{b}(\gamma)$ is the number of edges in the direction of the path minus the number of edges opposite to the direction of the path.

- A *cycle* on $\Gamma$ is a path $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ with $v_0 = v_n$.
- The *balance* of the graph $\Gamma$ is

$$\mathrm{b}(\Gamma) = \gcd\{|\mathrm{b}(\gamma)| \ : \ \gamma \text{ is a cycle on } \Gamma\}.$$

- A vertex $v \in V$ is said to be a *source* if it has no "ingoing" edges (i.e. $\nexists w \in V$ such that $(w, v) \in E$), and is said to be a *sink* if it has no "outgoing" edges (i.e. $\nexists w \in V$ such that $(v, w) \in E$).

To any graph we may associate two affine group schemes which will be the ones from which we extract the information to compute $\mathbf{Aut}(\mathcal{E})$. One of them is diagonalizable and the other one is a constant group scheme. Recall that we already talked about these in Chapter 1.

## 3.2 The diagonal group scheme

**Definition 3.4.** The *diagonal group* of a graph $\Gamma = (V, E)$ is the (diagonalizable) affine group scheme $\mathbf{Diag}(\Gamma)$ given by

$$\mathbf{Diag}(\Gamma)(R) := \{\varphi : V \to R^\times \mid \varphi(w) = \varphi(v)^2 \text{ for all } (v, w) \in E\},$$

with pointwise multiplication and the morphisms acting pointwise on vertices. Notice that since $V$ is finite we may identify $\mathbf{Diag}(\Gamma)(R)$ as a subgroup of $(R^\times)^{|V|} = (\mathbf{G}_m)^{|V|}(R)$, so $\mathbf{Diag}(\Gamma)$ is a subgroupscheme of $(\mathbf{G}_m)^{|V|}$. Since subgroupschemes of diagonalizable affine group schemes are themselves diagonalizable (see [1, Prop. A.31]) we conclude $\mathbf{Diag}(\Gamma)$ is diagonalizable.

The following results are presented here without proof, but all proofs can be found on [2].

**Lemma 3.5.** *Let $\Gamma = (V, E)$ be a graph, $\gamma = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ be a path in $\Gamma$. Let $R$ be a $k$-algebra and $\varphi \in \mathbf{Diag}(\Gamma)(R)$ such that $\varphi(v_i)$ has odd multiplicative order for all $i = 0, \ldots, n$. Then $\varphi(v_n) = \varphi(v_0)^{2^{b(\gamma)}}$.*

**Theorem 3.6.** *Let $\Gamma = (V, E)$ be connected graph with no sources. Then $\mathbf{Diag}(\Gamma) \cong \boldsymbol{\mu}_N$, where $N = 2^{b(\gamma)} - 1$.*

Notice that this result applies to the graphs associated to perfect evolution algebras, as in this case a source would correspond to a basis element $v_i$ with $v_i^2 = 0$.

**Corollary 3.7.** *Let $\Gamma = (V, E)$ be a connected graph with no sources and with a loop $e = (v, v)$. Then $\mathbf{Diag}(\Gamma) = 1$, the trivial affine group scheme.*

**Theorem 3.8.** *Let $\mathcal{E}$ be an evolution algebra with natural basis $B = \{v_1, \ldots, v_n\}$ and let $\Gamma$ be the associated graph. Then there is an injective homomorphism $\iota : \mathbf{Diag}(\Gamma) \to \mathbf{Aut}(\mathcal{E})$ such that, for any $k$-algebra $R$ and any $\varphi \in \mathbf{Diag}(\Gamma)(R)$, the image $\iota(\varphi)$ is the induced diagonal automorphism $v_i \mapsto \varphi(v_i)v_i$.*

*Proof.* Clearly the map $v_i \mapsto \varphi(v_i)v_i$ defines a linear automorphism of $\mathcal{E}_R = \mathcal{E} \otimes R$ because $\varphi(v_i)$ is invertible for all $i = 1, \dots, n$. Call this automorphism $\hat{\varphi}$. As before, denote $v_i^2 = \sum_{j=1}^n \alpha_{ij} v_j$, then

$$\hat{\varphi}(v_i^2) = \sum_{j=1}^n \alpha_{ij} \hat{\varphi}(v_j) = \sum_{j=1}^n \alpha_{ij} \varphi(v_j) v_j,$$
$$\hat{\varphi}(v_i)^2 = \varphi(v_i)^2 v_i^2 = \varphi(v_i)^2 \sum_{j=1}^n \alpha_{ij} v_j.$$

The only condition we need for $\hat{\varphi}$ to be an algebra homomorphism is that these two expressions are equal. Comparing coefficients, either $\alpha_{ij} = 0$ in which case the equality is trivial, or $\alpha_{ij} \neq 0$ in which case $(v_i, v_j) \in E$ and therefore $\varphi(v_j) = \varphi(v_i)^2$ so the equality holds.

The map $\iota$ is a well-defined homomorphism: the equality $\iota^R(\varphi\psi) = \iota^R(\varphi)\iota^R(\psi)$ holds because multiplication on $\mathbf{Diag}(\Gamma)(R)$ is defined pointwise and the matrices of $\iota^R(\varphi), \iota^R(\psi)$ are diagonal, so multiplication occurs componentwise. Finally, $\iota^R$ is injective for all $R$ because

$$\iota^R(\varphi) = id_{\mathcal{E}_R} \implies \varphi(v_i) = 1 \text{ for all } i = 1, \dots, n \implies \varphi = 1.$$

□

This is all that we need to know about $\mathbf{Diag}(\Gamma)$ for now. Let us move on to the second affine group scheme that we need to study $\mathbf{Aut}(\mathcal{E})$. As promised this is a constant group scheme, so all $k$-algebras with no idempotents other than 0 or 1 evaluate to the same group via this functor. Let us define this group.

## 3.3 The constant automorphism group scheme

**Definition 3.9.** Let $\Gamma = (V, E)$ be a graph, where $V = \{v_1, \dots, v_n\}$ is finite. We define the *automorphism group of* $\Gamma$, $\mathrm{Aut}(\Gamma)$, as the bijections of $V$ that respect the edges:

$$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid (v_i, v_j) \in E \implies (v_{\sigma(i)}, v_{\sigma(j)}) \in E \text{ for all } 1 \leq i, j \leq n\}.$$

The associated constant group scheme is called the *automorphism group scheme of* $\Gamma$ and we denote it by $\mathsf{Aut}(\Gamma)$.

Our goal is to show the existence of a morphism $\rho : \mathbf{Aut}(\mathcal{E}) \to \mathsf{Aut}(\Gamma)$, which will be useful because in the end this will produce a short exact sequence determining $\mathbf{Aut}(\mathcal{E})$. To do so let $R$ be a $k$-algebra and let $\varphi \in \mathbf{Aut}(\mathcal{E})(R) = \mathrm{Aut}(\mathcal{E}_R)$. Denote by $(r_{ij})$ the coefficients of $\varphi$ in the natural basis $B = \{v_1, \dots, v_n\}$, i.e. $\varphi(v_i) = \sum_{j=1}^n r_{ij} v_j$. Denote

$$r = \det(r_{ij}) = \sum_{\sigma \in S_n} (-1)^\sigma r_{\sigma(1)1} \dots r_{\sigma(n)n} \in R^\times.$$

On the other hand, since mixed products vanish we have $0 = \varphi(v_i v_j) = \varphi(v_i)\varphi(v_j) = \sum_{l=1}^n r_{il} r_{kl} v_k^2$. Since $\mathcal{E}^2 = \mathcal{E}$, $\{v_1^2, \dots, v_n^2\}$ is a basis of $\mathcal{E}$ and thus

$$r_{il} r_{jl} = 0 \quad \text{for all } 1 \leq i, j \leq n \text{ with } i \neq j.$$

In particular, if $\sigma, \tau \in S_n, \sigma \neq \tau$ then there exists some $l \in \{1, \dots, n\}$ such that $\sigma(l) \neq \tau(l)$ so $(r_{\sigma(1)1} \dots r_{\sigma(n)n})(r_{\tau(1)1} \dots r_{\tau(n)n}) = 0$. This is useful because if we consider the elements

$$e_\sigma^\varphi = (-1)^\sigma r^{-1} r_{\sigma(1)1} \dots r_{\sigma(n)n}$$

we have proven that they satisfy the following identities:

$$\sum_{\sigma\in S_n} e_\sigma^\varphi = 1, \quad e_\sigma^\varphi e_\tau^\varphi = 0 \text{ for } \sigma \neq \tau \in S_n, \quad e_\sigma^\varphi = e_\sigma^\varphi(\sum_{\tau\in S_n} e_\tau^\varphi) = (e_\sigma^\varphi)^2. \tag{3.1}$$

It is a well-known fact and easy to prove that in this case the $k$-algebra $R$ splits as a direct sum of principal ideals, $R = \bigoplus_{\sigma\in S_n} Re_\sigma^\varphi$. Additionally we have

$$r_{ij}e_\sigma^\varphi = 0 \text{ unless } i = \sigma(j),$$

which implies that the matrix $(r_{ij})$ can be written as a sum of monomial matrices with entries in the principal ideals from before. Each of these monomial matrices, $A_\sigma = e_\sigma^\varphi(r_{ij})$, can be regarded as an automorphism of $Re_\sigma^\varphi$ as long as $e_\sigma^\varphi \neq 0$. This shows that any nonzero $e_\sigma^\varphi$ must come from $\sigma \in \mathrm{Aut}(\Gamma)$. All of this will be made more explicit in the calculations that will come afterwards.

To define $\rho$ recall that the representing algebra of the constant group scheme $\mathrm{Aut}(\Gamma)$ is $k^{\mathrm{Aut}(\Gamma)}$, which has a basis $\{\epsilon_\sigma \mid \sigma \in \mathrm{Aut}(\Gamma)\}$ (we changed the $e$'s from Chapter 1 to $\epsilon$'s to avoid confusing $\epsilon_\sigma$ with $e_\sigma^\varphi$). The $R$ component of $\rho$ is defined by sending $\varphi \in \mathrm{Aut}(\mathcal{E}_R)$ to $\rho^R(\varphi) \in \mathsf{Aut}(\Gamma)(R) = \mathrm{Hom}_k(k^{\mathrm{Aut}(\Gamma)}, R)$, defined as

$$\begin{aligned} \rho^R(\varphi) : k^{\mathrm{Aut}(\Gamma)} &\to R \\ \epsilon_\sigma &\mapsto e_\sigma^\varphi. \end{aligned}$$

This is seen to be a $k$-algebra homomorphism, and thus $\rho$ is well defined. If $R$ has no idempotents other than 0 or 1, then $1 = e_\sigma^\varphi$ for a unique $\sigma \in \mathrm{Aut}(\Gamma)$, so the matrix of $\varphi$ is the monomial matrix attatched to $\sigma$. For such an $R$ we had an identification between $\mathsf{Aut}(\Gamma)(R)$ and the group $\mathrm{Aut}(\Gamma)$. Under this identification the map $\rho^R(\varphi)$ is just $\sigma$.

## 3.4 Exact sequences

In the previous two sections we have defined a morphism going *into* the affine group scheme $\mathbf{Aut}(\mathcal{E})$ and a morphism going *out* of it. The punchline is that these morphisms may be fitted into an exact sequence, therefore expresing $\mathbf{Aut}(\mathcal{E})$ in terms of $\mathbf{Diag}(\Gamma)$ and $\mathsf{Aut}(\Gamma)$.

**Theorem 3.10.** *Let $\mathcal{E}$ be a perfect evolution algebra with natural basis $B = \{v_1, \ldots, v_n\}$, and denote by $\Gamma$ its associated graph. Then the sequence*

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \xrightarrow{\iota} \mathbf{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathsf{Aut}(\Gamma) \tag{3.2}$$

*is exact.*

*Proof.* The sequence is exact at $\mathbf{Diag}(\Gamma)$ because we already showed $\iota$ is a monomorphism. To see that the sequence is exact at $\mathbf{Aut}(\mathcal{E})$ notice that $\ker(\rho^R)$ consists of the automorphisms $\varphi \in \mathrm{Aut}(\mathcal{E}_R)$ such that $e_\sigma^\varphi = 0$ for any $1 \neq \sigma \in \mathrm{Aut}(\Gamma)$. This is because the neutral element in $\mathsf{Aut}(\Gamma)(R) = \mathrm{Hom}_k(k^{\mathrm{Aut}(\Gamma)}, R)$ is the map sending all basis elements to 0 except the one corresponding to the neutral element of the group $\mathrm{Aut}(\Gamma)$ (which we denote by 1). In that case the equations (3.1) imply that we must have $e_1^\varphi = 1$ and the matrix of $\varphi$ is diagonal, so the elements of the natural basis $B$ are eigenvectors for $\varphi$. These are precisely the maps in the image of $\iota$. □

**Lemma 3.11.** *Let $G$ be a finite abelian group. Then the subgroupschemes of the constant group scheme $\mathsf{G}$ are precisely the constant group schemes $\mathsf{H}$ corresponding to subgroups $H \leq G$.*

*Proof.* As we know from Section 1.2, subgroupschemes of $\mathsf{G}$ are represented by quotients, or equivalently homomorphic images, of $k[\mathsf{G}] = k^G$. Our goal then is to prove that any homomorphic image of $k^G$ through a Hopf algebra homomorphism is isomorphic to $k^H$ for some subgroup $H \leq G$. To do so let $A$ be a Hopf algebra and let $f : k^G \to A$ be a Hopf algebra homomorphism.

The image of $f$ is generated as a $k$-algebra by the elements $\{f(e_g)\}_{g\in G}$ (recall that we defined $e_g(h) = \delta_{gh}$). These elements satisfy

$$f(e_g)^2 = f(e_g^2) = f(e_g), \quad f(e_g)f(e_h) = f(e_g e_h) = 0 \text{ for all } g \neq h \in G.$$

From here we deduce that for each $g \in G$, either $e_g$ is the only element of the family sent to $f(e_g)$ or $f(e_g) = 0$. An even stronger assertion holds: the nonzero elements of the generating family form a linearly independent set. Indeed, let

$$H := \{h \in G \mid f(e_h) \neq 0\}$$

and assume we have a nontrivial linear combination summing to 0,

$$0 = \sum_{h\in H} \lambda_h f(e_h), \quad \lambda_h \in k, \lambda_{h_0} \neq 0 \text{ for some } h_0 \in H.$$

In that case we would have $f(e_{h_0}) = f(e_{h_0})^2 = f(e_{h_0})\lambda_{h_0}^{-1} \sum_{h\neq h_0} \lambda_h f(e_h) = 0$, a contradiction.

Since the image of $f$ is generated by $\{f(e_h)\}_{h\in H}$ we just need to show that (i) $H$ is a subgroup of $G$ and (ii) the Hopf algebra structure on the image of $f$ is actually the "restriction" of the structure of $G$.

(i) Let $g, h \in H$. We want to see that $f(e_{gh}) \neq 0$. To do so we use the fact that $f$ is a Hopf algebra homomorphism.

$$\Delta_A(f(e_{gh})) = (f \otimes f)(\Delta_{k^G}(e_{gh})) = \sum_{\sigma\tau=gh} f(e_\sigma) \otimes f(e_\tau).$$

The linear independence of $\{f(e_h)\}_{h\in H}$ implies that $\{f(e_{h_1}) \otimes f(e_{h_2})\}_{h_1,h_2\in H}$ is also a linear independent family, and since the summand $f(e_g) \otimes f(e_h)$ is nonzero we conclude that the sum in the right hand side is nonzero. Consequently $f(e_{gh}) \neq 0$ and $gh \in H$. Similarly we see that $1 \in H$, where 1 denotes the neutral element of $G$. Indeed,

$$\epsilon_A(f(e_1)) = f(\epsilon_{k^G}(e_1)) = f(1) = 1 \neq 0,$$

so we conclude $f(e_1) \neq 0$ and $1 \in H$. Finally for any $h \in H$ we have

$$S_A(f(e_{h^{-1}})) = f(S_{k^G}(e_{h^{-1}})) = f(e_h) \neq 0, .$$

so $f(e_{h^{-1}}) \neq 0$ and $h^{-1} \in H$.

(ii) This is once again a consequence of $f$ being a Hopf algebra homomorphism:

$$\Delta_A(f(e_h)) = (f \otimes f)(\Delta_{k^G}(e_h)) = \sum_{\sigma\tau=h} f(e_\sigma) \otimes f(e_\tau),$$

showing that the group structure induced by the image is the same as the one induced by $G$. □

**Corollary 3.12.** *Let $\mathcal{E}$ be a perfect evolution algebra with natural basis $B = \{v_1, \ldots, v_n\}$, and denote by $\Gamma$ its associated graph. Then there is a subgroup $H$ of $\mathrm{Aut}(\Gamma)$ such that the sequence*

$$1 \longrightarrow \mathbf{Diag}(\Gamma) \xrightarrow{\iota} \mathbf{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathsf{H} \longrightarrow 1$$

*is exact, where again $\mathsf{H}$ denotes the constant group scheme associated to the group $H$.*

## 3.5 Examples

Now that we have developed the theory we are going to compute the automorphism group scheme of some two-dimensional evolution algebras. The classification of such algebras can be found on [3]. Looking at [3, Table 4] we see that some of the algebras do not satisfy the condition $\mathcal{E}^2 = \mathcal{E}$, so our results do not apply there. Let us begin by computing the automorphism group scheme for one of the non-perfect algebras of the list to see what are the computations that have to be done, which will of course get harder as the dimension of the algebra increases.

**Example 3.13.** Let $\mathcal{E}$ be an evolution algebra of type $A_6$ with natural basis $B = \{v_1, v_2\}$, that is, the multiplication of $\mathcal{E}$ is determined by

$$v_1^2 = 0, v_2^2 = v_1.$$

Let $R$ be an arbitrary $k$-algebra. Our goal is to compute $\mathrm{Aut}_R(\mathcal{E}_R)$, so let $\varphi$ be an $R$-linear automorphism of $\mathcal{E}_R$ and denote its matrix in the basis $B$ by $(r_{ij})_{i,j=1,2}$. We have to impose some conditions on the $r_{ij}$ so that the map is actually a homomorphism of algebras:

$$\begin{gathered} 0 = \varphi(v_1^2) = \varphi(v_1)^2 = (r_{11}v_1 + r_{21}v_2)^2 = r_{21}^2 v_2^2 = r_{21}^2 v_1, \\ r_{11}v_1 + r_{21}v_2 = \varphi(v_1) = \varphi(v_2^2) = \varphi(v_2)^2 = (r_{12}v_1 + r_{22}v_2)^2 = r_{22}^2 v_1, \\ 0 = \varphi(v_1v_2) = \varphi(v_1)\varphi(v_2) = (r_{11}v_1 + r_{21}v_2)(r_{12}v_1 + r_{22}v_2) = r_{21}r_{22}v_1. \end{gathered}$$

Comparing coefficients and removing the redundant equations we arrive at

$$r_{22}^2 = r_{11}, \quad r_{21} = 0.$$

Furthermore, since $\varphi$ is a linear isomorphism we know its determinant is invertible, so we have

$$\det(\varphi) = r_{11}r_{22} - r_{12}r_{21} = r_{22}^3 \in R^\times \implies r_{22} \in R^\times.$$

These conditions are thus necessary to have an algebra automorphism, and we claim that they are sufficient, that is

$$\mathrm{Aut}_R(\mathcal{E}_R) = \left\{ \begin{pmatrix} a^2 & b \\ 0 & a \end{pmatrix} \middle| \; a \in R^\times, b \in R \right\}.$$

Indeed, the matrices of this set have invertible determinant so they are invertible. Carrying out the computations for the basis vectors one sees that the linear maps corresponding to these matrices respect the multiplication of the algebra. Therefore they are $R$-algebra automorphisms. We want to point out that to find this expression we have to solve a system of polynomial equations. In this case that was easy to do, but of course the general situation can get much more complicated.

As a further remark, an easy argument from Group Theory gives that $\mathbf{Aut}(\mathcal{E})$ is, up to isomorphism, a semidirect product $\mathbf{G}_a \rtimes \mathbf{G}_\mathbf{m}$. Due to lack of space, these semidirect products have not been defined here in the category of affine group schemes, but its defintion is the natural one that could be expected.

Now we proceed to use the methods that we have developed in the previous sections to compute the automorphism group schemes of perfect evolution algebras. In some cases the situation is so straightforward that we can do several algebras at the same time.

**Example 3.14.** Consider the two-dimensional evolution algebras of types $A_{3,\alpha}$ and $A_{4,\alpha}$, once again we refer to [3, Table 4] for the classification. It is easy to see that they are perfect, i.e. the subspace spanned by the squares of the elements of the natural basis is the whole algebra. To apply the theorems we need to know the graphs associated to these algebras (see figure 3.1).

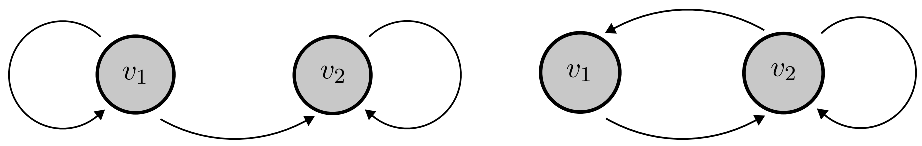


Figure 3.1: Associated graphs of algebras of type $A_{3,\alpha}$ and $A_{4,\alpha}$, respectively.

In both cases we may follow the same reasoning. Let $\Gamma$ be the graph of interest, let $R$ be a $k$-algebra, and let $\varphi \in \mathbf{Diag}(\Gamma)(R)$. Since there is a loop in $v_1$ we must have $\varphi(v_1)^2 = \varphi(v_1)$, and we conclude $\varphi(v_1) = 1$ because the image of $\varphi$ lies in $R^\times$. But there is also an edge from $v_1$ to $v_2$, so $\varphi(v_2) = \varphi(v_1)^2 = 1$, and hence $\mathbf{Diag}(\Gamma)(R)$ is the trivial group for all $R$, i.e. $\mathbf{Diag}(\Gamma) = 1$.

Similarly we see that for both graphs the only possible graph automorphism is the identity, as there is no edge $(v_2, v_1)$ in the graph of $A_{3,\alpha}$ and no loop $(v_1, v_1)$ in the graph of $A_{4,\alpha}$. Therefore $\mathrm{Aut}(\Gamma) = 1$. But then the short exact sequence (3.2) looks like

$$1 \longrightarrow \mathbf{Aut}(\mathcal{E}) \longrightarrow 1$$

and we conclude $\mathbf{Aut}(\mathcal{E}) = 1$ for both types of algebras.

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