Geometry of cubic surfaces in the three-dimensional complex projective space

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Prologue

Cubic surfaces are a particular case of projective varieties, that is, a geometrical object defined as the set of solutions of a system of homogeneous polynomial equations of degree three. The study of the geometric properties of cubic surfaces began to gain importance when mathematicians George Salmon (1819–1904) and Arthur Cayley (1821–1895) published the results of their correspondence in 1849.

Cayley proved, in a letter to Salmon, that there could be only a finite number of lines on a cubic surface, and Salmon then proved that there were exactly 27 such lines. Salmon described how both collaborated on stating the Cayley-Salmon theorem in his treatise *The Analytic Geometry of Three Dimensions* [4].

There exists an interesting geometrical structure called *double-six* on a general cubic surface. Ludwig Schläfli (1814–1895) found 36 double-sixes on cubic surfaces, which are related to the 27 lines.

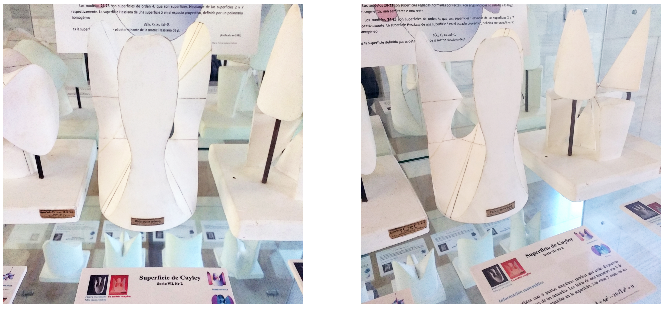

Figure 1: Model of the Cayley surface at the School of Science, University of Zaragoza.

In this work we will present some classical results on cubic surfaces and we will prove them using a modern approach. For the sake of completion, we will start giving a general context of projective geometry, then we will state some classical results and finally we will use modern mathematical tools to prove them. Although this is beyond the scope of this work, it is worth saying that the study of cubic surfaces is closely related to combinatorics and lattice structures.

Our goal in Chapter 1 is to introduce the basic concepts of projective geometry that will be used later. First we will present the concept of projective space, the Grassmannian of a projective space, homogeneous polynomial and projective variety.

In Chapter 2 we will present some general facts about cubic surfaces that will help to prove later results. Then a classical proof of the Cayley-Salmon theorem will be given together with the concepts of double-six and Eckardt point.

Chapter 3 contains the concepts required for a modern approach to the study of surfaces. We will introduce the concept of blow-up of a point in a complex surface. Then we present some basic notions of intersection theory and we build an alternative description of cubic surfaces as the blow-up of six points in general position on the projective plane. We end this chapter giving alternative proofs to some classical results. To conclude, we study some interesting examples of cubic surfaces and the geography of their Eckardt points.

Resumen

Las *superficies cúbicas* son un caso particular de variedades proyectivas, es decir, un objeto geométrico definido como el conjunto de soluciones de un sistema de ecuaciones polinómicas homogéneas de grado tres. En este trabajo vamos a estudiar algunos resultados clásicos sobre superficies cúbicas y los demostraremos utilizando un enfoque moderno.

En primer lugar pondremos en contexto este estudio hablando del *espacio proyectivo complejo n-dimensional*, que denotamos $\mathbb{P}^n$. Este espacio es el cociente del espacio vectorial complejo $(n+1)$-dimensional sin el origen, $\mathbb{C}^{n+1} \setminus \{0\}$, y una relación de equivalencia $\sim$, definida de forma que dos vectores u y v están relacionados ($u \sim v$) si y sólo si uno es múltiplo escalar del otro ($u = \lambda v$), con $u, v \in \mathbb{C}^{n+1} \setminus \{0\}$ y $\lambda \in \mathbb{C} \setminus \{0\}$. Entonces escribimos $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$.

Hemos visto que un punto $P \in \mathbb{P}^n$ cumple que $P = \lambda P$, pero entonces la ecuación $F(P) = 0$ no está bien definida para un polinomio en general F, ya que puede tener soluciones diferentes según el representante de P que se elija. Decimos que F es un *polinomio homogéneo* de grado d si cada uno de sus monomios tiene grado d. Estos polinomios cumplen la siguiente propiedad: un polinomio F es homogéneo si y sólo si $F(\lambda P) = \lambda^d F(P)$. De esta manera, aunque F no define una función sobre $\mathbb{P}^n$, al menos se tiene $F(P) = 0 \Leftrightarrow F(\lambda P) = 0$, por lo que el conjunto de soluciones de la ecuación $F(P) = 0$ está bien definido en $\mathbb{P}^n$. De este modo, el conjunto $\mathrm{V}(F) = \{P \in \mathbb{P}^n \mid F(P) = 0\}$ se denomina el *lugar de ceros* de F.

Cualquier subconjunto de $\mathbb{P}^n$ definido de esta manera se denomina *hipersuperficie proyectiva* de grado $d = \deg(F)$. El objetivo del trabajo es estudiar las hipersuperficies de grado 3 en $\mathbb{P}^3$, a las que llamaremos *superficies cúbicas*. Estas superficies son el lugar de ceros de un polinomio homogéneo de grado 3 en 4 variables. Una superficie cúbica se dice *lisa* si no contiene puntos de gradiente cero.

El primer resultado clásico que estudiamos es el teorema de Cayley-Salmon. Este teorema enuncia que las superficies cúbicas lisas contienen exactamente 27 rectas. Comenzaremos probando la existencia de al menos una recta ℓ_0 en la superficie. A partir de aquí se obtienen las 26 restantes de manera constructiva usando haces de hiperplanos basados en ℓ_0. Como resultado adicional, probaremos que hay exactamente 45 planos que contienen a tres rectas (los denominamos *planos tritangentes*).

Llamamos *sexteto* a un conjunto de seis rectas en $\mathbb{P}^3$ disjuntas dos a dos. La existencia de los sextetos y la no existencia de “septetos” es otro resultado clásico que estudiaremos con técnicas modernas más adelante. Un *doble sexteto* es un conjunto de doce rectas que puede dividirse en dos sextetos, de forma que cada una de las rectas de un sexteto interseca a cinco rectas del otro. Presentamos una clasificación de los dobles sextetos y más adelante la justificaremos.

Un punto en una superficie cúbica lisa se denomina *punto de Eckardt* si es la intersección de tres rectas de la superficie. El número de puntos de Eckardt permite distinguir superficies cúbicas que no son proyectivamente equivalentes entre sí.

Una vez hemos puesto en contexto el estudio de las superficies cúbicas y hemos enunciado algunos resultados clásicos, vamos a presentarlos con un enfoque moderno. Si bien antes construíamos las superficies a partir de sus ecuaciones en $\mathbb{P}^3$, ahora vamos a construirlas a partir de otras superficies por medio de una operación que llamaremos *explosión*. Al explotar una superficie en un punto, obtenemos

otra superficie cuya estructura compleja resulta de sustituir las cartas que contienen a ese punto por unas nuevas.

Sea $B \subset \mathbb{C}^2 \times \mathbb{P}^1$ la superficie compleja definida como

$$B = \{((x,y),[U:V]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xV = yU\}.$$

El punto proyectivo $[U:V] \in \mathbb{P}^1$ es la clase de una recta en $\mathbb{C}^2$ y la condición $xU = yV$ quiere decir que el punto complejo $(x,y) \in \mathbb{C}^2$ está en esa recta. Notemos que B tiene dos cartas: $B_U = \{p \in B \mid U \neq 0\}$ y $B_V = \{p \in B \mid V \neq 0\}$. Comprobando que el cambio de cartas es biholomorfo, llegamos a la conclusión de que B es efectivamente una superficie compleja. La proyección $B \to \mathbb{C}^2$ se denomina explosión de $\mathbb{C}^2$ en $(0,0)$. La preimagen de cualquier punto $(x,y) \neq (0,0)$ es un único punto de B. La preimagen del punto $(0,0)$ es la recta $\{(0,0)\} \times \mathbb{P}^1$. Esta recta se denomina *divisor excepcional* de la explosión. Esta definición se puede extender, usando un atlas, a cualquier superficie compleja.

Presentamos unas nociones básicas de teoría de intersección para poder utilizar el concepto de *autointersección*, que se puede interpretar como el número de veces que una curva se corta a sí misma cuando se desplaza ligeramente. Esta interpretación es una manera de ilustrar el concepto de forma sencilla y no siempre es válida. De hecho, un tipo especial de curvas que vamos a estudiar son aquellas que tienen autointersección -1, también llamadas (-1)-*curvas*. De manera intuitiva se deduce que estas rectas no pueden desplazarse en la superficie (esto caracteriza las superficies con un número finito de rectas).

Con estas herramientas ya podemos construir una superficie cúbica explotando el plano proyectivo $\mathbb{P}^2$ en seis puntos. Encajamos esta superficie en $\mathbb{P}^3$ y comprobamos que la nueva superficie que hemos hallado es, en efecto, cúbica calculando la autointersección de una sección hiperplana de esta superficie y viendo que es 3.

Además damos unas ideas generales sobre cómo demostrar que toda superficie cúbica puede construirse como la explosión de seis puntos de $\mathbb{P}^2$ en posición general, pero la demostración completa requiere resultados que escapan a los objetivos de este trabajo.

Con esta construcción de las superficies cúbicas a partir de las explosiones y con las herramientas de la teoría de la intersección demostraremos la existencia de los sextetos y llegaremos a la conclusión de que las 27 rectas de una superficie cúbica son sus (-1)-curvas. Además demostraremos que no pueden existir “septetos”. También volveremos a construir los dobles sextetos y los puntos de Eckardt a partir de las (-1)-curvas de la superficie. Para concluir, estudiaremos unos ejemplos de superficies cúbicas caracterizadas por la posición de sus puntos de Eckardt.

Contents

Chapter 1

Projective geometry

In this chapter we will remark some concepts that will be used in the study of cubic surfaces in the projective space. We refer to Fulton's book [3] for a more complete introduction to projective geometry.

1.1 Projective spaces

Let V be a vector space over the field $\mathbb{K}$ and the equivalence relation $\sim$ such that $u \sim v \Leftrightarrow u = \lambda v$ for $u, v \in V$, $\lambda \in \mathbb{K} \setminus \{0\}$. We define the *projective space* as the quotient $\mathbb{P}(V) = (V \setminus \{0\}) / \sim$, also called the *projectivization* of the vector space V. The dimension of the projectivization of a vector space V is given by $\dim \mathbb{P}(V) = \dim V - 1$.

Let us consider the case when $V = \mathbb{C}^{n+1}$. Then we denote $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ and we will call it the *n-dimensional complex projective space*. Let $\mathbb{A}^{n+1}$ be the $(n+1)$-dimensional affine space over $\mathbb{C}$, we can generalize what we said about the projectivization of vector spaces to affine spaces.

We call $P \in \mathbb{P}^n$ a *point*. Let P be described by the $(n+1)$-tuple $(x_0, \dots x_n) \in \mathbb{P}^n$, then $(x_0, \dots x_n) = \lambda(x_0, \dots x_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. The coordinates $(x_0, \dots, x_n)$ are called *homogeneous coordinates* and they are unique up to a scale factor. We will use the notation $P = [x_0 : \cdots : x_n]$ for homogeneous coordinates.

Since $(0, \dots, 0) \notin \mathbb{P}^n$, at least one of the x_i must be nonzero. Let $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$, then each $P \in U_i$ can be written uniquely in the form $P = [X_0 : \cdots : X_{i-1} : 1 : X_{i+1} : \cdots : X_n]$, with $X_j = x_j / x_i$. The coordinates $(X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ are called the *absolute* or *nonhomogeneous coordinates* for P with respect to U_i. Each subset U_i is isomorphic to $\mathbb{A}^n$, thus we have the affine covering of the projective space:

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

We consider the dehomogenization of the i-th coordinate, which is the map

$$\begin{gathered} U_i \to \mathbb{A}^n \\ [x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] \mapsto (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \end{gathered}$$

Unless otherwise stated, we will dehomogenize the n-th coordinate. The subset $H_\infty = \mathbb{P}^n \setminus U_n = \{[x_0 : \cdots : x_{n-1} : 0]\}$ is called the hyperplane at infinity. The elements of H_∞ are called *points at infinity* or *improper points* (the points in the affine space U_n are called *proper points*). The correspondence $[x_0 : \cdots : x_{n-1} : 0] \mapsto [x_0 : \cdots : x_{n-1}]$ shows that H_∞ may be identified with $\mathbb{P}^{n-1}$.

Therefore, $\mathbb{P}^n = U_n \cup H_\infty$, where $U_n \simeq \mathbb{A}^n$ and $H_\infty \simeq \mathbb{P}^{n-1}$.

1.2 Grassmannian of a vector space

Let V be an n-dimensional vector space and fix k an integer such that $0 < k \leq n$. We denote by $\mathrm{Gr}(k,n)$ the family of all k-dimensional vector subspaces of V. For instance, the *complex Grassmannian* $\mathrm{Gr}(1,n)$ parametrizes the space of all lines in $\mathbb{C}^n$, that is, $\mathrm{Gr}(1,n) \sim \mathbb{P}^{n-1}$. It is well known that $\mathrm{Gr}(k,n)$ is a projective variety, but not always a projective space, and its dimension is $\dim \mathrm{Gr}(k,n) = k(n-k)$.

The *Grassmannian of lines* of a projective space is the space of all lines in $\mathbb{P}^n$. A line in $\mathbb{P}^n$ and a plane in $\mathbb{C}^{n+1}$ are equivalent. Then the Grassmannian of all lines in $\mathbb{P}^n$ is $\mathrm{Gr}(2,n+1)$, which has dimension $\dim \mathrm{Gr}(2,n+1) = 2(n-1)$. Therefore, the Grassmannian of all lines in $\mathbb{P}^3$ is $\mathrm{Gr}(2,4)$ and has dimension $2(4-2) = 4$.

For convenience, we will denote the Grassmannian of a projective space as $\mathrm{Gr}_{\mathbb{P}}(k,n) = \mathrm{Gr}(k+1,n+1)$ and its dimension will be $\dim \mathrm{Gr}_{\mathbb{P}}(k,n) = (k+1)(n-k)$. Therefore, the Grassmannian of lines of $\mathbb{P}^3$ is $\mathrm{Gr}_{\mathbb{P}}(1,3) = \mathrm{Gr}(2,4)$.

1.3 Homogeneous polynomials

Let $\mathbb{C}[X_0,\ldots,X_n]$ be the ring of polynomials in $n+1$ complex variables and $d \in \mathbb{N}$. We say that $F \in \mathbb{C}[X_0,\ldots,X_n]$ is a *homogeneous polynomial* of degree d if

$$F = \sum_{i_0+\cdots+i_n=d} a_{i_0,\ldots,i_n} X_0^{i_0} \ldots X_n^{i_n}.$$

F is a homogeneous polynomial of degree d if and only if $F(\lambda X_0,\ldots,\lambda X_n) = \lambda^d F(X_0,\ldots,X_n)$ for any $\lambda \in \mathbb{C}$. Following the standard notation, we will denote by $|\mathscr{O}_{\mathbb{P}^n}(d)|$ the vector space of homogeneous polynomials of degree d in $n+1$ variables. Note that

$$\mathbb{C}[X_0,\ldots,X_n] = \sum_{d=0}^{\infty} |\mathscr{O}_{\mathbb{P}^n}(d)|$$

which gives the ring of polynomials $\mathbb{C}[X_0,\ldots,X_n]$ a graded structure.

This graduation allows for any polynomial $F \in \mathbb{C}[X_0,\ldots,X_n]$ to have a decomposition such as:

$$F = \sum_{i=0}^{d} F_i,$$

where $F_i \in |\mathscr{O}_{\mathbb{P}^n}(i)|$.

Let $f \in \mathbb{C}[X_0,\ldots,X_{n-1}]$ be a polynomial of degree d, the *homogenization* of f with respect to the variable X_n is the homogeneous polynomial $f^* \in \mathbb{C}[X_0,\ldots,X_n]$ of the same degree, given by

$$f^*(X_0,\ldots X_n) = X_n^d f\left(\frac{X_0}{X_n},\ldots,\frac{X_{n-1}}{X_n}\right).$$

Let $F \in \mathbb{C}[X_0,\ldots,X_n]$ be a homogeneous polynomial, the *dehomogenization* of F with respect to the variable X_n is the polynomial $F_* \in \mathbb{C}[X_0,\ldots,X_{n-1}]$ given by

$$F_*(X_0,\ldots,X_{n-1}) = F(X_0,\ldots,X_{n-1},1).$$

It is easy to see that $(f^*)_* = f$, but $(F_*)^* \neq F$ in general. For example, let $F(X_0,\ldots,X_n) = X_0X_1\cdots X_n$, which is a polynomial of degree $n+1$. Then $F_*(X_0,\ldots,X_{n-1}) = F(X_0,\ldots,X_{n-1},1) = X_0X_1\ldots X_{n-1}$, which is a polynomial of degree n. Then $(F_*)^*$ will be a homogeneous polynomial of degree n (it must have the same degree as F_*), so $(F_*)^*(X_0,\ldots,X_n) = X_0X_1\cdots X_{n-1}$. Therefore $(F_*)^* \neq F$.

1.4 Projective varieties

Let $P \in \mathbb{P}^n$ be a point in the projective space and $F \in \mathbb{C}[X_0, \ldots, X_n]$ be a polynomial, then $F(P)$ is not well defined in general, since $F(P) \neq F(\lambda P)$. If we consider F a homogeneous polynomial of degree d and $F(P) = 0$, then the set

$$\mathrm{V}(F) = \{P \in \mathbb{P}^n \mid F(P) = 0\}$$

is well defined, since $F(\lambda P) = \lambda^d F(P) = 0$. We call the set $\mathrm{V}(F)$ the *zero locus* of the homogeneous polynomial F. Let $\mathscr{F}$ be a family of homogeneous polynomials, we define the zero locus of $\mathscr{F}$ as

$$\mathrm{V}(\mathscr{F}) = \{P \in \mathbb{P}^n \mid F(P) = 0, \forall F \in \mathscr{F}\} \subset \mathbb{P}^n.$$

Any subset of $\mathbb{P}^n$ defined this way is called a *projective algebraic variety* (or simply *projective variety*).

Conversely, given any projective variety S, the family of homogeneous polynomials $I(S) = \{F \mid F(P) = 0, \ \forall P \in S\}$ represents the system of equations vanishing on S. Note that the set $I(S)$ is a homogeneous ideal in $\mathbb{C}[X_0, \ldots, X_n]$.

A *hypersurface* S of degree d in $\mathbb{P}^n$ is the zero locus of a non-zero homogeneous polynomial F of degree d in $n+1$ variables, that is, $S = \mathrm{V}(F)$.

An important difference between projective varieties and differentiable manifolds is given by the concept of singularity. First note that if F is homogeneous, then $\partial F / \partial X_i$ is also homogeneous for any $i \in \{0, \ldots, n\}$. A point $P \in \mathbb{P}^n$ in a hypersurface $S = \mathrm{V}(F)$ is called *singular* if $F(P) = 0$ and $\partial F / \partial X_i(P) = 0$, for all $i \in \{0, \ldots, n\}$, that is, $\mathrm{grad} F(P) = (0, \ldots, 0)$. The set of singular points of S is denoted by $\mathrm{Sing}(S)$. We say that a hypersurface is *nonsingular* if $\mathrm{Sing}(S) = \emptyset$. The set of regular points is the complement of $\mathrm{Sing}(P)$ in S, that is, $\mathrm{Reg}(S) = S \setminus \mathrm{Sing}(S)$.

From the point of view of differential geometry, a projective variety is not necessarily a manifold since it can have singularities. However, $\mathrm{Reg}(S)$ is a manifold. The concept of dimension of an irreducible projective variety can be defined as the dimension of its regular part.

Chapter 2

Classical results on cubic surfaces

In this chapter we will study hypersurfaces of degree 3 in $\mathbb{P}^3$, which we call *cubic surfaces*. Therefore, a cubic surface is the zero locus of a homogeneous polynomial of degree 3 in 4 variables.

2.1 General facts on cubic surfaces

Let us denote the projectivization of the vector space of homogeneous polynomials of degree d in $n+1$ variables as $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^n}(d)|)$. By Hilbert's Nullstellensatz, $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^n}(d)|)$ defines the space of hypersurfaces of degree d in $\mathbb{P}^n$.

Proposition 1. *The set of all cubic surfaces in* $\mathbb{P}^3$, $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$, *is a projective space of dimension* 19.

Proof. Let $F \in |\mathscr{O}_{\mathbb{P}^3}(3)|$ be a homogeneous polynomial of degree 3 in 4 variables, that is,

$$F(x_0,x_1,x_2,x_3) = \sum_{0\leq i+j+k\leq 3} a_{ijk} x_0^i x_1^j x_2^k x_3^{3-i-j-k}.$$

Let us compute the dimension of the vector space $|\mathscr{O}_{\mathbb{P}^3}(3)|$:

$$\dim|\mathscr{O}_{\mathbb{P}^3}(3)| = \#\{a_{ijk}\} = \#\{(i,j,k) \mid 0 \leq i,j,k \leq 3,\ i+j+k \leq 3\}.$$

- For $i = 3$, we have one way to choose j,k such that $j+k \leq 0$.
- For $i = 2$, we have $\binom{3}{2}$ ways to choose j,k such that $j+k \leq 1$.
- For $i = 1$, we have $\binom{4}{2}$ ways to choose j,k such that $j+k \leq 2$.
- For $i = 0$, we have $\binom{5}{2}$ ways to choose j,k such that $j+k \leq 3$.

Since $1 = \binom{2}{2}$ and

$$\binom{n}{2} + \binom{n+1}{2} = \frac{n(n-1)+n(n+1)}{2} = n^2,$$

then

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 2^2 + 4^2 = 20.$$

Therefore $\dim|\mathscr{O}_{\mathbb{P}^3}(3)| = 20$ and

$$\dim\mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|) = 20 - 1 = 19. \qquad \square$$

Since all projective spaces of the same finite dimension are isomorphic, then $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|) \cong \mathbb{P}^{19}$.

Proposition 2. *The space of smooth cubic surfaces is a connected space.*

Proof. Let $S \in \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$ be a singular cubic surface which is given by $S = \mathrm{V}(F)$, where F is a homogeneous polynomial of degree 3. Since S is singular, there exists a point $P \in S$ such that the Jacobian determinant of F vanish, that is, $\det J_F(P) = 0$. This condition makes the set of singular cubic surfaces to be a subspace of (complex) codimension 1 in $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$.

Complex codimension 1 means real codimension 2. Then the subspace of smooth (that is, nonsingular) cubic surfaces is the complement of a subspace of real codimension 2 and hence connected. □

This result implies that there exists a homotopy equivalence between all smooth cubic surfaces, but the proof of this implication requires results that go beyond the scope of this work.

Proposition 3. *The Euler characteristic of a smooth cubic surface is 9.*

Proof. Let us consider the following smooth cubic surface

$$S = \{[x:y:z:w] \in \mathbb{P}^3 \mid x^3 + y^3 + z^3 - w^3 = 0\}.$$

Since the Euler characteristic is a homeomorphism invariant, it is enough to compute the Euler characteristic of S instead of the general case (recall that after Proposition 2 we have concluded that smooth cubic surfaces are homeomorphic).

Let $P = [0:0:0:1] \notin S$ and consider the projection

$$\begin{array}{rccc} \varphi: & \mathbb{P}^3 \setminus \{P\} & \to & \mathbb{P}^2 \\ & [x:y:z:w] & \mapsto & [x:y:z]. \end{array}$$

This map is well defined everywhere, since we don't consider the point P. Also, since $P \notin S$, the restriction $\varphi|_S : S \to \mathbb{P}^2$ is well defined too. Let $q = [x_0 : y_0 : z_0] \in \mathbb{P}^2$, then its preimage in S is

$$\varphi|_S^{-1}(q) = \{[x_0 : y_0 : z_0 : w] \in \mathbb{P}^3 \mid w^3 = x_0^3 + y_0^3 + z_0^3\} \neq \emptyset.$$

Then $\varphi|_S$ is onto and there exist three different preimages for each $q \in \mathbb{P}^2$ as long as $x_0^3 + y_0^3 + z_0^3 \neq 0$. In other words, if we denote by C the curve in $\mathbb{P}^2$ given by the equation $x^3 + y^3 + z^3 = 0$, the map $\varphi|_S$ is 3:1 over $\mathbb{P}^2 \setminus C$ and 1:1 over C. Therefore, $\chi(S) = 3\chi(\mathbb{P}^2 \setminus C) + \chi(C)$. This can be rearranged as $3\chi(\mathbb{P}^2) - 2\chi(C)$.

It is a known fact that the Euler characteristic of the projective plane is $\chi(\mathbb{P}^2) = 3$. Notice that C is an elliptic curve of genus $g = 1$, then $\chi(C) = 2g - 2 = 2 - 2 = 0$.

Considering all of this:

$$\chi(S) = 3\chi(\mathbb{P}^2 \setminus C) + \chi(C) = 3\chi(\mathbb{P}^2) - 2\chi(C) = 3 \cdot 3 - 2 \cdot 0 \ = \ 9.$$ □

2.2 The Cayley-Salmon theorem

An interesting property of cubic surfaces is that they contain a well determined finite number of lines. Following Dolgachev's proof [2] (which in fact follows the ideas of the original proof by Cayley) we will see that this number is 27 and we will study properties that derive from that fact.

Proposition 4. *Any cubic surface contains at least one line.*

Proof. Let X be the incidence variety

$$X = \{(S,\ell) \in \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|) \times \mathrm{Gr}_{\mathbb{P}}(1,3) \mid \ell \subset S\},$$

where $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^n}(d)|)$ is the projectivization of the vector space of homogeneous polynomials of degree d in $n+1$ variables and $\mathrm{Gr}_{\mathbb{P}}(1,n)$ is the Grassmannian of all lines in $\mathbb{P}^n$ (see Chapter 1).

We will consider the projections pr_1 and pr_2 such that

$$\mathrm{pr}_1: \quad X \quad \rightarrow \quad \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|), \quad \mathrm{pr}_2: \quad X \quad \rightarrow \quad \mathrm{Gr}_{\mathbb{P}}(1,3).$$

Let S be a cubic surface, $S \in \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$, then $\mathrm{pr}_2\,\mathrm{pr}_1^{-1}(S)$ is the set of all lines in S. We can deduce that $S \in \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$ will contain a line $\ell \in \mathrm{Gr}_{\mathbb{P}}(1,3)$ if pr_1 is surjective. It is easy to see that the fibers of the second projection pr_2 are projective subspaces of codimension 4: without loss of generality, by a linear change of coordinates we can assume that a line ℓ is given by

$$\ell = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \mid x_2 = x_3 = 0\}$$

and let F be a homogeneous polynomial such that $S = \mathrm{V}(F)$, given by

$$F(x_0,x_1,x_2,x_3) = \sum_{0 \leq i+j+k \leq 3} a_{ijk} x_0^i x_1^j x_2^k x_3^{3-i-j-k}.$$

As $\ell \subset S$, we evaluate $x_2 = x_3 = 0$. Then $F(x_0,x_1,0,0) \equiv 0$ if and only if $a_{i(3-i)} = 0$ for all $i \in \{0,1,2,3\}$, so there are 4 monomials in F that do not contain either x_2 or x_3. These monomials are

$$a_{i(3-i)0} x_0^i x_1^{3-i} \quad \text{for all } i \in \{0,1,2,3\}.$$

Then $\mathrm{pr}_2^{-1}(\ell)$ has codimension 4. Recall that we proved that $\dim \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|) = 19$ in Proposition 1. Therefore, $\dim \mathrm{pr}_2^{-1}(\ell) = 19 - 4 = 15$. Also, since $\dim \mathrm{Gr}_{\mathbb{P}}(1,3) = 4$, we have that

$$\dim X = \dim \mathrm{pr}_2^{-1}(\ell) + \dim \mathrm{Gr}_{\mathbb{P}}(1,3) = 15 + 4 = 19 = \dim \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|).$$

It is enough to find a single cubic surface with only finitely many lines on it to prove the projection pr_1 is finite onto its image, hence (as $\dim X = \dim \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$) it is surjective.

We have to find $S \in \mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$ such that $\mathrm{pr}_1^{-1}(S)$ is a finite set. Let S be a surface given by the equation

$$x_1 x_2 x_3 - x_0^3 = 0.$$

Let ℓ be a line that lies on S and let $[a_0 : a_1 : a_2 : a_3] \in \ell \subset S$. If $a_0 \neq 0$, then $a_0^3 = a_1 a_2 a_3 \neq 0$ and $a_i \neq 0\ \forall i$. On the other hand, every line hits the hyperplanes $x_i = 0$. This shows that ℓ is contained in the hyperplane $x_0 = 0$. But there are only three lines on S contained in this hyperplane:

$$\ell_i = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \mid x_i = x_0 = 0\}, \quad i = 1,2,3.$$

Therefore S contains only three lines, so $\mathrm{pr}_1^{-1}(S)$ is a finite set and the projection pr_1 is surjective. Thus, every cubic surface contains at least one line. □

Recall from Section 1.4 that a cubic surface $S = \mathrm{V}(F)$ is *nonsingular* if it has no singular points, that is, $\mathrm{Reg}(S) = S$.

Theorem 1 (Cayley-Salmon theorem)**.** *A general*[1] *nonsingular cubic surface contains* 27 *lines and* 45 *tritangent planes.*

Proof. By Proposition 4, we already know that every cubic surface $S = \mathrm{V}(F)$ has at least one line. Let us denote such a line by ℓ_0. After an appropriate projective change of coordinates, we may assume that ℓ_0 is given by

$$\ell_0 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \mid x_2 = x_3 = 0\}.$$

Thus we can write F as

$$F = x_2 q_0(x_0,x_1,x_2,x_3) + x_3 q_1(x_0,x_1,x_2,x_3) = 0,$$

where q_0 and q_1 are quadratic forms, as f has degree 3.

Let us denote by $\Pi_{\lambda,\mu} = \mathrm{V}(\lambda x_2 - \mu x_3)$ the pencil of hyperplanes through the line ℓ_0. Any hyperplane H in $\Pi_{\lambda,\mu}$ cuts out S in a curve of degree 3. Since $\ell_0 \subset H \cap S$, the remaining intersection must be a (non-necessarily smooth) conic. Therefore the pencil $\Pi_{\lambda,\mu}$ cuts out a pencil of conics on S. Whenever the conic is degenerated we find more lines in S. Assume $\mu_0 \neq 0$ and take $H = \mathrm{V}(\lambda_0 x_2 - \mu_0 x_3)$ in $\Pi_{\lambda,\mu}$. To compute the intersection $\mathrm{V}(F) \cap H$ we can substitute $x_3 = (\lambda_0/\mu_0)x_2$ in F, so

$$x_2 q_0\left(x_0,x_1,x_2,\frac{\lambda_0}{\mu_0}x_2\right) + \frac{\lambda_0}{\mu_0}x_2 q_1\left(x_0,x_1,x_2,\frac{\lambda_0}{\mu_0}x_2\right) = 0.$$

Since q_0, q_1 are quadratic forms, the maximal degree of μ_0 in the denominator is 3 in the second term. We will factor out μ_0^3 as follows:

$$\frac{x_2}{\mu_0^3}\left(\mu_0^3 q_0\left(x_0,x_1,x_2,\frac{\lambda_0}{\mu_0}x_2\right) + \lambda_0\mu_0^2 q_1\left(x_0,x_1,x_2,\frac{\lambda_0}{\mu_0}x_2\right)\right) = 0.$$

The set of zeroes of x_2/μ_0^3 is a line, which is multiplying the conic. We can expand the conic factor:

$$\frac{x_2}{\mu_0^3}\left(A_{00}(\lambda_0,\mu_0)(\mu_0 x_0)^2 + A_{11}(\lambda_0,\mu_0)(\mu_0 x_1)^2 + 2A_{01}(\lambda_0,\mu_0)(\mu_0 x_0)(\mu_0 x_1) + \right.$$
$$\left. + 2A_{02}(\lambda_0,\mu_0)(\mu_0 x_0)x_2 + 2A_{12}(\lambda_0,\mu_0)(\mu_0 x_1)x_2 + A_{22}(\lambda_0,\mu_0)x_2^2\right) = 0.$$

We apply the transformation $[x_0 : x_1 : x_2] \mapsto [x_0/\mu_0 : x_1/\mu_0 : x_2]$. Then, the equation of the conic in the hyperplane H is given by

$$A_{00}(\lambda_0,\mu_0)x_0^2 + A_{11}(\lambda_0,\mu_0)x_1^2 + A_{22}(\lambda_0,\mu_0)x_2^2 +$$
$$+ 2A_{01}(\lambda_0,\mu_0)x_0x_1 + 2A_{12}(\lambda_0,\mu_0)x_1x_2 + 2A_{02}(\lambda_0,\mu_0)x_0x_2 = 0,$$

where A_{ij} is either zero or a two-variable homogeneous polynomial. Moreover, if A_{ij} is not zero, then A_{00}, A_{11}, A_{01} has degree 1, A_{02}, A_{12} has degree 2, and A_{22} has degree 3.

The conic is given by

$$(x_0,x_1,x_2)\begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

so the discriminant equation of a general conic in the pencil is equal to

$$P(\lambda,\mu) = \det\begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{pmatrix} = A_{00}A_{11}A_{22} + 2A_{01}A_{12}A_{02} - A_{11}A_{02}^2 - A_{22}A_{01}^2 - A_{00}A_{12}^2.$$

It is easy to see that each term is homogeneous of degree 5, so this is a homogeneous equation of degree 5 in variables λ, μ. Thus, we expect $P(\lambda,\mu) = 0$ to have five different roots, which would give us five reducible conics.

---

[1] In algebraic geometry, *general* means outside a projective variety of positive codimension.

We do not know whether the equation has five distinct roots. First, we can show a nonsingular cubic surface and a line on it and check that the equation indeed has five distinct roots.

For example, let us consider the cubic surface given by

$$F(x_0,x_1,x_2,x_3) = 2x_0x_1x_2 + x_3(x_0^2+x_1^2+x_2^2+x_3^2) = 0.$$

First, let us prove it is nonsingular. Note that

$$\operatorname{grad}(F) = (2x_1x_2+2x_0x_3, 2x_0x_2+2x_1x_3, 2x_0x_1+2x_2x_3, x_0^2+x_1^2+x_2^2+3x_3^2).$$

The gradient of F is zero only when $x_i = 0$ for all $i \in 0,1,2,3$, which is not a point in $\mathbb{P}^3$. Therefore we conclude $\mathrm{V}(F)$ is nonsingular. In this case, $q_0 = 2x_0x_1$ and $q_1 = x_0^2+x_1^2+x_2^2+x_3^2$. Intersecting with a hyperplane of $\Pi_{\lambda,\mu}$ we have

$$2x_0x_1x_2 + \frac{\lambda}{\mu}x_2\left(x_0^2+x_1^2+x_2^2+\frac{\lambda^2}{\mu^2}x_2^2\right) = 0.$$

Then

$$\frac{x_2}{\mu^3}\left(2\mu^3x_0x_1 + \lambda\mu^2\left(x_0^2+x_1^2+x_2^2+\frac{\lambda^2}{\mu^2}x_2^2\right)\right) = 0,$$

$$\frac{x_2}{\mu^3}(2\mu^3x_0x_1 + \lambda\mu^2x_0^2 + \lambda\mu^2x_1^2 + (\lambda\mu^2+\lambda^3)x_2^2) = 0,$$

$$\frac{x_2}{\mu^3}(2\mu(\mu x_0)(\mu x_1) + \lambda(\mu x_0)^2 + \lambda(\mu x_1)^2 + (\lambda\mu^2+\lambda^3)x_2^2) = 0.$$

Applying the transformation $[x_0 : x_1 : x_2] \mapsto [x_0/\mu : x_1/\mu : x_2]$:

$$\frac{x_2}{\mu^3}(2\mu x_0x_1 + \lambda x_0^2 + \lambda x_1^2 + (\lambda\mu^2+\lambda^3)x_2^2) = 0,$$

so the conic is given by

$$(x_0,x_1,x_2)\begin{pmatrix} \lambda & \mu & 0 \\ \mu & \lambda & 0 \\ 0 & 0 & \lambda^3+\lambda\mu^2 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

so the discriminant of this conic is equal to

$$\begin{aligned} P(\lambda,\mu) &= \det\begin{pmatrix} \lambda & \mu & 0 \\ \mu & \lambda & 0 \\ 0 & 0 & \lambda^3+\lambda\mu^2 \end{pmatrix} \\ &= \lambda^5 + \lambda^3\mu^2 - \lambda^3\mu^2 - \lambda\mu^4 \\ &= \lambda^5 - \lambda\mu^4 = \lambda(\lambda^4-\mu^4). \end{aligned}$$

Note that $P(\lambda,\mu) = 0$ has five distinct roots: $[0:1]$, $[\pm 1:1]$ and $[\pm\sqrt{-1}:1]$. This implies that, for general[2] nonsingular cubic surfaces, we have five reducible residual conics. We have seen that the determinant is zero for some λ,μ, so the rank of the quadratic form must be less than 3 and the conic must be degenerated. No conic is a double line, since the cubic surface would be singular, so the rank of the quadratic form must be greater than 1. Therefore, the rank is 2 and the conic is a product of two lines.

Each solution (λ,μ) of the quintic equation $P(\lambda,\mu) = 0$ defines a plane $\Pi_i \in \Pi_{\lambda,\mu}$ such that $\Pi_i \cap S = \ell_0 \cup q$, where q is a conic (the product of two lines). This way, the plane Π_i contains three lines: ℓ_0, ℓ_i and ℓ_i'. These lines intersect at three points, which are singular points of $\Pi_i \cap S$. Therefore, by Proposition 16 (see Annex), the plane Π_i is tangent to S at those three points (we call it a *tritangent plane*).

---

[2] Note that *general* in this context means outside the zero set of the discriminant of the polynomials $P(\lambda,\mu) = 0$.

Then we have found 11 lines in S: the line ℓ_0 and five pairs ℓ_i, ℓ_i', one of each plane Π_i. Let us take one plane, for example, Π_1. We have three lines in Π_1: ℓ_0, ℓ_1 and ℓ_1'. Replacing ℓ_0 by ℓ_1 we get four planes which contain ℓ_1, but not ℓ_0. We can do the same for ℓ_1', so we get four planes which contain ℓ_1' but not ℓ_0. Each plane contains a pair of additional lines, so we get 8 more when replacing by ℓ_1 and 8 more when replacing by ℓ_1'. Adding them to the 11 lines we got before, we count $11+8+8=27$ lines in S.

Let us see that we have counted all the possible lines. Any line that intersects either ℓ_0, ℓ_1 or ℓ_2 lies in one of the planes we have considered before and we have counted them.

Let ℓ be any line. We can find a plane Π that contains three lines: ℓ, ℓ' and ℓ''. This plane intersects Π_1, which contains ℓ_0, ℓ_1 and ℓ_1' , at a line. This line intersects S at some point of ℓ and on one of the lines ℓ_0, ℓ_1, ℓ_1'. Then ℓ intersects one of the lines ℓ_0, ℓ_1, ℓ_1' and we have counted it.

Finally, let us count the tritangent planes. Each line belongs to five tritangent planes and each tritangent plane contains three lines. Then the number of tritangent planes is $27 \cdot 5/3 = 45$. $\square$

2.3 Double-sixes

From the proof of Theorem 1, note that any line in S must be secant to 10 other lines in S. In other words each line is skewed with respect to 16 lines. A classical result states that the maximum number of pairwise skewed lines on a smooth cubic surface is 6. This justifies the following definition.

We call a *sixer* to a set of 6 lines in $\mathbb{P}^3$ that do not cut each other. The existence of sixers and the proof that no “7-er” is possible will be given in Proposition 11 after we give a more modern interpretation of smooth cubic surfaces.

A *Schläfli double-six configuration* or simply *double-six* is a set of 12 lines in $\mathbb{P}^3$ which can be partitioned into two subsets of six lines: each line do not cross the lines in its own subset of six lines (so each subset is a sixer) and intersects all but one of the lines in the other subset of six lines. A double-six is often written as a matrix

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}$$

which has the property that two lines among those 12 are secant if and only if they are not placed in the same row or column of the matrix. The notation for lines, double-sixes and tritangent planes used in this chapter is from Dolgachev's book [2] and the article from Betten, Hirschfeld and Karaoglu [1], but we will introduce another one in Chapter 3, which will be more convenient for the purpose of this work.

The 30 pairs of secant lines (a_i, b_j) such that $i \neq j$, generate 30 planes Π_{ij}. The planes Π_{ij} and Π_{ji} intersect in one line $c_{ij} = \Pi_{ij} \cap \Pi_{ji}$. Using the fact that S contains exactly 27 lines one can prove that c_{ij} is in fact a line in S. So there are 15 lines c_{ij} for the 15 pairs of planes $\{\Pi_{ij}, \Pi_{ji}\}$.

Thus, the 27 lines on a cubic surface are the 6 lines a_i, the 6 lines b_i and the 15 lines c_{ij}, where $i, j \in \{1, \ldots, 6\}$.

Also recall from the previous proof that each line of a cubic surface meets 10 other lines. Let us describe these 10 lines for the different types of lines a_i, b_i, c_{ij} of our original double-six.

- a_i meets b_j and c_{ik}, with $i \neq j, k$;
- b_i meets a_j and c_{ik}, with $i \neq j, k$; and
- c_{ij} meets a_i, a_j, b_i, b_j and c_{kl}, with $k, l \neq i, j$.

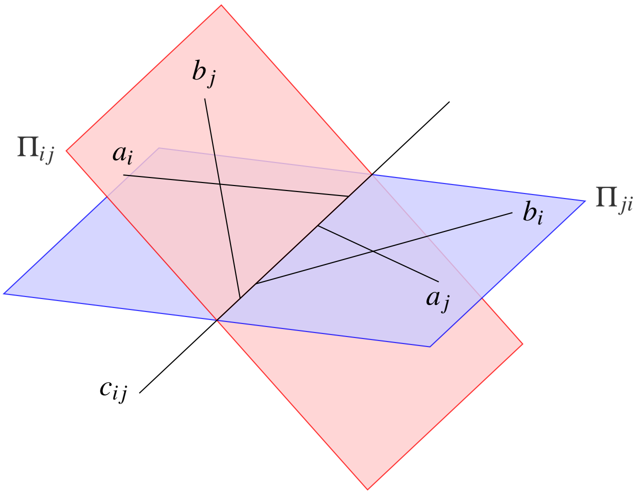

Figure 2.1: Lines in the planes Π_{ij} and Π_{ji}.

Proposition 5. *There are* 36 *double-sixes in a cubic surface, which will be labeled as type D, type* D_{ij} *and type* D_{ijk}*. These double-sixes can be constructed from the original one and are listed below*[3]*:*

- 1 *of type D:*

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix},$$

- 15 *of type* D_{ij}*:*

$$\begin{matrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{matrix},$$

- 20 *of type* D_{ijk}*:*

$$\begin{matrix} a_i & a_j & a_k & c_{lm} & c_{mn} & c_{ln} \\ c_{jk} & c_{ik} & c_{ij} & b_n & b_l & b_m \end{matrix}.$$

All tritangent planes[4] *are listed below:*

- 30 *of type:*

$$[a_i, b_j, c_{ij}], \text{ with } i \neq j,$$

- 15 *of type:*

$$[c_{ij}, c_{kl}, c_{mn}], \text{ with } \{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}.$$

Proof. This Proposition is proved in Subsection 3.6.2 after we have introduced some modern mathematical tools. □

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[3] There are $\binom{6}{0} = 1$ double-sixes of type D, $\binom{6}{2} = 15$ double-sixes of type D_{ij} and $\binom{6}{3} = 20$ double-sixes of type D_{ijk} (we can see the dependence with respect to the number of indices).

[4] There are $6 \cdot 5 = 30$ tritangent planes of type $[a_i, b_j, c_{ij}]$ and $\binom{6}{2} = 15$ tritangent planes of type $[c_{ij}, c_{kl}, c_{mn}]$.

2.4 Eckardt points

In this section we will study a special set of points on a smooth cubic surface. A point on a smooth cubic surface S is called an *Eckardt point* if it is the intersection of three concurrent lines on S. The following result provides an interesting insight on the position of such points and the lines containing them.

Proposition 6. *Let S be a smooth cubic surface, then:*

(i) *Four lines in S cannot be concurrent.*

(ii) *The three lines defining an Eckardt point must be coplanar.*

(iii) *There are at most 45 Eckardt points in S.*

Proof. Assume three lines in S are concurrent at $P \in S$. We will first show they must be coplanar. Otherwise, each plane containing two of them should be a tangent plane T_PS of S at P by Proposition 17. Since P is a smooth point on S, the tangent plane T_PS must be unique and hence the result follows.

Also, by the previous paragraph four concurrent lines should also be coplanar. However, since the intersection of a plane with S is a curve of degree 3, this is not possible.

For the last part, note that different Eckardt points belong in different tritangent planes. Since there are 45 tritangent planes, as we computed at the end of the proof of Theorem 1, the result follows. □

To conclude this section we classify Eckardt points in two types. The proof of this result will be given in Proposition 14.

Proposition 7. *Eckardt points may be of two types:*

- *At most* 30 *of type:*

$$a_i \cap b_j \cap c_{ij}, \text{ with } i \neq j,$$

- *At most* 15 *of type:*

$$c_{ij} \cap c_{kl} \cap c_{mn}, \text{ with } \{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}.$$

An isomorphism between two cubic surfaces is a projective transformation, therefore it maps lines to lines, so it must preserve the number of Eckardt points. Thus, we say that the number of Eckardt points is an isomorphism invariant.

This means that the number of Eckardt points can be used to distinguish cubic surfaces up to isomorphism. If two surfaces have different numbers of Eckardt points, they cannot be isomorphic. However, the converse is false, since there are cubic surfaces with the same number of Eckardt points which are not isomorphic.

Chapter 3

Blow-up model of a cubic surface

Previously we have built surfaces from their equations in $\mathbb{P}^3$. There is another way to build them by operating on other known surfaces. That operation, known as the *blow-up*, will define an atlas of the new surface.

3.1 Blow-up of a point in $\mathbb{C}^2$

Let $B \subset \mathbb{C}^2 \times \mathbb{P}^1$ be a complex surface defined as

$$B = \{((x,y),[U:V]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xV = yU\}. \tag{3.1}$$

The projective point $[U:V]$ is the class of a line in $\mathbb{C}^2$ and the condition $xV = yU$ means that the complex point (x,y) lies inside that line.

Note that B has two charts: $B_U = \{p \in B \mid U \neq 0\}$ and $B_V = \{p \in B \mid V \neq 0\}$. The maps

$$\begin{array}{rrcl} \varphi_U : & B_U & \to & \mathbb{C}^2 \\ & ((x,y),[U:V]) & \mapsto & \left(x,\dfrac{V}{U}\right) \end{array} \qquad \begin{array}{rrcl} \varphi_V : & B_V & \to & \mathbb{C}^2 \\ & ((x,y),[U:V]) & \mapsto & \left(y,\dfrac{U}{V}\right) \end{array} \tag{3.2}$$

are isomorphisms between B_U, (resp. B_V) and $\mathbb{C}^2$. Let $\varphi_{U,V} = \varphi_V \circ \varphi_U^{-1}$ be the change of charts. We can see the way it works:

$$\begin{tikzcd}
\mathbb{C}^2 \supset \varphi_U(B_U \cap B_V) \arrow[r, "\varphi_U^{-1}"] \arrow[rr, bend left, "\varphi_{U,V}"] & B_U \cap B_V \arrow[r, "\varphi_V"] & \varphi_V(B_U \cap B_V) \subset \mathbb{C}^2 \\
(x,v) \arrow[r, mapsto] & ((x,xv),[1:v]) \arrow[r, mapsto] & \left(xv,\frac{1}{v}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(yu,\frac{1}{u}\right) & (y,u). \arrow[l, mapsto, "\psi_{V,U}"']
\end{tikzcd}$$

This makes sense because $v \neq 0$ (resp. $u \neq 0$) since $((x,xv),[1:v]) \in B_V$ (resp. $((yu,y),[u:1]) \in B_U$). We can see that $\psi_{V,U}$ is the inverse map of the change of charts $\varphi_{U,V}$, since

$$(\psi_{V,U} \circ \varphi_{U,V})(x,v) = \psi_{V,U}\left(xv,\frac{1}{v}\right) = \left(xv\frac{1}{v},\frac{1}{1/v}\right) = (x,v).$$

Then B is a complex surface, since it has an atlas made of charts homeomorphic to $\mathbb{C}^2$ whose change of charts $\varphi_{U,V}$ is biholomorphic.

Let us consider the projection on the first component in $\mathbb{C}^2 \times \mathbb{P}^1$:

$$\begin{array}{rcccccc} \mathrm{pr}_1 : & \mathbb{C}^2 & \times & \mathbb{P}^1 & \to & \mathbb{C}^2 \\ & ((x,y) & , & [U:V]) & \mapsto & (x,y). \end{array}$$

Then we can compute the preimage in B (as defined in (3.1)) of a point in $\mathbb{C}^2$:

$$B \cap \mathrm{pr}_1^{-1}(x,y) = \begin{cases} ((0,y),[0:1]), & \text{if } x=0,\ y\neq 0, \\ ((x,0),[1:0]), & \text{if } x\neq 0,\ y=0, \\ ((x,y),[x:y]), & \text{if } x\neq 0,\ y\neq 0, \\ \{(0,0)\}\times\mathbb{P}^1, & \text{if } x=0,\ y=0. \end{cases} \tag{3.3}$$

Since $[x:y]=[1:y/x]$ for $x\neq 0$, one can interpret the second coordinate as the slope of the line joining $(0,0)$ and (x,y). In case $x=0$, the projective point $[0:y]$ represents the vertical line, that is, the line joining $(0,0)$ and $(0,y)$. Thus, we can think of $\mathbb{P}^1$ as the set of all possible lines joining $(0,0)$ and (x,y).

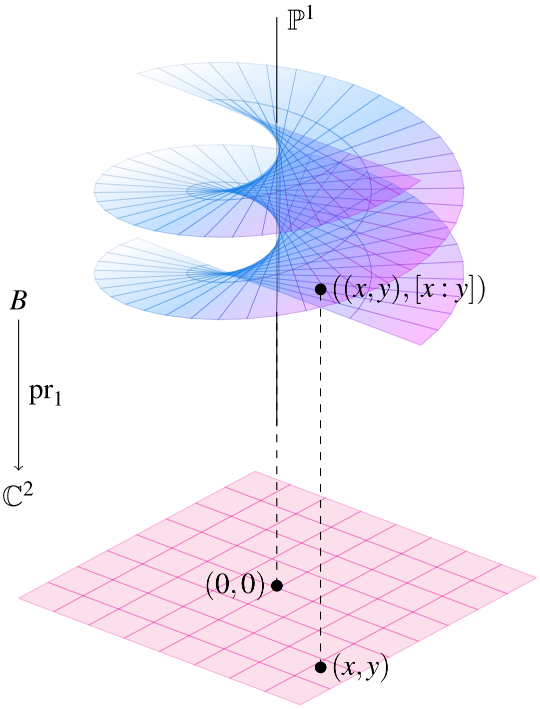

Figure 3.1: Blow-up of $\mathbb{C}^2$ at $(0,0)$.

The map

$$\begin{array}{rcl} \mathbb{P}^1 & \to & \mathrm{Gr}(1,2) \\ [U:V] & \mapsto & \ell = \{(x,y)\in\mathbb{C}^2 \mid xV = yU\} \end{array}$$

is injective, so it is an isomorphism over its image.

Let us denote $\pi = \mathrm{pr}_1|_B$. We will call π the *blow-up of the point* $(0,0)\in\mathbb{C}^2$.

3.2 Blow-up of a point on any surface

Let S be a surface and $p \in S$ a point. Let us take $\mathscr{U} = \{U_i\}$ an atlas of S such that there exists a unique open set $A \in \mathscr{U}$ such that $p \in A$

Since π is biholomorphic in $B \setminus \pi^{-1}(0,0)$, then $B \setminus \pi^{-1}(0,0) \cong \mathbb{C}^2 \setminus \{(0,0)\}$. Given a chart $\varphi_A : A \to \mathbb{C}^2$ of S, then the composition $\varphi_A^{-1} \circ \pi$ is a homeomorphism between $B \setminus \pi^{-1}(0,0)$ and $S \setminus \{p\}$. In particular, it is an identification.

Then we define

$$\hat{S} = \frac{(S \setminus \{p\}) \cup B}{\sim},$$

where $\sim$ is the identification given by $\varphi_A^{-1} \circ \pi|_{B \setminus \pi^{-1}(0,0)}$. We will see that $\hat{S}$ is a complex surface.

Let $\hat{\mathscr{U}} = (\mathscr{U} \setminus \{A\}) \cup \{B_U, B_V\}$ be a cover of $\hat{S}$. We define the following charts of $\hat{S}$:

$$\begin{array}{rcccl} \hat{\varphi}_W = \varphi_W : & W & \to & \mathbb{C}^2 & \text{if } W \in \mathscr{U},\ W \neq A, \\ \hat{\varphi}_U = \varphi_U : & B_U & \to & \mathbb{C}^2 & \text{as in formula (3.2)}, \\ \hat{\varphi}_V = \varphi_V : & B_V & \to & \mathbb{C}^2 & \text{as in formula (3.2)}. \end{array}$$

Let us check that the change of charts is biholomorphic. If $W_1, W_2 \in \mathscr{U} \setminus \{A\}$ are charts in S, then the change of charts $\hat{\varphi}_{W_1,W_2} = \varphi_{W_1,W_2}$ is biholomorphic, since S is a complex surface. Analogously, we have already seen that, for $W_1, W_2 \in \{B_U, B_V\}$, the change of charts $\hat{\varphi}_{W_1,W_2} = \varphi_{W_1,W_2}$ is biholomorphic. Now we have to check what happens if $W_1 \in \mathscr{U} \setminus \{A\}$ is a chart of S and $W_2 \in \{B_U, B_V\}$ is a chart of B. The change of charts will be $\hat{\varphi}_{W_1} \circ \hat{\varphi}_{W_2}^{-1}$. In order to check that this is biholomorphic, one has to consider a point $z \in \varphi_{W_1}(W_1 \cap W_2) \subset \mathbb{C}^2$. Since $p \notin W_1$, one has $q = \varphi_{W_1}^{-1}(z) \neq p$ and hence there is a neighbourhood of q in $S \setminus \{p\}$, say $W \subset S \setminus \{p\}$, such that $q \in W \subset W_1 \cap W_2$. Therefore $z \in \varphi_{W_1}(W)$ is a neighbourhood of z where $\hat{\varphi}_{W_2} \circ \hat{\varphi}_{W_1}^{-1}|_{\varphi_{W_1}(W)}$ is holomorphic.

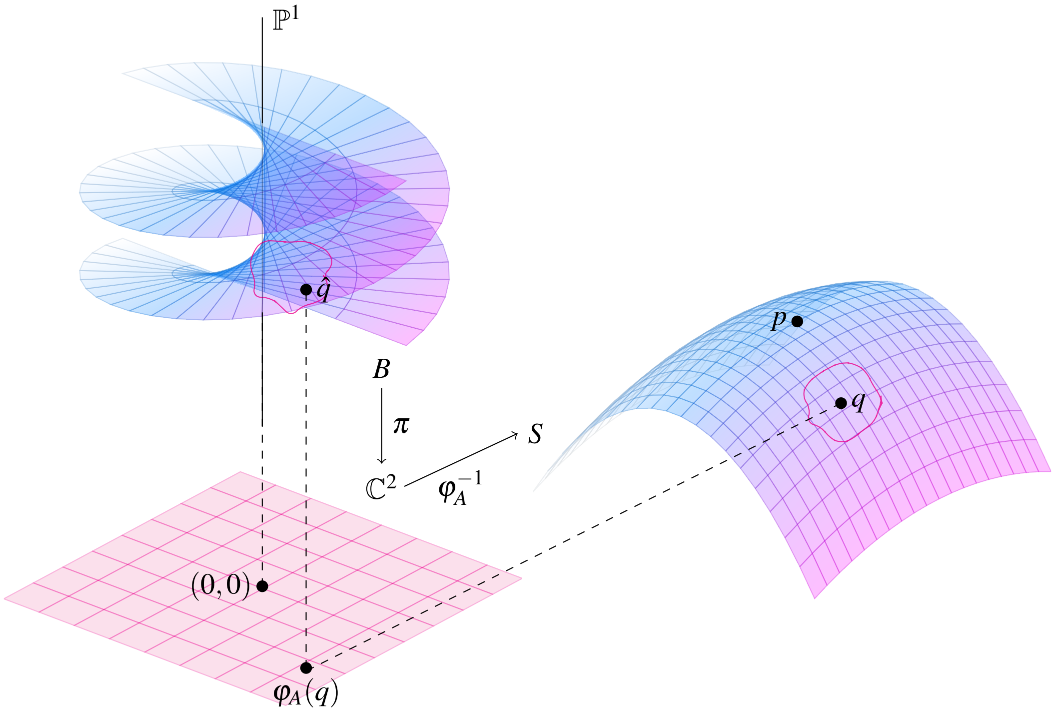

Figure 3.2: Identification of $q \in S$ and $\hat{q} \in B$.

3.3 Blow-up of a point on the complex proyective plane $\mathbb{P}^2$

Since our goal is to describe cubic surfaces as the blow-up of $\mathbb{P}^2$ at six points, we will first prove that $\mathbb{P}^2$ is a complex surface and then apply the previous section to construct the blow-up of one of its points.

3.3.1 The complex structure of $\mathbb{P}^2$

Let $[X:Y:Z] \in \mathbb{P}^2$. Since $(0,0,0) \notin \mathbb{P}^2$, some of the three coordinates must be nonzero. Let us assume $Z \neq 0$ and consider the set of points

$$U_Z = \{[X:Y:Z] \in \mathbb{P}^2 \mid Z \neq 0\} = \left\{ \left[\frac{X}{Z} : \frac{Y}{Z} : 1\right] \in \mathbb{P}^2 \,\middle|\, Z \neq 0 \right\} = \{[x:y:1] \in \mathbb{P}^2\},$$

which is an open set, since for every point such that $Z \neq 0$ there is a neighbourhood of points such that $Z \neq 0$.

Notice that there is only one representative $[x:y:1]$ in each class $[X:Y:Z]$ for $Z \neq 0$. Then the map

$$\begin{array}{rccc} \varphi_Z : & U_Z & \to & \mathbb{C}^2 \\ & [x:y:1] & \mapsto & (x,y) \end{array}$$

is well defined and it is a homeomorphism (since it is a projection) that takes an open set $U_Z \subset \mathbb{P}^2$ to $\mathbb{C}^2$. Therefore the pair (U_Z, φ_Z) defines a chart.

Analogously, one can build the (U_X, φ_X) and (U_Y, φ_Y) charts. It is easy to see that $U_X \cup U_Y \cup U_Z = \mathbb{P}^2$, so these three charts cover all $\mathbb{P}^2$ and they form an atlas. Notice that no two of them cover $\mathbb{P}^2$: for example, $[0:0:1] \notin U_X \cup U_Y$.

Let $\varphi_{X,Y} = \varphi_Y \circ \varphi_X^{-1}$ be the change of charts. We can see the way it works:

$$\begin{tikzcd}
\mathbb{C}^2 \supset \varphi_X(U_X \cap U_Y) \arrow[r, "\varphi_X^{-1}"] \arrow[rr, bend left, "\varphi_{X,Y}"] & U_X \cap U_Y \arrow[r, "\varphi_Y"] & \varphi_Y(U_X \cap U_Y) \subset \mathbb{C}^2 \\
(u,v) \arrow[r, mapsto] & {[1:u:v] = \left[\frac{1}{u} : 1 : \frac{v}{u}\right]} \arrow[r, mapsto] & \left(\frac{1}{u}, \frac{v}{u}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\frac{1}{w}, \frac{z}{w}\right) & & & (w,z). \arrow[lll, mapsto, "\psi_{Y,X}"']
\end{tikzcd}$$

This makes sense because $u \neq 0$ (resp. $w \neq 0$), since $[1:u:v] \in U_Y$ (resp. $[w:1:z] \in U_X$). We can see that $\psi_{Y,X}$ is the inverse map of the change of charts $\varphi_{X,Y}$, since

$$(\psi_{Y,X} \circ \varphi_{X,Y})(u,v) = \psi_{Y,X}\left(\frac{1}{u}, \frac{v}{u}\right) = \left(\frac{1}{1/u}, \frac{v/u}{1/u}\right) = (u,v).$$

Note that $\mathbb{P}^2$ is a complex surface, since it has an atlas made of charts homeomorphic to $\mathbb{C}^2$ whose changes of charts $\varphi_{X,Y}, \varphi_{Y,Z}, \varphi_{Z,X}$ are biholomorphic.

3.3.2 The blow-up of a point of $\mathbb{P}^2$

In this section we want to describe the blow-up of a point $P \in \mathbb{P}^2$. First note that after an affine change of coordinates, one may assume that $P = [0:0:1]$. Consider $\mathscr{U} = \{U_X, U_Y, U_Z\}$ as above. Note that U_Z is the only chart containing P. Thus

$$\hat{\mathbb{P}}^2 = \frac{(\mathbb{P}^2 \setminus \{P\}) \cup B}{\sim},$$

where $\sim$ is the identification given by $\varphi_Z^{-1} \circ \pi|_{B\setminus\pi^{-1}(0,0)}$, that is, we identify $((x,y),[x:y]) \in B$ with $[x:y:1] \in \mathbb{P}^2$ whenever $(x,y) \neq (0,0)$. The new atlas is given as $\hat{\mathscr{U}} = \{U_X, U_Y, B_U, B_V\}$. Note for instance that $\pi(B_U) = \mathbb{C}^2 \setminus \{x=0\} = U_Z \cap U_X$ and analogously $\pi(B_V) = U_Z \cap U_Y$.

The projection map $\pi : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ is also called the blow-up of $\mathbb{P}^2$ at a point. We denote by E the fiber of $P \in \mathbb{P}^2$, that is $E = \pi^{-1}(P)$, and we call it the *exceptional divisor*. This is a copy of $\mathbb{P}^1$ lying over P. As mentioned above (see (3.3)) the map π restricted to $\hat{\mathbb{P}}^2 \setminus E$ is biregular. A map that is biregular when restricted to the complement of a hypersurface is called a *birational* map. Hence, blow-ups of surfaces are birational maps.

3.4 Intersection theory

Intersection theory is defined in a very general setting, but we will start describing it in the simplest case of algebraic curves in $\mathbb{C}^2$. Let $C = \mathrm{V}(f)$ and $C' = \mathrm{V}(g)$ be two curves in $\mathbb{C}^2$ defined by polynomials $f, g \in \mathbb{C}[x,y]$ which are free of squares and have no common components. Consider a point $P \in \mathbb{C}^2$, after a change of coordinates we can assume $P = (0,0)$. We define the *intersection multiplicity* of C and C' at P as:

$$m_P(C,C') = \dim_{\mathbb{C}}(\mathbb{C}\{x,y\}/(f,g)), \tag{3.4}$$

where $\mathbb{C}\{x,y\}$ is the *local ring of* $\mathbb{C}[x,y]$ *at* $P = (0,0)$, that is, the subring of the field of fractions $\mathbb{K}(\mathbb{C}[x,y])$ given by the localization at the maximal ideal, that is,

$$\mathbb{C}\{x,y\} = \left\{ \frac{h_1(x,y)}{h_2(x,y)} \,\middle|\, h_1, h_2 \in \mathbb{C}[x,y],\ h_2(P) \neq 0 \right\}.$$

These functions are also called *regular*.

The *intersection number* of C and C' in $\mathbb{C}^2$ is defined by

$$C \cdot C' = \sum_{P \in \mathbb{C}^2} m_P(C,C').$$

In fact, this number can also be defined for two curves (or *divisors*, as we will call them later) on any compact smooth surface using local charts.

The *self-intersection* of $C \subset S$ is the intersection number of the curve with itself when it is moved slightly apart in S, and it is denoted by C^2.

We will call $D \subset S$ an *irreducible divisor* if D is defined locally as the set of zeroes of irreducible regular functions. Given two irreducible divisors D_1 and D_2 on S, the number $m_P(D_1, D_2)$ is defined using the formula (3.4) for any regular point $P \in S$ of the surface S. If S is for instance smooth and compact, then $D_1 \cap D_2$ must be a finite number of points as long as $D_1 \neq D_2$. Then one can define

$$D_1 \cdot D_2 = \sum_{P \in S} m_P(D_1, D_2).$$

Since $m_P(D_1, D_2) = 0$ whenever $P \notin D_1 \cap D_2$ this sum is finite by the previous discussion. If $D = D_1 = D_2$ the idea is similar after *deforming* D on S. The abelian group formally generated by sums of irreducible divisors is called the group of divisors of S and denoted as $\mathrm{Div}(S)$. The intersection number of irreducible divisors extends by linearity to the following map

$$\begin{array}{ccccc} \mathrm{Div}(S) & \times & \mathrm{Div}(S) & \to & \mathbb{Z} \\ (D_1 & , & D_2) & \mapsto & D_1 \cdot D_2. \end{array}$$

In the particular case when $D_1, D_2 \subset \mathbb{P}^2$ are two projective curves of degrees d_1 and d_2 respectively, by Bézout's Theorem, the map described above is given by $D_1 \cdot D_2 = d_1 d_2$.

Consider $\pi\colon \hat{S} \to S$ the blow-up of S at a smooth point P. Recall that the fiber of $P \in S$, that is $E = \pi^{-1}(P)$, is the exceptional divisor of π. The exceptional divisor is a copy of $\mathbb{P}^1$ in S lying over P.

Proposition 8. *The self-intersection of the exceptional divisor E is $E^2 = -1$.*

Proof. Let $L \subset S$ be a divisor that passes through P and such that P is a smooth point of L. Since π is injective outside of P, if we move L slightly away from P one can see that $L^2 = \pi^{-1}(L)^2 = \hat{L}^2 + 1$.

The preimage of L is $E + \hat{L}$ in B (see figure 3.1). That is, $\pi^{-1}(L) = \hat{L} + E$.

Since the intersection number is bilinear and symmetric, then $(\hat{L}+E)^2 = \hat{L}^2 + 2\hat{L}\cdot E + E^2$. Also, it is easy to see that $\hat{L}$ and E intersect in exactly one point, so $\hat{L} \cdot E = 1$.

To recap, we have got the following equalities

$$L^2 = (\pi^{-1}(L))^2 = (\hat{L}+E)^2 = \hat{L}^2 + 2\hat{L}\cdot E + E^2 = L^2 - 1 + 2 \cdot 1 + E^2.$$

Solving for E^2, we have

$$E^2 = -1.$$

□

This fact characterizes exceptional divisors: if a surface contains a line with self-intersection -1, then it can be blown down to a point. Curves with self-intersection -1 are often called (-1)-*curves*.

As mentioned in the previous proof, the preimage by π of a line L passing through P is $\pi^{-1}(L) = \hat{L} + E$. This is a general fact, namely, if C is a plane curve having P as a smooth point, then $\pi^{-1}(C) = \hat{C} + E$, where $\hat{C} \cdot E = 1$ and $E^2 = -1$. The curve $\hat{C}$ is called the *strict transform* of C by π.

We can define an equivalence relation in the set of divisors. Two divisors D_1 and D_2 of a surface S are linearly equivalent if there is a meromorphic function whose set of zeroes (resp. poles) define D_1 (resp. D_2). The standard notation for this equivalence relation is $D_1 \sim D_2$. An important consequence of this relation is that $D \cdot D_1 = D \cdot D_2$.

3.5 Blow-up description of a cubic surface

Let $P_1, \dots, P_6$ be six points in $\mathbb{P}^2$ in general position. A cubic homogeneous polynomial on $\mathbb{P}^2$ can be written as

$$F(x,y,z) = \sum_{i+j+k=3} a_{ijk} x^i y^j z^k.$$

Since there are 10 parameters a_{ijk}, the vector space of cubic homogeneous polynomials in three variables has dimension 10. Each condition $F(P_i) = 0$ generates a linear equation in the variables a_{ijk}. The condition about general position of the points implies that the matrix of coefficients of $F(P_1) = 0, \dots, F(P_6) = 0$ has rank 6. This means that there will be exactly four independent cubic polynomials $F_0, \dots, F_3$ that vanish at the six points $P_1, \dots, P_6$.

We consider the map

$$\begin{array}{rccc} \phi : & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ & [x:y:z] & \mapsto & [F_0(x,y,z):\cdots:F_3(x,y,z)]. \end{array}$$

Since the polynomials F_i are homogeneous and have the same degree, this map is almost well defined. It is not defined at the points in $\mathrm{V}(F_0,\dots,F_3)=\{P_1,\dots,P_6\}$. Therefore, it is a rational map.

Let X be the blow-up of $\mathbb{P}^2$ at the six points $P_1,\dots,P_6$, then there is a natural map

$$\begin{array}{rccc} \varepsilon : & X & \to & \mathbb{P}^2 \\ & E_i & \mapsto & P_i \end{array}$$

that contracts the six exceptional divisors $E_1,\dots,E_6$ into the points $P_1,\dots,P_6$.

The rational map ϕ extends to a morphism $\Phi : X \to \mathbb{P}^3$, as it is defined everywhere. Then Φ is an embedding, so the image $S=\Phi(X)\subset\mathbb{P}^3$ is a smooth surface isomorphic to X.

Let us calculate the degree of S as a hypersurface of $\mathbb{P}^3$ given by an irreducible homogeneous polynomial F. We can interpret its degree, say d, as the number of points of intersection of S with a projective line ℓ as follows. Let us parametrize the line ℓ as $[t:s]\mapsto[\alpha_0t+\beta_0s:\dots:\alpha_3t+\beta_3s]\in\mathbb{P}^3$. A point $P=[p_0:\dots:p_3]\in\mathbb{P}^3$ is in the intersection $\ell\cap S$ if and only if $p_i=\alpha_it+\beta_is$ is a solution of $\tilde{F}(t,s)=F(\alpha_0t+\beta_0s,\dots,\alpha_3t+\beta_3s)=0$ for some $[t:s]\in\mathbb{P}^1$. Thus the number of solutions of $\tilde{F}(t,s)=0$ coincides with the cardinality of $\ell\cap S$ counted with multiplicity. By a general result (see Proposition 18 in Annex) $\tilde{F}(t,s)$ decomposes as a product of d linear factors counted with multiplicity. Each factor $(\alpha_it-\beta_is)$ contributes with a root $[\beta_i:\alpha_i]\in\mathbb{P}^1$. Therefore d is the cardinality of $\ell\cap S$ for a generic ℓ.

We can write the line ℓ as the intersection of two hyperplanes, $\ell=H_1\cap H_2$. Then, the intersection of ℓ and S is $\ell\cap S=(H_1\cap H_2)\cap S=(H_1\cap S)\cap(H_2\cap S)$. Since we can take the two hyperplanes H_1 and H_2 as close as we want, we can think of $\ell\cap S$ as the self-intersection of the curve $H_S=H\cap S$, obtained by intersecting S with a hyperplane. This curve is called a *hyperplane section* of S. In other words $\ell\cap S=H_S^2$.

The hyperplane section H_S is contained in $\mathbb{P}^3$, but in order to calculate H_S^2 it is more convenient to work in X. Recall that

$$\begin{tikzcd} X \arrow[r, "\varepsilon"] \arrow[rr, bend left, "\Phi"] & \mathbb{P}^2 \arrow[r, dashed, "\phi"] & \mathbb{P}^3 \\ & {[x:y:z]} \arrow[r, mapsto] & {[F_0(x,y,z):\cdots:F_3(x,y,z)].} \end{tikzcd}$$

Since Φ is an embedding one has that $H_S^2=(\Phi^{-1}(H_S))^2$. One can interpret $\Phi^{-1}(H_S)$ as the strict transform $\hat{C}$ of a cubic C in $\mathbb{P}^2$ passing through the six points $P_1,\dots,P_6$. In other words $C=\phi^{-1}(H_S)$ is a cubic curve in $\mathbb{P}^2$ that passes through the points $P_1,\dots,P_6$ and $\varepsilon^{-1}(C)=\hat{C}+E_1+\cdots+E_6$. By the equivalence relation on the set of divisors defined at the end of Section 3.4, a cubic curve is equivalent to three projective lines. We denote this as $C\sim 3L$, where L can be assumed not to contain any P_i. Hence the strict transform $\hat{L}$ and L are in one-to-one correspondence via ε, which implies $\hat{L}^2=L^2=1$.

Summarizing,

$$\Phi^{-1}(H_S)=\hat{C}\sim\varepsilon^{-1}(C)-\sum_{i=1}^{6}E_i\sim 3\hat{L}-\sum_{i=1}^{6}E_i.$$

Now we can calculate the self-intersection ${\phi'}^{-1}(H_S)^2$ as follows

$$\begin{aligned}\Phi^{-1}(H_S)^2 = \hat{C}^2 &= (3\hat{L} - \sum_{i=1}^{6} E_i)^2 \\ &= 9\hat{L}^2 + \sum_{i=1}^{6} E_i^2 - 2\sum_{i=1}^{6} \hat{L}\cdot E_i - 2\sum_{1\leq i<j\leq 6} E_i \cdot E_j \\ &= 9L^2 + \sum_{i=1}^{6} E_i^2 = 9-6=3,\end{aligned}$$

since $L^2 = 1$, $\hat{L}\cdot E_i = E_i \cdot E_j = 0$ (the intersection of these curves is empty) and $E_i^2 = -1$ (see Proposition 8).

Hence $H_S^2 = \Phi^{-1}(H_S)^2 = \hat{C}^2 = 3$, which can be interpreted geometrically as the fact that a line ℓ intersects S three times. This implies that the degree of S is 3. This proves the following.

Theorem 2. *The blow-up of* $\mathbb{P}^2$ *at six points in general position is a smooth cubic surface.*

Proof. It follows from the discussion above. □

In fact, the converse of this result is also true, that is, any smooth cubic surface S can be constructed as the blow-up of six different points in $\mathbb{P}^2$. However, this proof requires a deeper result of algebraic geometry that states that any (-1)-curve on a smooth surface can be blown down obtaining again a smooth surface. Contracting a set of six (-1)-curves in S pairwise disjoint (this is a sixer, as introduced in page 10) one obtains a smooth surface $\tilde{S}$. The Euler characteristic of S is 9 (recall Proposition 3) and hence the Euler characteristic of $\tilde{S}$ is 3 (each time we blow down a (-1)-curve, the Euler characteristic decreases by 1). Finally, it remains to show that $\mathbb{P}^2$ is the only rational surface whose Euler characteristic is 3. Since this requires results that go beyond the scope of this work, we don't present a full proof.

3.6 Lines, double-sixes and Eckardt points

Let us consider a smooth cubic surface S as the blow-up $\varepsilon : S \to \mathbb{P}^2$ of six points $P_1, \ldots, P_6 \in \mathbb{P}^2$ in general position. Our purpose in this section is to count the 27 lines as the (-1)-curves of S. Denote $E_i = \varepsilon^{-1}(P_i)$ the six exceptional divisors of ε, which are (-1)-curves by Proposition 8.

As an immediate consequence of Theorem 2 we have the following result announced in Chapter 2.

Proposition 9. *Any smooth cubic surface contains a sixer.*

Proof. The exceptional divisors E_i obtained above describe 6 pairwise skewed lines on S. □

3.6.1 The geometry of the 27 lines translated to $\mathbb{P}^2$

Our purpose now is to describe the projection on $\mathbb{P}^2$ of the lines on S.

Consider a line L_{ij} joining the points P_i and P_j. Note that $\varepsilon^{-1}(L_{ij}) = \hat{L}_{ij} + E_i + E_j$, where $\hat{L}_{ij}$ is the strict transform of L_{ij}. If we compute the self intersection:

$$\begin{aligned}L_{ij}^2 &= (\varepsilon^{-1}(L_{ij}))^2 = (\hat{L}_{ij} + E_i + E_j)^2 \\ &= \hat{L}_{ij}^2 + E_i^2 + E_j^2 + 2\hat{L}_{ij}\cdot E_i + 2\hat{L}_{ij}\cdot E_j + 2E_i \cdot E_j \\ &= \hat{L}_{ij}^2 - 1 - 1 + 2\cdot 1 + 2\cdot 1 + 2\cdot 0 = \hat{L}_{ij}^2 + 2.\end{aligned}$$

Since $L_{ij}^2 = 1$, solving for $\hat{L}_{ij}^2$, we have:

$$\hat{L}_{ij}^2 = L_{ij}^2 - 2 = 1 - 2 = -1.$$

Therefore, $\hat{L}_{ij}$ is also a (-1)-curve.

Let us consider the conic C_k determined by the other five points that are not P_k. If we blow up this conic, we have

$$\varepsilon^{-1}(C_k) = \hat{C}_k + \sum_{i \neq k} E_i,$$

where $\hat{C}_k$ is the strict transform of the conic C_k. If we compute its self-intersection:

$$\begin{aligned} C_k^2 = (\varepsilon^{-1}(C_k))^2 &= \left(\hat{C}_k + \sum_{i \neq k} E_i\right)^2 \\ &= \hat{C}_k^2 + \sum_{i \neq k} E_i^2 + 2\sum_{i \neq k} \hat{C}_k \cdot E_i + 2\sum_{\substack{i,j \neq k \\ i<j}} E_i \cdot E_j \\ &= \hat{C}_k^2 - 5 + 2 \cdot 5 + 2 \cdot 0 = \hat{C}_k^2 + 5. \end{aligned}$$

Since a conic has degree 2, its self-intersection is $C_k^2 = 2^2 = 4$. Solving for $\hat{C}_k^2$:

$$\hat{C}_k^2 = C_k^2 - 5 = 4 - 5 = -1.$$

Therefore, $\hat{C}_k$ is also a (-1)-curve.

A straightforward calculation shows that these are the only (-1)-curves in S. Let us count how many we have constructed: 6 of type E_i, $\binom{6}{2} = 15$ of type L_{ij} and 6 of type C_k, that is, there are 27 (-1)-curves in S.

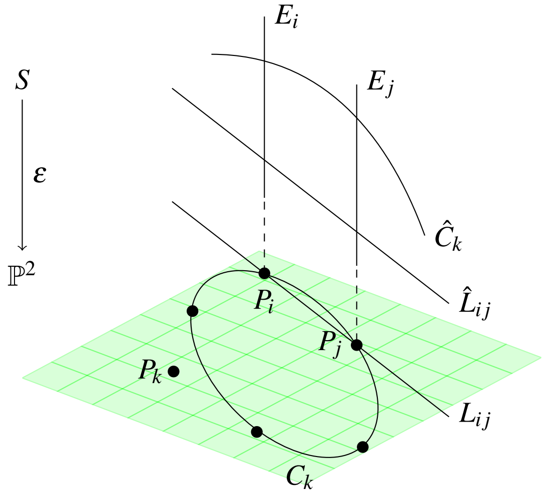

Figure 3.3: (-1)-curves on S: type E_i, type $\hat{L}_{ij}$ and type $\hat{C}_k$.

Proposition 10. *Following the construction above, the 27 lines on S are the following curves:*

- *the 6 exceptional divisors E_i,*
- *the strict transform of the 6 conics C_i, and*
- *the strict transform of the 15 lines L_{ij}.*

Proof. After the discussion above all it is left to check is that these 27 (-1)-curves in S are exactly the 27 lines in S. Let ℓ_1, ℓ_2 and ℓ_3 be three lines of a smooth cubic surface such that $(\ell_1+\ell_2+\ell_3)$ is a hyperplane section of S. In page 20 we proved that hyperplane sections of cubic surfaces have self-intersection 3. Then $(\ell_1+\ell_2+\ell_3)^2=3$ and

$$3=\left(\sum_i \ell_i\right)^2=\sum_i \ell_i^2+2\sum_{i\neq j}\ell_i\cdot\ell_j=\sum_i \ell_i^2+6.$$

Therefore

$$\sum_i \ell_i^2=-3.$$

As seen in Theorem 1, there are 27 possible variables ℓ_i (lines) and 45 possible equations (tritangent planes). Therefore, the only solution is $\ell_i^2=-1$ for $i\in\{1,\dots,27\}$. Then the 27 lines of a smooth cubic surface coincide with the (-1)-curves obtained before. □

Proposition 11. *There are no* 7 *pairwise skewed lines on S.*

Proof. Assume $\ell_1,...,\ell_6,\ell_7$ are such lines. Since they are all (-1)-curves we can contract $\ell_1,...,\ell_6$. The target surface must be $\mathbb{P}^2$ and the image of ℓ_7 has to be one of the remaining curves described in Proposition 10. However, both L_{ij} and C_i pass through at least one P_j and then ℓ_7 must intersect ℓ_j. □

3.6.2 The geography of the 27 lines

The purpose of this section is to describe how the (-1)-curves intersect. In particular we will present a proof of Proposition 5. We will use the notation E_i, $\hat{C}_i$ and $\hat{L}_{ij}$ for the lines in S as described at the beginning of this section and analogously for P_i, C_i and L_{ij} as the points and curves in $\mathbb{P}^2$ resulting from their projection by the blow-up ε.

Proposition 12. *There are* 72 *possible sixers in a cubic surface, namely,*

- 1 *of kind* $\{E_1,E_2,E_3,E_4,E_5,E_6\}$,
- 1 *of kind* $\{\hat{C}_1,\hat{C}_2,\hat{C}_3,\hat{C}_4,\hat{C}_5,\hat{C}_6\}$,
- 30 *of kind* $\{E_i,\hat{C}_i,\hat{L}_{jk},\hat{L}_{jl},\hat{L}_{jm},\hat{L}_{jn}\}$,
- 20 *of kind* $\{E_i,E_j,E_k,\hat{L}_{lm},\hat{L}_{mn},\hat{L}_{ln}\}$,
- 20 *of kind* $\{\hat{L}_{jk},\hat{L}_{ik},\hat{L}_{ij},\hat{C}_n,\hat{C}_l,\hat{C}_m\}$.

Proof. Starting with a (-1)-curve, one can easily build a tree graph with possible candidates to obtain a sixer. It is enough to keep in mind that

$$E_i\cap E_j=\emptyset,\quad E_i\cap\hat{C}_i=\emptyset,\quad E_i\cap\hat{L}_{jk}=\emptyset,\quad \hat{C}_i\cap\hat{C}_j=\emptyset,\quad \hat{C}_i\cap\hat{L}_{jk}=\emptyset,\quad \hat{L}_{ij}\cap\hat{L}_{kl}=\emptyset,$$

are the only empty intersections and any other cannot be considered to build a sixer.

Since $E_i\cap E_j=\emptyset$, and $\hat{C}_i\cap\hat{C}_j=\emptyset$ for $i\neq j$, it is immediate that both $\{E_1,...,E_6\}$ and $\{C_1,...,C_6\}$ form sixers.

For any pair (i,j), $i\neq j$, we can form another sixer using E_i, $\hat{C}_i$, and $\hat{L}_{jk}$ for $k\neq i,j$. Since (i,j) form a different sixer than (j,i), there are $2\cdot\binom{6}{2}=2\cdot 15=30$ of them.

Finally, let us fix $\{i,j,k\}$, where $i\neq j\neq k\neq i$. We can form another sixer using E_i, E_j, E_k and $\hat{L}_{mn}$ for $m,n\neq i,j,k$. There are $\binom{6}{3}=20$ of them. Also, one can build an analogous sixer fixing $\{l,m,n\}$, where

$l \neq m \neq n \neq l$, and using $\hat{C}_l$, $\hat{C}_m$, $\hat{C}_n$ and $\hat{L}_{ij}$ for $i, j \neq l, m, n$. This choice of indices will be justified later when we build double-sixes. There are 20 of them too.

One can check that there are no more possible combinations. □

Now let us build all the possible double-sixes. Recall that a double-six could be written as an array where each row is a sixer and each element is skewed with everyone in its same row and column. Now we can prove the result on the geography of double-sixes announced in Proposition 5. Since notation is slightly different, we state here again for convenience.

Proposition 13. *There are* 36 *double-sixes in a cubic surface, namely*

- 1 *of type D:*
$$\begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \\ \hat{C}_1 & \hat{C}_2 & \hat{C}_3 & \hat{C}_4 & \hat{C}_5 & \hat{C}_6 \end{array},$$
- 15 *of type D_{ij}:*
$$\begin{array}{cccccc} E_i & \hat{C}_i & \hat{L}_{jk} & \hat{L}_{jl} & \hat{L}_{jm} & \hat{L}_{jn} \\ E_j & \hat{C}_j & \hat{L}_{ik} & \hat{L}_{il} & \hat{L}_{im} & \hat{L}_{in} \end{array},$$
- 20 *of type D_{ijk}:*
$$\begin{array}{cccccc} E_i & E_j & E_k & \hat{L}_{lm} & \hat{L}_{mn} & \hat{L}_{ln} \\ \hat{L}_{jk} & \hat{L}_{ik} & \hat{L}_{ij} & \hat{C}_n & \hat{C}_l & \hat{C}_m \end{array}.$$

Proof. Taking the sixers obtained in Proposition 12, one can easily see that these are the only possible combinations that form double-sixes. □

Recall that we defined the Eckardt points in Section 2.4 as the point of intersection of three lines in a tritangent plane. Checking case by case one can prove the following about the possible realizations of Eckardt points, which proves Proposition 7. Again, since notation is slightly different, we include the statement for convenience.

Proposition 14. *There are only two types of Eckardt points:*

- *At most 15 of type:*
$$(\hat{L}_{ij}, \hat{L}_{kl}, \hat{L}_{mn}), \text{ with } \{i,j,k,l,m,n\} = \{1,2,3,4,5,6\},$$
where the lines L_{ij}, L_{kl} and L_{mn} intersect in a triple point different from $P_1, \dots, P_6$.
- *At most 30 of type:*
$$(E_i, \hat{C}_j, \hat{L}_{ij}), \text{ with } i \neq j,$$
where, the line L_{ij} and the conic C_j are tangent to each other at P_i.

3.6.3 Example

Recall from Proposition 1 that the space of all cubic surfaces in $\mathbb{P}^3$, denoted by $\mathbb{P}(|\mathscr{O}_{\mathbb{P}^3}(3)|)$, is a projective space of dimension 19. Let us denote by $M^{(i)}$ the subspace of the smooth cubic surfaces that have at least i Eckardt points. Note that $M^{(0)}$ is the space of all smooth cubic surfaces. Also $M^{(i+1)} \subset M^{(i)}$.

Since general cubic surfaces are smooth, this means $\dim M^{(0)} = 19$. The space of smooth cubic surfaces with at least one Eckardt point, $M^{(1)}$, requires one additional condition to make a line pass through the intersection of other two (see Figure 3.4). Then $\dim M^{(1)} = 18$. Analogously $\dim M^{(2)} = 17$. However,

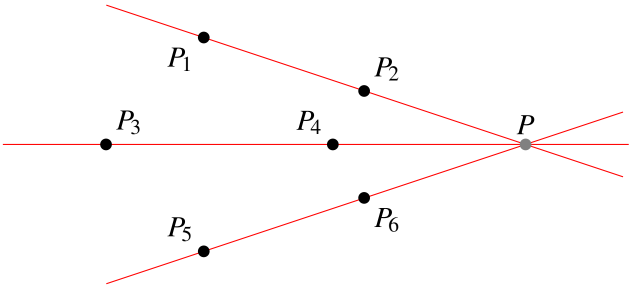

Figure 3.4: Description on $\mathbb{P}^2$ of a cubic surface $S \in M^{(1)}$ with one Eckardt point P.

the general formula $\dim M^{(i)} = 19 - i$ is not true since the additional conditions might not be independent.

We claim the space $M^{(2)}$ is not irreducible. To prove this it is enough to show that the space of general elements of $M^{(2)}$ is not connected. To see this, note that the configuration of two Eckardt points has two possibilities:

- Two Eckardt points $Q, Q' \in S$ that share a line in S, that is, $\overline{QQ'} \subset S$. The irreducible component of $M^{(2)}$ satisfying this property is denoted by $M_1^{(2)}$.
- Two Eckardt points $Q, Q' \in S$ that do not share a line in S, that is, $\overline{QQ'} \not\subset S$. The irreducible component of $M^{(2)}$ satisfying this property is denoted by $M_2^{(2)}$.

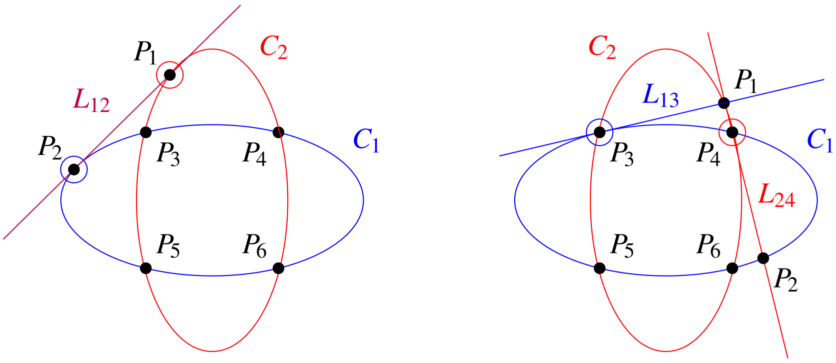

Figure 3.5: Description on $\mathbb{P}^2$ of the cubic surfaces $S_1 \in M_1^{(2)}$ (left) and $S_2 \in M_2^{(2)}$ (right).

Proposition 15. *The space $M^{(2)}$ is not irreducible.*

Proof. Consider two smooth conics C_1 and C_2 in $\mathbb{P}^2$ intersecting transversally and denote by P_3, P_4, P_5, P_6 their intersection points. Consider a bitangent line L_{12} to both C_1 and C_2. Denote by P_2 (resp. P_1) the intersection $C_1 \cap L_{12}$ (resp. $C_2 \cap L_{12}$) (see Figure 3.5). The cubic surface S_1 resulting from the blow-up of $P_1, \dots, P_6$ has two Eckardt points $\{Q_1\} = \hat{L}_{12} \cap E_1 \cap \hat{C}_2$ and $\{Q_1'\} = \hat{L}_{12} \cap E_2 \cap \hat{C}_1$. Hence $S_1 \in M_1^{(2)}$ since $\overline{Q_1 Q_1'} = \hat{L}_{12} \subset S_1$.

Consider now a line L_{13} tangent to C_1 at P_3 and L_{24} tangent to C_2 at P_4. Denote by P_1 (resp. P_2) the intersection $C_2 \cap L_{13}$ (resp. $C_1 \cap L_{24}$) (see Figure 3.5). The cubic surface S_2 resulting from the blow-up of $P_1, \dots, P_6$ has two Eckardt points $\{Q_2\} = \hat{L}_{13} \cap E_3 \cap \hat{C}_1$ and $\{Q_2'\} = \hat{L}_{24} \cap E_4 \cap \hat{C}_2$. Hence $S_2 \in M_2^{(2)}$ since $\overline{Q_2 Q_2'} \not\subset S_2$.

Finally, any projective transformation ϕ of $\mathbb{P}^3$ such that $\phi(S_1) = S_2$ should send Eckardt points to Eckardt points, in particular, $\{Q_1, Q_1'\} \mapsto \{Q_2, Q_2'\}$, which means $\phi(\overline{Q_1 Q_1'}) = \overline{Q_2 Q_2'}$. However, $\phi(\overline{Q_1 Q_1'}) = \phi(\hat{L}_{12}) \subset \phi(S_1) = S_2$ which contradicts $\overline{Q_2 Q_2'} \not\subset S_2$. □

Annex

Here we present some general facts about algebraic geometry that will be useful for our work.

Proposition 16. *Let* $S = \mathrm{V}(F)$*, be a hypersurface,* $P \in S$ *a smooth point in* S*, and* H *a hyperplane in* $\mathbb{P}^n$*. Then* $P \in H \cap S$ *is singular point of* $H \cap S$ *as a hypersurface in* H *if and only if* H *is tangent to* S *at* P*.*

Proof. Let $P \in H \cap S$. After an appropriate change of coordinates we can write $P = [1 : 0 : \cdots : 0] \in S$. We can dehomogenize F in the first coordinate and denote it by f. This way we can work in the affine chart $\{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$, which is isomorphic to the affine space $\mathbb{C}^n$. Thus, P corresponds to $p = (0, \ldots, 0) \in \mathrm{V}(f)$. Let f be the polynomial

$$f(x_1, \ldots, x_n) = a_{10\cdots0}x_1 + \cdots + a_{0\cdots01}x_n + a_{20\cdots0}x_1^2 + \cdots$$

Notice that, as $p \in \mathrm{V}(f)$, then $f(0, \ldots, 0) = 0$, so there is no independent term. It is easy to see that $\mathrm{grad}(f)|_p = (a_{10\cdots0}, \ldots, a_{0\cdots01}) \neq (0, \ldots, 0)$, as $P \in S$ is a smooth point by hypothesis.

Since the gradient $\mathrm{grad}(f)|_p$ is the normal vector of the hypersurface $\mathrm{V}(f)$ at p, it defines the tangent hyperplane T_f of $\mathrm{V}(f)$ at p, whose equation is

$$T_f = \mathrm{V}(a_{10\cdots0}x_1 + \cdots + a_{0\cdots01}x_n).$$

This shows that P must be a singular point at $T_f \cap V(f)$.

Now for the proof of the "if" part, let H be a general hyperplane, which can be written as

$$H = \mathrm{V}(\lambda_1 x_1 + \cdots + \lambda_{n-1}x_{n-1} + x_n)$$

after maybe reordering the variables. Moreover, since $\mathrm{grad}(f)|_p \neq 0$ we can assume $a_{0\cdots01} = 1$. If $H \neq T_f$, then $\mathrm{V}(f) \cap H$ is defined by the equation

$$g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, -\lambda_1 x_1 - \cdots - \lambda_{n-1}x_{n-1})$$

Let $C = \mathrm{V}(f) \cap H = \mathrm{V}(g) \subset H$ be an affine variety in $n-1$ variables $(x_1, \ldots, x_{n-1})$. As above, the gradient of g at $p = (0, \ldots, 0)$ is given by the linear part of g. Using the chain's rule one has

$$\mathrm{grad}(g)|_p = \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_{n-1}}\right)\Bigg|_p = (a_{10\cdots0} - \lambda_1, \ldots, a_{0\cdots010} - \lambda_{n-1}).$$

Therefore, C is singular at the point $p = (0, \ldots, 0)$ if and only if $\lambda_1 = a_{10\cdots0}, \ldots, \lambda_{n-1} = a_{0\cdots010}$. In that case

$$\lambda_1 x_1 + \cdots + \lambda_{n-1}x_{n-1} + x_n = a_{10\cdots0}x_1 + \cdots + a_{0\cdots010}x_{n-1} + x_n$$

which means $H = T_f$. This proves the result. □

This result can be applied to smooth cubic surfaces directly providing the following description of tangent planes.

Proposition 17. *Let S be a nonsingular cubic surface, let P be a point of S and let T_PS denote the tangent plane of S at P.*

(i) *If P is on no line of S, then T_PS meets S in an irreducible cubic with a double point at P.*

(ii) *If P is on exactly one line of S, then T_PS meets S in the line plus a conic through P.*

(iii) *If P is on exactly two lines of S, then T_PS meets S in these two lines plus another line forming a triangle.*

(iv) *If P is on exactly three lines of S, then T_PS meets S in these three concurrent lines.*

Proof. This is an immediate consequence of Proposition 16. □

Proposition 18. *Any homogeneous polynomial in two variables $F \in \mathbb{C}[X,Y]$ of degree d decomposes as a product of d linear factors counted with multiplicity*

$$F(X,Y) = \prod(\alpha_i X - \beta_i Y)^{m_i}$$

where $\sum_i m_i = d$.

Proof. Let $F(X,Y)$ be a polynomial of degree d and $F_*(X,Y) = F(X,1)$ be its dehomogenization. Recall that the dehomogenization does not always preserve the degree of a polynomial, as seen in Section 1.3.

If $F(X,0) \neq 0$, then $F_*(X,Y) = F(X,1)$ has degree d. By the Fundamental Theorem of Algebra, $F(X,1)$ have exactly d roots (counted with multiplicity). Therefore, we can decompose $F(X,1)$ as the product of d linear factors $F(X,1) = \gamma(X-\gamma_1)\cdots(X-\gamma_d)$. If we homogenize $F(X,1)$, then we get $F(X,Y) = \gamma(X-\gamma_1 Y)\cdots(X-\gamma_d Y)$, which also has d roots.

If $F(X,0) = 0$, then $F_*(X,Y) = F(X,1)$ have degree less than d. However, in this case, F can be written as $F(X,Y) = Y^k \tilde{F}(X,Y)$, where $\tilde{F}$ is a homogeneous polynomial of degree $d-k$ such that $\tilde{F}(X,0) \neq 0$. Then we can apply the previous reasoning for $\tilde{F}$, of degree $d-k$, and consider the k additional roots that were lost in dehomogenization. □

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