THE SQUARE NEGATIVE CORRELATION PROPERTY ON ℓ_p^n-BALLS

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Abstract. In this paper we prove that for any $p \in [2, \infty)$ the ℓ_p^n unit ball, B_p^n, satisfies the square negative correlation property with respect to every orthonormal basis, while we show it is not always the case for $1 \leq p \leq 2$. In order to do that we regard B_p^n as the orthogonal projection of B_p^{n+1} onto the hyperplane $e_{n+1}^\perp$. We will also study the orthogonal projection of B_p^n onto the hyperplane orthogonal to the diagonal vector $(1, \ldots, 1)$. In this case, the property holds for all $p \geq 1$ and n large enough.

1. Introduction and notation

A random vector X on $\mathbb{R}^n$ is said to satisfy the square negative correlation property (SNCP) with respect to the orthonormal basis $\{\eta_i\}_{i=1}^n$ if for every $i \neq j$

$$\mathbb{E}\langle X, \eta_i\rangle^2\langle X, \eta_j\rangle^2 - \mathbb{E}\langle X, \eta_i\rangle^2\mathbb{E}\langle X, \eta_j\rangle^2 \leq 0,$$

where $\mathbb{E}$ denotes the expectation and $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^n$.

The study of the SNCP of random vectors uniformly distributed on convex bodies with respect to *some* orthonormal basis appeared in [ABP] in the context of the central limit problem for convex bodies, where the authors showed that for any $p \geq 1$ a random vector uniformly distributed on $B_p^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\|_p^p = \sum_{i=1}^n |x_i|^p \leq 1\}$ satisfies the SNCP with respect to the canonical basis $\{e_i\}_{i=1}^n$. In [W], this result was extended to random vectors uniformly distributed on generalized Orlicz balls, also with respect to the canonical basis. These two papers actually show a much stronger result than the SNCP with respect to the canonical basis, they establish the negative association (see [ABP] and [W]). A straightforward consequence is that, by the rotational invariance of B_2^n, a random vector uniformly distributed on B_2^n satisfies the SNCP with respect to *every* orthonormal basis. The first non-trivial example in this new situation appeared in [AB1], where it was proved that any random vector uniformly distributed on any hyperplane projection of B_∞^n satisfies the SNCP with respect to *every* orthonormal basis. In particular, the SNCP with respect to *every* orthonormal basis is satisfied by B_∞^n itself. On the other hand, it is not hard to show that a random vector uniformly distributed on B_1^n does not satisfy the SNCP with respect to every orthonormal basis (see the second part of Corollary 1.1 below).

The relation between the SNCP and the central limit problem comes from the fact (see [ABP]) that if the Euclidean norm of a random vector uniformly distributed on an isotropic convex body is highly concentrated, then most of the marginals are

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approximately Gaussian and the fact (see, for instance, [AB2, Proposition 1.8]) that if a zero-mean random vector uniformly distributed on a convex body K in $\mathbb{R}^n$ satisfies the SNCP with respect to *some* orthonormal basis, then it verifies the so called *General Variance Conjecture* which states:

There exists an absolute constant C such that for every zero-mean random vector X uniformly distributed on a convex body

$$Var\|X\|_2^2 \leq C\lambda_X^2 \mathbb{E}\|X\|_2^2,$$

where Var denotes the variance, $\lambda_X^2 = \max_{\xi\in S^{n-1}} \mathbb{E}\langle X,\xi\rangle^2$ is the largest eigenvalue of the covariance matrix of X and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$.

Furthermore, [AB2, Proposition 1.9], if a zero-mean random vector uniformly distributed on K satisfies the SNCP with respect to *every* orthonormal basis, then TK verifies the General Variance Conjecture for every linear isomorphism T in $\mathbb{R}^n$.

This is a particular case of a well-known conjecture due to Kannan-Lovász-Simonovits (see [AB2], for detailed explanations on this topic).

In Section 3 we study the SNCP on random vectors uniformly distributed on $B_p^n, p \geq 1$, with respect to any orthonormal basis. The main result is the following

Theorem 1.1. *Let X be a random vector uniformly distributed on B_p^n, $p \geq 1$, and write $\xi_1 = \frac{e_1+e_2}{\sqrt{2}}$, $\xi_2 = \frac{e_1-e_2}{\sqrt{2}}$. Let $f : S^{n-1} \times S^{n-1} \to \mathbb{R}$ be the function*

$$f(\eta_1,\eta_2) = \mathbb{E}\langle X,\eta_1\rangle^2\langle X,\eta_2\rangle^2 - \mathbb{E}\langle X,\eta_1\rangle^2\mathbb{E}\langle X,\eta_2\rangle^2.$$

Then for every $\eta_1,\eta_2 \in S^{n-1}$ such that $\langle\eta_1,\eta_2\rangle = 0$ we have,

$$\begin{aligned} f(\xi_1,\xi_2) \leq f(\eta_1,\eta_2) \leq f(e_1,e_2), \qquad & if \quad p \geq 2 \\ f(e_1,e_2) \leq f(\eta_1,\eta_2) \leq f(\xi_1,\xi_2), \qquad & if \quad p \leq 2. \end{aligned}$$

Clearly, the choice of e_1, e_2 is not relevant as $f(e_1,e_2) = f(e_i,e_j), \forall\, i \neq j$. The analogous observation applies also to ξ_1, ξ_2.

We will compute $f(e_1,e_2)$ and $f(\xi_1,\xi_2)$ in Lemma 3.2 and express them in terms of the Γ function in order to obtain the following

Corollary 1.1. *Let $p \geq 1$ and X be a random vector uniformly distributed on B_p^n.*

- *If $p \geq 2$, X satisfies the SNCP with respect to every orthonormal basis.*
- *If $1 \leq p < 2$, there exists $n_0(p) \in \mathbb{N}$ such that for any $n \geq n_0$ there is an orthonormal basis $\{\eta_i\}_{i=1}^n$ such that X does not satisfy the SNCP with respect to $\{\eta_i\}_{i=1}^n$.*

Moreover, we will show that $f(e_1,e_2) < 0$ for all $p \geq 1$, providing a new proof of the aforementioned result from [ABP], which was established there as a simple corollary of the negative association.

In order to prove Theorem 1.1 we will view B_p^n as the projection of B_p^{n+1} onto the coordinate hyperplane $e_{n+1}^\perp$ orthogonal to e_{n+1} and we will make use of the techniques developed in [BN] and [AB3]. The details of this approach are explained in Section 2.

In Section 4 we apply the same strategy to a random vector uniformly distributed on $P_{\theta_0^\perp}B_p^n$, the orthogonal projection of B_p^n onto $\theta_0^\perp$, where $\theta_0 = \left(\frac{1}{\sqrt{n}},\dots,\frac{1,}{\sqrt{n}}\right)$. However, the computations become more involved, due to the fact that some of random variables that appear are no longer independent.

Denoting $S_{\theta_0^\perp} = S^{n-1} \cap \theta_0^\perp$, we prove the following

Theorem 1.2. *Let X be a random vector uniformly distributed on $P_{\theta_0^\perp} B_p^n$, $p \geq 1$ and write $\xi_1 = \frac{e_1-e_2+e_3-e_4}{2}$, $\xi_2 = \frac{e_1-e_2-e_3+e_4}{2}$, $\overline{\xi}_1 = \frac{e_1-e_2}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{e_3-e_4}{\sqrt{2}} \in S_{\theta_0^\perp}$. Let $f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$ be the function*

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

For every fixed $p \geq 2$, there exists $n_0(p) \in \mathbb{N}$ such that if $n \geq n_0$, then for every $\eta_1, \eta_2 \in S_{\theta_0^\perp}$ such that $\langle \eta_1, \eta_2\rangle = 0$, we have that,

$$f(\xi_1, \xi_2) \leq f(\eta_1, \eta_2) \leq f(\overline{\xi}_1, \overline{\xi}_2),$$

and for every $1 \leq p \leq 2$, there exists $n_1(p) \in \mathbb{N}$ such that if $n \geq n_1$, then

$$f(\overline{\xi}_1, \overline{\xi}_2) \leq f(\eta_1, \eta_2) \leq f(\xi_1, \xi_2).$$

Studying the sign of $f(\overline{\xi}_1, \overline{\xi}_2)$ and $f(\xi_1, \xi_2)$, (see Lemmas 4.8 and 4.10) we obtain the following corollary:

Corollary 1.2. *Let X be a random vector uniformly distributed on $P_{\theta_0^\perp} B_p^n$, $p \geq 1$. There exists $n_2(p) \in \mathbb{N}$ such that for all $n \geq n_2$, X satisfies the SNCP with respect to every orthonormal basis in $\theta_0^\perp$.*

As a consequence of [AB2, Proposition 1.9]

Corollary 1.3. *Let X be a random vector uniformly distributed on $T(P_{\theta_0^\perp} B_p^n)$, $p \geq 1$, $T\colon \mathbb{R}^n \to \mathbb{R}^n$ linear isomorphism. There exists $C(p) > 0$ (depending only on p) such that X satisfies the General Variance Conjecture with $C = C(p)$.*

2. Preliminary results

In this section we will introduce the preliminary results that we need in order to prove Theorems 1.1 and 1.2. We briefly review the tools developed in [BN] and [AB3]. Let σ_p^n be the surface measure (Hausdorff measure) on ∂B_p^n, the boundary of $B_p^n, p \geq 1$, and denote by μ_p^n the cone probability measure on ∂B_p^n, defined by $\mu_p^n(A) = \frac{1}{\mathrm{Vol}(B_p^n)} \mathrm{Vol}(\{ta \in \mathbb{R}^n; a \in A, 0 \leq t \leq 1\})$, $A \subseteq \partial B_p^n$, where Vol denotes the Lebesgue measure.

The following relation between the surface measure and the cone measure on ∂B_p^n was stated in [NR] (see also [AB3]): For almost every point $x \in \partial B_p^n$

$$\frac{d\sigma_p^n(x)}{d\mu_p^n(x)} = n\mathrm{Vol}(B_p^n)\, |\nabla(\|\cdot\|_p)(x)|\,.$$

The cone measure on ∂B_p^n was proved in [SZ] to have the following probabilistic description: Let $g_1, \dots, g_n$ be independent copies of a random variable g with density with respect to Lebesgue measure given by $\dfrac{e^{-|t|^p}}{2\Gamma(1+1/p)}, t \in \mathbb{R}$, $p \geq 1$ and denote $S := \left(\sum_{i=1}^n |g_i|^p\right)^{\frac{1}{p}}$. Then

- The random vector $\frac{G}{S} := \left(\dfrac{g_1}{S}, \dots, \dfrac{g_n}{S}\right)$ and the random variable S are independent.
- $\frac{G}{S}$ is distributed on ∂B_p^n according to the cone measure μ_p^n.

Now, in order to compute the expectation of a suitable function $f(X)$ when X is a random vector uniformly distributed on the orthogonal projection of B_p^n onto some hyperplane orthogonal to $\theta \in S^{n-1}$, $P_{\theta^\perp} B_p^n$, we first use Cauchy's formula and pass to an integration on ∂B_p^n with respect to the surface measure, then use the relation between the surface measure and the cone measure and finally the latter probabilistic representation of the cone measure (see [AB3] for the details). The final result is the starting point for the proof of our main results:

Lemma 2.1. [AB3] *Let $\theta \in S^{n-1}$. If X is a random vector uniformly distributed on $P_{\theta^\perp} B_p^n$, $g_1, \dots, g_n$ are independent copies of g as above and $S = (\sum_{i=1}^n |g_i|^p)^{\frac{1}{p}}$, then for every integrable function $f : P_{\theta^\perp} B_p^n \to \mathbb{R}$*

$$\mathbb{E} f(X) = \frac{\mathbb{E} f\left(P_{\theta^\perp}\left(\frac{g_1}{S}, \dots, \frac{g_n}{S}\right)\right)\left|\sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}} \operatorname{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}} \operatorname{sgn}(g_i)\theta_i\right|}.$$

where $\theta = (\theta_1, \dots, \theta_n)$ and $\operatorname{sgn}(g_i)$ denotes the sign of g_i.

The following lemma computes the expectation of the random variables involved in terms of the Gamma function:

Lemma 2.2. [AB3][BN] *Let $\alpha \geq 0$, let $g_1, \dots, g_n$ be independent copies of g as above and $S = (\sum_{i=1}^n |g_i|^p)^{\frac{1}{p}}$. Then*

$$\mathbb{E}|g|^\alpha = \frac{\Gamma\left(\frac{\alpha+1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} \qquad \text{and} \qquad \mathbb{E} S^\alpha = \frac{\Gamma\left(\frac{n+\alpha}{p}\right)}{\Gamma\left(\frac{n}{p}\right)}.$$

Our last lemma concerns the so called Gurland's ratio for the Gamma function (see more details in [M]) and it will be crucial in our estimates.

Lemma 2.3. *The function*

$$F(x) := \Gamma(5x)\Gamma(x)/\Gamma(3x)^2$$

is strictly increasing in $(0, 1]$ and satisfies $F(\frac{1}{2}) = 3$.

Proof. The function F is increasing if and only if its logarithm is increasing. Therefore, let us see that the function

$$h(x) = \log \Gamma(5x) + \log \Gamma(x) - 2 \log \Gamma(3x)$$

is increasing. Denoting by ψ the logarithmic derivative of the Gamma function, which satisfies (see, for instance [ABR])

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt,$$

we have that

$$\begin{aligned} h'(x) &= 5\psi(5x) + \psi(x) - 6\psi(3x) = 5\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-5xt}}{1 - e^{-t}}\right) dt + \\ &+ \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt - 6\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-3xt}}{1 - e^{-t}}\right) dt \\ &= \int_0^\infty \frac{1}{1 - e^{-t}}(-5e^{-5xt} - e^{-xt} + 6e^{-3xt}) dt. \end{aligned}$$

Then,

$$\begin{aligned} xh'(x) &= \int_0^\infty \frac{1}{1-e^{-t}}(-5xe^{-5xt} - xe^{-xt} + 6xe^{-3xt})dt \\ &= \int_0^\infty \frac{1}{1-e^{-t}}\frac{d}{dt}(e^{-5xt} + e^{-xt} - 2e^{-3xt})dt \\ &= \int_0^\infty \frac{e^{-t}}{(1-e^{-t})^2}\left((e^{-5xt} - e^{-3xt}) - (e^{-3xt} - e^{-xt})\right)dt. \end{aligned}$$

Since the function e^{-y} is convex, we have $\dfrac{e^{-5xt} - e^{-3xt}}{2xt} \geq \dfrac{e^{-3xt} - e^{-xt}}{2xt}$, $\forall\, x, t > 0$ and so the last integral is positive. Thus, for every $x > 0$, $h'(x) > 0$ and we obtain the result. It is clear that $F(\frac{1}{2}) = 3$. □

3. The SCNP on B_p^n

In this section we will prove Theorem 1.1. We state the first result in the more general context of a random vector uniformly distributed on a 1-symmetric convex body. A convex body $K \subseteq \mathbb{R}^n$ is called 1-symmetric if for every $(x_1, \dots, x_n) \in \mathbb{R}^n$, $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ and permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, we have $(x_1, \dots, x_n) \in K$ if and only if $(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}) \in K$. Clearly, B_p^n is a 1-symmetric convex body in $\mathbb{R}^n$.

Proposition 3.1. *Let X be a random vector uniformly distributed on a 1-symmetric convex body $K \subseteq \mathbb{R}^n$ and write $\xi_1 = \frac{e_1+e_2}{\sqrt{2}}$, $\xi_2 = \frac{e_1-e_2}{\sqrt{2}}$. Let $f : S^{n-1} \times S^{n-1} \to \mathbb{R}$ be the function*

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

For every $\eta_1, \eta_2 \in S^{n-1}$ such that $\langle \eta_1, \eta_2\rangle = 0$, we have

$$f(\eta_1, \eta_2) = f(e_1, e_2) + 2\Big(f(\xi_1, \xi_2) - f(e_1, e_2)\Big)\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2,$$

where $\eta_j = (\eta_j(1), \dots \eta_j(n))$, $j = 1, 2$.

Proof. Since $X = (X_1, \dots, X_n)$ is uniformly distributed on a 1-symmetric convex body, we have $\mathbb{E}X_iX_j = 0, \forall\, i \neq j$, and for every $\eta \in S^{n-1}$,

$$\mathbb{E}\langle X, \eta\rangle^2 = \mathbb{E}\Big(\sum_{i=1}^n \eta(i)X_i\Big)^2 = \mathbb{E}X_1^2.$$

Therefore, $\mathbb{E}\langle X, \eta_1\rangle^2\mathbb{E}\langle X, \eta_2\rangle^2 = \mathbb{E}X_1^2\mathbb{E}X_2^2 = \mathbb{E}\langle X, e_1\rangle^2\mathbb{E}\langle X, e_2\rangle^2$. Again, since K is a 1-symmetric convex body and $\langle \eta_1, \eta_2\rangle = 0$,

$$\begin{aligned} &\mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 \\ =&\mathbb{E}X_1^4\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2 + \mathbb{E}X_1^2X_2^2\sum_{i\neq j}\eta_1(i)^2\eta_2(j)^2 + 2\mathbb{E}X_1^2X_2^2\sum_{i\neq j}\eta_1(i)\eta_2(i)\eta_1(j)\eta_2(j) \\ =&\mathbb{E}X_1^4\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2 + \mathbb{E}X_1^2X_2^2\Big(1 - \sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2\Big) - 2\mathbb{E}X_1^2X_2^2\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2 \\ =&\mathbb{E}\langle X, e_1\rangle^2\langle X, e_2\rangle^2 + \Big(\mathbb{E}X_1^4 - 3\mathbb{E}X_1^2X_2^2\Big)\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2. \end{aligned}$$

On the other hand, it is easy to check that

$$\begin{aligned}\mathbb{E}X_1^4 - 3\mathbb{E}X_1^2X_2^2 &= 2\left(\mathbb{E}\left(\frac{X_1+X_2}{\sqrt{2}}\right)^2\left(\frac{X_1-X_2}{\sqrt{2}}\right)^2 - \mathbb{E}X_1^2X_2^2\right)\\ &= 2\left(\mathbb{E}\langle X,\xi_1\rangle^2\langle X,\xi_2\rangle^2 - \mathbb{E}\langle X,e_1\rangle^2\langle X,e_2\rangle^2\right).\end{aligned}$$

The fact that $\mathbb{E}\langle X,\eta\rangle^2$ is independent of $\eta\in S^{n-1}$ finishes the proof. □

Lemma 3.1. *Let $\eta_1,\eta_2\in S^{n-1}$ with $\langle\eta_1,\eta_2\rangle=0$. Then*

$$0\le\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2\le\frac{1}{2}.$$

The lower bound is attained at any two vectors of the canonical basis. The upper bound is attained at the vectors $\xi_1=\frac{e_i+e_j}{\sqrt{2}}$ and $\xi_2=\frac{e_i-e_j}{\sqrt{2}}$ for any $i\neq j$.

Proof. The lower bound is trivial. For the upper bound consider the function $F\colon\mathbb{R}^{2n}\to\mathbb{R}$ given by $F(\eta_1,\eta_2)=\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2$ which we want to maximize under the conditions $\sum_{i=1}^n\eta_1(i)\eta_2(i)=0$ and $\sum_{i=1}^n\eta_1(i)^2=\sum_{i=1}^n\eta_2(i)^2=1$. Observe that if (η_1,η_2) is an extremal point so is $(\pm\eta_1,\pm\eta_2)$ and (η_2,η_1). The proof is a consequence of the Lagrange multipliers theorem. □

Proof of Theorem 1.1. Let X be a random vector uniformly distributed on $B_p^n=P_{e_{n+1}^\perp}B_p^{n+1}$. We have seen in the proof of Proposition 3.1 that

$$2(f(\xi_1,\xi_2)-f(e_1,e_2))=\mathbb{E}\langle X,e_1\rangle^4-3\mathbb{E}\langle X,e_1\rangle^2\langle X,e_2\rangle^2.$$

We first apply Lemma 2.1 to the function $\langle X,e_1\rangle^4$. Notice that since $e_1\in\mathbb{R}^n$ we can omit $P_{e_{n+1}^\perp}$. Recall $G=(g_1,\dots,g_{n+1})$ and $S=\left(\sum_{i=1}^{n+1}|g_i|^p\right)^{\frac{1}{p}}$, where $g_1,\dots g_{n+1}$ are independent copies of g as in Section 2. Since G and $\frac{G}{S}$ are also independent,

$$\mathbb{E}\langle X,e_1\rangle^4=\frac{\mathbb{E}\left\langle\frac{G}{S},e_1\right\rangle^4\frac{|g_{n+1}|^{p-1}}{S^{p-1}}}{\mathbb{E}\frac{|g_{n+1}|^{p-1}}{S^{p-1}}}=\frac{\mathbb{E}S^{p-1}\mathbb{E}\left\langle G,e_1\right\rangle^4|g_{n+1}|^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}|g_{n+1}|^{p-1}}=\frac{\mathbb{E}S^{p-1}\mathbb{E}g_1^4}{\mathbb{E}S^{p+3}}.$$

In the same way, we apply Lemma 2.1 to $\langle X,e_1\rangle^2\langle X,e_2\rangle^2$

$$\begin{aligned}\mathbb{E}\langle X,e_1\rangle^2\langle X,e_2\rangle^2 &= \frac{\mathbb{E}\left\langle\frac{G}{S},e_1\right\rangle^2\left\langle\frac{G}{S},e_2\right\rangle^2\frac{|g_{n+1}|^{p-1}}{S^{p-1}}}{\mathbb{E}\frac{|g_{n+1}|^{p-1}}{S^{p-1}}}\\ &= \frac{\mathbb{E}S^{p-1}\mathbb{E}\left\langle G,e_1\right\rangle^2\left\langle G,e_2\right\rangle^2|g_{n+1}|^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}|g_{n+1}|^{p-1}}=\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}\mathbb{E}g_1^2g_2^2\\ &= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}(\mathbb{E}g_1^2)^2.\end{aligned}$$

Therefore, the sign of $f(\xi_1,\xi_2)-f(e_1,e_2)$ is equal to the sign of $\mathbb{E}g_1^4-3(\mathbb{E}g_1^2)^2$ and by Lemma 2.2,

$$\mathbb{E}g_1^4-3(\mathbb{E}g_1^2)^2=\frac{\Gamma\left(\frac{5}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}-3\frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}=\frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right)-3\right),$$

where $F(x)=\Gamma(5x)\Gamma(x)/\Gamma(3x)^2$. By Lemma 2.3 its sign is negative if $p\ge2$ and positive if $1\le p\le2$.

By Lemma 3.1 and Proposition 3.1 the function f attains its maximum (resp. minimum) at (e_1, e_2) and its minimum (resp. maximum) at (ξ_1, ξ_2) depending on whether the sign of $f(\xi_1, \xi_2) - f(e_1, e_2)$ is negative (resp. positive). $\square$

Remark. Alternatively to the use of Lemma 2.3, the sign of $\mathbb{E}g_1^4 - 3(\mathbb{E}g_1^2)^2$ can be determined by results from [ENT, Corollary 23] (for $1 \le p \le 2$) and [ENT2, Theorem 2] (for $p \ge 2$).

In order to prove Corollary 1.1, we compute the value of f at the extremal pairs,

Lemma 3.2.

$$f(e_1, e_2) = \frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(1+\frac{n+4}{p}\right)\Gamma\left(\frac{1}{p}\right)^2}\left(1 - \frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n+4}{p}\right)}{\Gamma\left(1+\frac{n+2}{p}\right)^2}\right),$$

and

$$f(\xi_1, \xi_2) = \frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(\frac{3}{p}\right)^2\left(F\left(\frac{1}{p}\right) - 1 - 2\frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n+4}{p}\right)}{\Gamma\left(1+\frac{n+2}{p}\right)^2}\right)}{2\Gamma\left(1+\frac{n+4}{p}\right)\Gamma\left(\frac{1}{p}\right)^2}.$$

Proof. Using Lemma 2.1, we have that

$$f(e_1, e_2) = \mathbb{E}X_1^2X_2^2 - (\mathbb{E}X_1^2)^2 = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}(\mathbb{E}g_1^2)^2 - \left(\frac{\mathbb{E}S^{p-1}\mathbb{E}g_1^2}{\mathbb{E}S^{p+1}}\right)^2$$

and, as we have seen in the proof of Proposition 3.1, that

$$f(\xi_1, \xi_2) = \frac{1}{2}(\mathbb{E}X_1^4 - \mathbb{E}X_1^2X_2^2) - (\mathbb{E}X_1^2)^2 = \frac{\mathbb{E}S^{p-1}}{2\mathbb{E}S^{p+3}}\left(\mathbb{E}g_1^4 - (\mathbb{E}g_1^2)^2\right) - \left(\frac{\mathbb{E}S^{p-1}\mathbb{E}g_1^2}{\mathbb{E}S^{p+1}}\right)^2.$$

Now substitute the expressions from Lemma 2.2, where $S = \left(\sum_{i=1}^{n+1} |g_i|^p\right)^{\frac{1}{p}}$. $\square$

Proof of Corollary 1.1. Since $n+2 = \frac{n+(n+4)}{2}$ and $\log\Gamma(x)$ is strictly convex ([ABR]), $f(e_1, e_2) < 0$ for every $p \ge 1$. If $1 \le p < 2$, Lemma 2.3 implies $F\left(\frac{1}{p}\right) > 3$ and by Stirling's formula ([ABR], [M]),

$$\lim_{n\to\infty} \frac{\Gamma\left(1+\frac{n}{p}\right)\Gamma\left(1+\frac{n+4}{p}\right)}{\Gamma\left(1+\frac{n+2}{p}\right)^2} = 1.$$

Thus, for every $1 \le p < 2$ there exists $n_0(p) \in \mathbb{N}$ so that if $n \ge n_0$, $f(\xi_1, \xi_2) > 0$. $\square$

Remark. A statement fixing n first yields: For $p = 2$, $f(\xi_1, \xi_2) = f(e_1, e_2) < 0$ and so, by continuity, for every $n \in \mathbb{N}$ there exists $p_0(n) \in (1, 2)$ such that for every $p \ge p_0$ a random vector uniformly distributed on B_p^n satisfies the SNCP with respect to every orthonormal basis.

4. The SNCP on a projection of B_p^n

In this section we will prove Theorem 1.2. The general scheme is analogous to the one used in the previous section. The first Proposition below corresponds to Proposition 3.1 for B_p^n.

Recall that $\theta_0 = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$ denotes the diagonal direction and $P_{\theta_0^\perp}$ denotes the orthogonal projection onto the hyperplane $\theta_0^\perp$.

Proposition 4.1. *Let X be a random vector uniformly distributed on $P_{\theta_0^\perp} B_p^n$, $n \geq 4$, $p \geq 1$. Let $f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$ be the function*

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2 \langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2 \mathbb{E}\langle X, \eta_2\rangle^2.$$

and write $\xi_1 = \frac{e_1 - e_2 + e_3 - e_4}{2}$, $\xi_2 = \frac{e_1 - e_2 - e_3 + e_4}{2}$, $\bar{\xi}_1 = \frac{e_1 - e_2}{\sqrt{2}}$, $\bar{\xi}_2 = \frac{e_3 - e_4}{\sqrt{2}} \in S_{\theta_0^\perp}$. For every $\eta_1, \eta_2 \in S_{\theta_0^\perp}$ such that $\langle \eta_1, \eta_2 \rangle = 0$ we have

$$f(\eta_1, \eta_2) = f(\bar{\xi}_1, \bar{\xi}_2) + 4\Big(f(\xi_1, \xi_2) - f(\bar{\xi}_1, \bar{\xi}_2)\Big) \sum_{i=1}^n \eta_1(i)^2 \eta_2(i)^2.$$

In order to prove this proposition we first state two lemmas. Recall $g_1, \dots, g_n$ denote independent copies of a random variable g, with density with respect to Lebesgue measure $\frac{e^{-|t|^p}}{2\Gamma(1+1/p)}$, $G = (g_1, \dots, g_n)$, and $S = (\sum_{i=1}^n |g_i|^p)^{\frac{1}{p}}$. We define $\psi_{\theta_0} := \frac{1}{\sqrt{n}} \left|\sum_{i=1}^n |g_i|^{p-1} \operatorname{sgn}(g_i)\right|$.

Lemma 4.1. *For every $\eta \in S_{\theta_0^\perp}$, $\mathbb{E}\langle X, \eta \rangle^2 = \dfrac{\mathbb{E}S^{p-1} \mathbb{E} g_1(g_1 - g_2)\psi_{\theta_0}}{\mathbb{E}S^{p+1}\mathbb{E}\psi_{\theta_0}}$ and, in particular, it is independent of η.*

Proof. $\eta = (\eta(1), \dots, \eta(n)) \in S_{\theta_0^\perp}$ is equivalent to $\sum_{i=1}^n \eta(i)^2 = 1$, $\sum_{i=1}^n \eta(i) = 0$. Apply Lemma 2.1 to the function $\langle X, \eta \rangle^2$

$$\begin{aligned}
\mathbb{E}\langle X, \eta\rangle^2 &= \frac{\mathbb{E}\left\langle \frac{G}{S}, \eta \right\rangle^2 \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}} \operatorname{sgn}(g_i) \right|}{\mathbb{E}\left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}} \operatorname{sgn}(g_i)\right|} = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}} \frac{\mathbb{E}\langle G, \eta\rangle^2 \psi_{\theta_0}}{\mathbb{E}\psi_{\theta_0}} \\
&= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}\mathbb{E}\psi_{\theta_0}} \left(\sum_{i=1}^n \mathbb{E} g_i^2 \psi_{\theta_0} \eta(i)^2 + \sum_{i \neq j}^n \mathbb{E} g_i g_j \psi_{\theta_0} \eta(i)\eta(j) \right) \\
&= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}\mathbb{E}\psi_{\theta_0}} \left(\mathbb{E} g_1^2 \psi_{\theta_0} \sum_{i=1}^n \eta(i)^2 + \mathbb{E} g_1 g_2 \psi_{\theta_0} \sum_{i \neq j}^n \eta(i)\eta(j) \right) \\
&= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}\mathbb{E}\psi_{\theta_0}} \left(\mathbb{E} g_1^2 \psi_{\theta_0} + \mathbb{E} g_1 g_2 \psi_{\theta_0} \left(\left(\sum_{i=1}^n \eta(i) \right)^2 - 1 \right) \right).
\end{aligned}$$

□

In the next lemma we rewrite several expressions in terms of $\displaystyle\sum_{i=1}^n \eta_1(i)^2 \eta_2(i)^2$.

Lemma 4.2. *Let $\eta_1, \eta_2 \in S_{\theta_0^\perp}$ with $\langle \eta_1, \eta_2 \rangle = 0$. Then*

- $\displaystyle\sum_{i \neq j}^n \eta_1(i)\eta_1(j)\eta_2(j)^2 = -\sum_{i=1}^n \eta_1(i)^2 \eta_2(i)^2$,

- $\displaystyle\sum_{i\neq j}\eta_1(i)^2\eta_2(j)^2 = 1-\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2,$
- $\displaystyle\sum_{i\neq j}^n \eta_1(i)\eta_2(i)\eta_1(j)\eta_2(j) = -\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2,$
- $\displaystyle\sum_{i\neq j\neq k}^n \eta_1(i)^2\eta_2(j)\eta_2(k) = -1+2\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2$, *where the sum on the left runs over the tripples of distinct indices* (i,j,k).
- $\displaystyle\sum_{i\neq j\neq k} \eta_1(i)\eta_2(i)\eta_1(j)\eta_2(k) = 2\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2$, *where the sum on the left runs over the tripples of distinct indices* (i,j,k).
- $\displaystyle\sum_{i\neq j\neq k\neq l}^n \eta_1(i)\eta_1(j)\eta_2(k)\eta_2(l) = 1-6\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2$, *where the sum on the left runs over the tuples of distinct indices* (i,j,k,l).

Proof. The first three identities are obtained by adding and substracting the sum with $i=j$ and taking into account that $\|\eta_1\|_2=\|\eta_2\|_2=1$ and $\langle\eta_1,\eta_2\rangle=0$. For the fourth one, notice that since $\|\eta_1\|_2=1$,

$$\begin{aligned}\sum_{i\neq j\neq k}^n \eta_1(i)^2\eta_2(j)\eta_2(k) &= \sum_{k\neq j}\Big(\eta_2(j)\eta_2(k)-\eta_1(j)^2\eta_2(j)\eta_2(k)-\eta_1(k)^2\eta_2(j)\eta_2(k)\Big)\\ &= -\sum_{j=1}^n \eta_2(j)^2 - \sum_{j\neq k}^n\Big(\eta_1(j)^2\eta_2(j)\eta_2(k)+\eta_1(k)^2\eta_2(j)\eta_2(k)\Big)\end{aligned}$$

and then use the first identity. For the fifth one, notice that since $\langle\eta_1,\eta_2\rangle=0$

$$\sum_{i\neq j\neq k}^n \eta_1(i)\eta_2(i)\eta_1(j)\eta_2(k) = \sum_{k\neq j}\Big(0-\eta_1(j)^2\eta_2(j)\eta_2(k)-\eta_1(j)\eta_1(k)\eta_2(k)^2\Big)$$

and then use the first identity. For the last one, we use $\sum_{i=1}^n\eta_1(i)=0$

$$\begin{aligned}&\sum_{i\neq j\neq k\neq l}^n \eta_1(i)\eta_1(j)\eta_2(k)\eta_2(l)\\ &= \sum_{j\neq k\neq l}\Big(0-\eta_1(j)^2\eta_2(k)\eta_2(l)-\eta_1(j)\eta_1(k)\eta_2(k)\eta_2(l)-\eta_1(j)\eta_2(k)\eta_1(l)\eta_2(l)\Big)\end{aligned}$$

and then use the fourth and the fifth identities. □

Proof of Proposition 4.1. By Lemma 2.1 we have that for every $\eta_1,\eta_2\in S_{\theta_0^\perp}$, with $\langle\eta_1,\eta_2\rangle=0$,

$$\begin{aligned}\mathbb{E}\langle X,\eta_1\rangle^2\langle X,\eta_2\rangle^2 &= \frac{\mathbb{E}\left\langle\frac{G}{S},\eta_1\right\rangle^2\left\langle\frac{G}{S},\eta_2\right\rangle^2\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\right|}{\mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\right|}\\ &= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}\frac{\mathbb{E}\langle G,\eta_1\rangle^2\langle G,\eta_2\rangle^2\psi_{\theta_0}}{\mathbb{E}\psi_{\theta_0}}.\end{aligned}$$

Expanding the product and since the $g_i's$ are identically distributed, we have

$$\mathbb{E}\langle G,\eta_1\rangle^2\langle G,\eta_2\rangle^2\psi_{\theta_0}=\sum_{i,j,k,l=1}^n \mathbb{E}g_ig_jg_kg_l\psi_{\theta_0}\eta_1(i)\eta_1(j)\eta_2(k)\eta_2(l)$$

$$\begin{aligned}
&= \mathbb{E}g_1^4\psi_{\theta_0}\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2\\
&+ \mathbb{E}g_1^3g_2\psi_{\theta_0}\Big(2\sum_{i\neq j}^n\eta_1(i)\eta_1(j)\eta_2(j)^2+2\sum_{i\neq j}^n\eta_1(i)^2\eta_2(i)\eta_2(j)\Big)\\
&+ \mathbb{E}g_1^2g_2^2\psi_{\theta_0}\Big(\sum_{i\neq j}^n\eta_1(i)^2\eta_2(j)^2+2\sum_{i\neq j}^n\eta_1(i)\eta_2(i)\eta_1(j)\eta_2(j)\Big)\\
&+ \mathbb{E}g_1^2g_2g_3\psi_{\theta_0}\Big(\sum_{i\neq j\neq k}^n\eta_1(i)^2\eta_2(j)\eta_2(k)+4\sum_{i\neq j\neq k}^n\eta_1(i)\eta_2(i)\eta_1(j)\eta_2(k)\\
&\qquad\qquad+\sum_{i\neq j\neq k}\eta_2(i)^2\eta_1(j)\eta_1(k)\Big)\\
&+ \mathbb{E}g_1g_2g_3g_4\psi_{\theta_0}\sum_{i\neq j\neq k\neq l}\eta_1(i)\eta_1(j)\eta_2(k)\eta_2(l).
\end{aligned}$$

and by the identities in Lemma 4.2 we obtain

$$\begin{aligned}
&\mathbb{E}\langle G,\eta_1\rangle^2\langle G,\eta_2\rangle^2\psi_{\theta_0}=\mathbb{E}\big(g_1^2g_2^2-2g_1^2g_2g_3+g_1g_2g_3g_4\big)\psi_{\theta_0}\\
&+\ \big(\mathbb{E}\big(g_1^4-4g_1^3g_2-3g_1^2g_2^2+12g_1^2g_2g_3-6g_1g_2g_3g_4\big)\psi_{\theta_0}\big)\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2.
\end{aligned}$$

We can express the first summand as

$$\mathbb{E}(g_1^2g_2^2-2g_1^2g_2g_3+g_1g_2g_3g_4)\psi_{\theta_0}=\frac{1}{4}\mathbb{E}(g_1-g_2)^2(g_3-g_4)^2\psi_{\theta_0}=\mathbb{E}\langle G,\overline{\xi}_1\rangle^2\langle G,\overline{\xi}_2\rangle^2\psi_{\theta_0}.$$

and the factor $\mathbb{E}(g_1^4-4g_1^3g_2-3g_1^2g_2^2+12g_1^2g_2g_3-6g_1g_2g_3g_4)\psi_{\theta_0}$ in the second summand as

$$\begin{aligned}
&4\mathbb{E}\left(\frac{g_1-g_2+g_3-g_4}{2}\right)^2\left(\frac{g_1-g_2-g_3+g_4}{2}\right)^2\psi_{\theta_0}-4\mathbb{E}\left(\frac{g_1-g_2}{\sqrt{2}}\right)^2\left(\frac{g_3-g_4}{\sqrt{2}}\right)^2\psi_{\theta_0}\\
&\qquad=4\mathbb{E}\langle G,\xi_1\rangle^2\langle G,\xi_2\rangle^2\psi_{\theta_0}-4\mathbb{E}\langle G,\overline{\xi}_1\rangle^2\langle G,\overline{\xi}_2\rangle^2\psi_{\theta_0}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\langle X,\eta_1\rangle^2\langle X,\eta_2\rangle^2 &= \mathbb{E}\langle X,\overline{\xi}_1\rangle^2\langle X,\overline{\xi}_2\rangle^2\\
&+ 4\left(\mathbb{E}\langle X,\xi_1\rangle^2\langle X,\xi_2\rangle^2-\mathbb{E}\langle X,\overline{\xi}_1\rangle^2\langle G,\overline{\xi}_2\rangle^2\right)\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2
\end{aligned}$$

and, since by Lemma 4.1 the value of $\mathbb{E}\langle X,\eta\rangle^2$ does not depend on the vector $\eta\in S_{\theta_0^\perp}$, we obtain the result. □

The following lemma is analogous to Lemma 3.1

Lemma 4.3. *Let $\eta_1,\eta_2\in S_{\theta_0^\perp}$ with $\langle\eta_1,\eta_2\rangle=0$, $n\geq 4$. Then*

$$0\leq\sum_{i=1}^n\eta_1(i)^2\eta_2(i)^2\leq\frac{1}{4}.$$

The lower bound is attained at the vectors $\overline{\xi}_1 = \frac{e_1-e_2}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{e_3-e_4}{\sqrt{2}}$. *The upper bound is attained at the vectors* $\xi_1 = \frac{e_1-e_2+e_3-e_4}{2}$ *and* $\xi_2 = \frac{e_1-e_2-e_3+e_4}{\sqrt{2}}$.

Proof. The lower bound is trivial. For the upper bound consider the function $F\colon \mathbb{R}^{2n} \to \mathbb{R}$ given by $F(\eta_1,\eta_2) = \sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2$ which we want to maximize under the conditions $\sum_{i=1}^n \eta_1(i) = \sum_{i=1}^n \eta_2(i) = \sum_{i=1}^n \eta_1(i)\eta_2(i) = 0$ and $\sum_{i=1}^n \eta_1(i)^2 = \sum_{i=1}^n \eta_2(i)^2 = 1$. Observe that if (η_1,η_2) is an extremal point so is $(\pm\eta_1,\pm\eta_2)$ and (η_2,η_1). Applying the Lagrange multipliers theorem, there exist $A,B,C \in \mathbb{R}$ such that the extremal points satisfy

$$\eta_1(i)\eta_2(i)^2 - A\eta_1(i) - B\eta_2(i) - C = 0 \qquad \forall\ i = 1\dots n$$

and, by the observation above, also satisfy the equality exchanging η_j and $\pm\eta_j$ $(j = 1,2)$ and η_2 and η_1.

That implies $B = C = 0$ and $\eta_1(i)^2 = \eta_2(i)^2$ $\forall\ i = 1\dots n$. Write k, $0 \le k \le n$, for the number of non zero coordinates of η_1 (or η_2). Since $\sum_{i=1}^n \eta_1(i)^2\eta_2(i)^2 = A$, we have $k\eta_1(i)^4 = k\eta_2(i)^4 = A$ for every non zero coordinate. $k = 0,1,2,3$ do not satisfy the conditions, so the maximum value is attained at $k = 4$ and corresponds to the vectors ξ_1 and ξ_2. □

We now proceed to determining the sign of $f(\xi_1,\xi_2) - f(\overline{\xi}_1,\overline{\xi}_2)$. We will use the following probabilistic argument:

Lemma 4.4. *Denote* $Y = (g_1,\dots,g_k)$ *and* $Y_k = \sum_{i=1}^k \operatorname{sgn}(g_i)|g_i|^{p-1}$. *Let* Z *be a symmetric real random variable independent of* Y *and let* $h : \mathbb{R}^k \to \mathbb{R}$ *be integrable. Then*

$$\mathbb{E}h(g_1,\dots,g_k)|Y_k + Z| = \mathbb{E}h(g_1,\dots,g_k)\mathbb{E}|Z| + \mathbb{E}h(g_1,\dots,g_k)(|Y_k| - |Z|)\chi_{\{|Y_k|\ge|Z|\}}$$

and

$$\left|\mathbb{E}h(g_1,\dots,g_k)(|Y_k| - |Z|)\chi_{\{|Y_k|\ge|Z|\}}\right| \le \left(\mathbb{E}|h(g_1,\dots,g_k)|^2\right)^{1/2}\left(\mathbb{E}|Y_k|^2\right)^{1/2}.$$

Proof. Write $h = h(g_1,\dots,g_k)$ for short. Our hypotheses readily imply,

$$\mathbb{E}h\cdot|Y_k+Z| = \mathbb{E}h\cdot\frac{1}{2}(|Y_k+Z| + |Y_k-Z|) = \mathbb{E}h\cdot\max\{|Y_k|,|Z|\}.$$

Fix $(g_1,\dots,g_k)$ and compute the expectation with respect to Z. We have,

$$\begin{aligned}
&\mathbb{E}_Z\ \ h(g_1,\dots,g_k)\cdot\max\{|Y_k|,|Z|\} = h(g_1,\dots,g_k)\cdot\int_0^\infty \mathbb{P}_Z\{\max\{|Y_k|,|Z|\} > t\}\,dt\\
&= h\cdot\big(|Y_k| + \int_{|Y_k|}^\infty \mathbb{P}_Z\{|Z|>t\}\,dt\big) = h\cdot\big(|Y_k| - \int_0^{|Y_k|}\mathbb{P}_Z\{|Z|>t\}\,dt\big) + h\cdot\mathbb{E}|Z|\\
&= h\cdot\mathbb{E}|Z| + h\cdot\int_0^{|Y_k|}\mathbb{P}_Z\{|Z|\le t\}\,dt.
\end{aligned}$$

Finally, notice that by Fubini's theorem

$$\begin{aligned}
\int_0^{|Y_k|}\mathbb{P}_Z\{|Z|\le t\}\,dt &= |Y_k|\int_0^1 \mathbb{P}_Z\{|Z|\le t|Y_k|\}\,dt = |Y_k|\ \mathbb{E}_Z\int_0^1 \chi_{\{\frac{|Z|}{|Y_k|}\le t\}}\,dt\\
&= |Y_k|\mathbb{E}_Z\left(1 - \min\left\{1, \frac{|Z|}{|Y_k|}\right\}\right).
\end{aligned}$$

Taking now expectation $\mathbb{E}_Y$ finishes the proof of the first statement. For the second one notice that $\left|\mathbb{E}h(g_1,\dots,g_k)(|Y_k|-|Z|)\chi_{\{|Y_k|\geq|Z|\}}\right| \leq \mathbb{E}|h(g_1,\dots,g_k)|\cdot|Y_k|$ and use the Cauchy-Schwarz inequality. □

The following estimate will be useful in the sequel,

Lemma 4.5. ([AB3, Lemma 3.4]) *For some absolute constants $c, C > 0$,*

$$\frac{c}{\sqrt{p}} \leq \mathbb{E}\psi_{\theta_0} \leq \frac{C}{\sqrt{p}} \qquad if \qquad 1 \leq p \leq n.$$

We have the following result regarding the sign of $f(\xi_1,\xi_2) - f(\overline{\xi}_1,\overline{\xi}_2)$, which shall give Theorem 1.2 as a consequence.

Proposition 4.2. *Let X be a random vector uniformly distributed on $P_{\theta_0}B_p^n$ and let $f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$ given by*

$$f(\eta_1,\eta_2) = \mathbb{E}\langle X,\eta_1\rangle^2\langle X,\eta_2\rangle^2 - \mathbb{E}\langle X,\eta_1\rangle^2\mathbb{E}\langle X,\eta_2\rangle^2.$$

Write $\xi_1 = \frac{e_1-e_2+e_3-e_4}{2}$, $\xi_2 = \frac{e_1-e_2-e_3+e_4}{2}$, $\overline{\xi}_1 = \frac{e_1-e_2}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{e_3-e_4}{\sqrt{2}} \in S_{\theta_0^\perp}$. For every $p > 2$ there exists $n_0(p) \in \mathbb{N}$ such that if $n \geq n_0$,

$$f(\xi_1,\xi_2) - f(\overline{\xi}_1,\overline{\xi}_2) \leq 0$$

and for every $1 \leq p \leq 2$ there exists $n_1(p) \in \mathbb{N}$ such that if $n \geq n_1$,

$$f(\xi_1,\xi_2) - f(\overline{\xi}_1,\overline{\xi}_2) \geq 0.$$

Proof. Since f is constant for $p = 2$, we will focus on the cases $p \neq 2$. We have seen in the proof of Proposition 4.1 that

$$4(f(\xi_1,\xi_2)-f(\overline{\xi}_1,\overline{\xi}_2)) = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}}\mathbb{E}(g_1^4-4g_1^3g_2-3g_1^2g_2^2+12g_1^2g_2g_3-6g_1g_2g_3g_4)\psi_{\theta_0}$$

and so, the sign of $f(\xi_1,\xi_2) - f(\overline{\xi}_1,\overline{\xi}_2)$ is equal to the sign of $\mathbb{E}(h\ \psi_{\theta_0})$ where $h(g_1,g_2,g_3,g_4) = g_1^4 - 4g_1^3g_2 - 3g_1^2g_2^2 + 12g_1^2g_2g_3 - 6g_1g_2g_3g_4$.

We apply then Lemma 4.4 to $Y_4 = \sum_{i=1}^{4}\operatorname{sgn}(g_i)|g_i|^{p-1}$, $Z = \sum_{i=5}^{n}\operatorname{sgn}(g_i)|g_i|^{p-1}$ and h as above and

$$\sqrt{n}\psi_{\theta_0} = \left|\sum_{i=1}^{4}\operatorname{sgn}(g_i)|g_i|^{p-1} + \sum_{i=5}^{n}\operatorname{sgn}(g_i)|g_i|^{p-1}\right| = |Y_4 + Z|.$$

On one hand,

$$\mathbb{E}h(g_1,\dots,g_4)\mathbb{E}|Z| = \frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right)-3\right)\mathbb{E}\left|\sum_{i=5}^{n}|g_i|^{p-1}\operatorname{sgn}(g_i)\right|,$$

where $F(x) = \Gamma(5x)\Gamma(x)/\Gamma(3x)^2$, since the $g_i's$ are i.i.d. symmetric random variables and by the computation in the proof of Theorem 1.1,

$$\mathbb{E}h(g_1,g_2,g_3,g_4) = \mathbb{E}g_1^4 - 3(\mathbb{E}g_1^2)^2 = \frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right)-3\right).$$

Also, by Lemma 4.5 we have

$$\frac{c\sqrt{n}}{\sqrt{p}} \le \mathbb{E}\left|\sum_{i=5}^{n} |g_i|^{p-1}\operatorname{sgn}(g_i)\right| \le \frac{C\sqrt{n}}{\sqrt{p}}$$

provided that $1 \le p \le n$ for some absolute constants $c, C > 0$.

On the other hand, by straightforward computations

$$\mathbb{E}|h(g_1,\dots,g_4)|^2 = \mathbb{E}g_1^8 + 10\mathbb{E}g_1^6\mathbb{E}g_1^2 + 9(\mathbb{E}g_1^4)^2 + 144\mathbb{E}g_1^4(\mathbb{E}g_1^2)^2 + 36(\mathbb{E}g_1^2)^4$$

which, by Lemma 2.2 is bounded by an absolute constant. Thus,

$$\left(\mathbb{E}|h(g_1,\dots,g_4)|^2\right)^{1/2}\mathbb{E}(|Y_4|^2)^{1/2} \le C\left(\mathbb{E}|g_1|^{2p-2}\right)^{\frac{1}{2}} \le \frac{C'}{\sqrt{p}}$$

again by Lemma 2.2 for some absolute constant $C' > 0$.

Recall that factor $F\left(\frac{1}{p}\right) - 3$ is positive for $1 \le p < 2$ and negative for $p > 2$. We put all estimates together and obtain for some absolute constants $c_1, c_2, C_1, C_2 > 0$:

Let $p > 2$, then for $n \ge n_0(p)$

$$\sqrt{n}\ \mathbb{E}\ h(g_1,g_2,g_3,g_4)\psi_{\theta_0} \le \frac{C_1\sqrt{n}\left(\frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right) - 3\right)\right) + C_2}{\sqrt{p}} < 0.$$

Let $1 \le p < 2$, then for $n \ge n_1(p)$

$$\sqrt{n}\mathbb{E}h(g_1,g_2,g_3,g_4)\psi_{\theta_0} \ge \frac{c_1\sqrt{n}\left(\frac{\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right) - 3\right)\right) - c_2}{\sqrt{p}} > 0.$$

□

Proof of Theorem 1.2. Proceed as in the proof of Theorem 1.1, using Lemma 4.3 and Propositions 4.1 and 4.2. □

Finally, in order to deduce Corollary 1.2 we shall compute the sign of $f(\overline{\xi}_1, \overline{\xi}_2)$ for $p \ge 2$ and $f(\xi_1, \xi_2)$ for $1 \le p \le 2$. For that matter, we denote $\overline{g}_1, \dots \overline{g}_n$ i.i.d. copies of g and the $g_i's$, and $\overline{\psi}_{\theta_0} = \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \operatorname{sgn}(\overline{g}_i)|\overline{g}_i|^{p-1}\right|$.

Lemma 4.6. *Let X a be random vector uniformly distributed on $P_{\theta_0^\perp}B_p^n$, $p \ge 1$ and $f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$ given by*

$$f(\eta_1,\eta_2) = \mathbb{E}\langle X,\eta_1\rangle^2\langle X,\eta_2\rangle^2 - \mathbb{E}\langle X,\eta_1\rangle^2\mathbb{E}\langle X,\eta_2\rangle^2.$$

Write $\overline{\xi}_1 = \frac{e_1 - e_2}{\sqrt{2}}$, $\overline{\xi}_2 = \frac{e_3 - e_4}{\sqrt{2}} \in S_{\theta_0^\perp}$. Then

$$f(\overline{\xi}_1,\overline{\xi}_2) = \frac{\mathbb{E}S^{p-1}\mathbb{E}h(g_1,g_2,g_3,g_4,\overline{g}_1,\overline{g}_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}},$$

where $h : \mathbb{R}^6 \to \mathbb{R}$ is defined by

$$\begin{aligned} h(x_1,\dots,x_6) &= x_1x_2^2 - 2x_1^2x_2x_3 + x_1x_2x_3x_4 \\ &- \frac{\mathbb{E}S^{p-1}\mathbb{E}S^{p+3}}{(\mathbb{E}S^{p+1})^2}(x_1^2x_5^2 - 2x_1x_2x_5^2 + x_1x_2x_5x_6). \end{aligned}$$

Proof. We have seen in the proof of Proposition 4.1 that

$$\begin{aligned}\mathbb{E}\langle X,\bar\xi_1\rangle^2\langle X,\bar\xi_2\rangle^2 &= \frac{\mathbb{E}S^{p-1}\mathbb{E}(g_1g_2^2-2g_1^2g_2g_3+g_1g_2g_3g_4)\psi_{\theta_0}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}}\\ &= \frac{\mathbb{E}S^{p-1}\mathbb{E}(g_1g_2^2-2g_1^2g_2g_3+g_1g_2g_3g_4)\psi_{\theta_0}\overline{\psi}_{\theta_0}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}.\end{aligned}$$

and in the proof of Lemma 4.1,

$$\mathbb{E}\langle X,\bar\xi_1\rangle^2\mathbb{E}\langle X,\bar\xi_2\rangle^2=\frac{(\mathbb{E}S^{p-1})^2\mathbb{E}(g_1^2-g_1g_2)(\bar g_1^2-\bar g_1\bar g_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}}{(\mathbb{E}S^{p+1})^2\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}.$$

Since $\mathbb{E}g_1g_2\bar g_1^2\psi_{\theta_0}\overline{\psi}_{\theta_0}=\mathbb{E}\bar g_1\bar g_2 g_1^2\psi_{\theta_0}\overline{\psi}_{\theta_0}$, we obtain the result. □

Therefore, the sign of $f(\bar\xi_1,\bar\xi_2)$ coincides with the sign of $\mathbb{E}h\psi_{\theta_0}\overline{\psi}_{\theta_0}$. We split the latter quantity in four terms by using Lemma 4.4.

Lemma 4.7. *Denote* $\overline{Y}_2=\sum_{i=1}^{2}|\bar g_i|^{p-1}\operatorname{sgn}(\bar g_i), Y_4=\sum_{i=1}^{4}|g_i|^{p-1}\operatorname{sgn}(g_i)$ *and* $\overline{Z}=\sum_{i=3}^{n}|\bar g_i|^{p-1}\operatorname{sgn}(\bar g_i), Z=\sum_{i=5}^{n}|g_i|^{p-1}\operatorname{sgn}(g_i)$. *Then,*

$$\begin{aligned}&n\ \mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}=\mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\mathbb{E}\,|Z|\,\mathbb{E}\,|\overline{Z}|\\&+\mathbb{E}\,|\overline{Z}|\,\mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\left(|Y_4|-|Z|\right)\chi_{\{|Y_4|\ge|Z|\}}\\&+\mathbb{E}\,|Z|\,\mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\left(|\overline{Y_2}|-|\overline{Z}|\right)\chi_{\{|\overline{Y_2}|\ge|\overline{Z}|\}}\\&+\mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\left(|Y_4|-|Z|\right)\chi_{\{|Y_4|\ge|Z|\}}\left(|\overline{Y_2}|-|\overline{Z}|\right)\chi_{\{|\overline{Y_2}|\ge|\overline{Z}|\}}.\end{aligned}$$

Proof. First condition on the random variables $g_1,\dots g_n$ and apply Lemma 4.4 with $\overline{Y_2}$ and $\overline{Z}$. Then take expectations with respect to $g_1,\dots g_n$, use Fubini's theorem and, conditioning on $\bar g_1,\dots\bar g_n$, apply again Lemma 4.4 with Y_4 and Z. □

Lemma 4.8. *Let* $p\ge1$ *and* $\bar\xi_1=\frac{e_1-e_2}{\sqrt2}$, $\bar\xi_2=\frac{e_3-e_4}{\sqrt2}\in S_{\theta_0^\perp}$. *For every* $n\ge n_0(p)$ *for some* $n_0(p)\in\mathbb{N}$,

$$f(\bar\xi_1,\bar\xi_2)<0.$$

Proof. We proceed as in the proof of Proposition 4.2:

By Lemma 4.6, we will compute the sign of $n\mathbb{E}h(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}$. For that matter we apply Lemma 4.7 and estimate each summand. By definition of h and Lemma 2.2,

$$\mathbb{E}\ h=-\frac{\mathbb{E}S^{p-1}\mathbb{E}S^{p+3}}{(\mathbb{E}S^{p+1})^2}(\mathbb{E}g_1^2)^2=-\frac{\Gamma\left(1+\frac{n-1}{p}\right)\Gamma\left(1+\frac{n+3}{p}\right)\Gamma\left(\frac{3}{p}\right)^2}{\Gamma\left(1+\frac{n+1}{p}\right)^2\Gamma\left(\frac{1}{p}\right)^2}<-c$$

for some absolute constant $c>0$, since $\Gamma(1+x)=x\Gamma(x), x>0$, implies $c<\mathbb{E}g_1^2$ and the convexity of $\log\Gamma$ yields $\Gamma\left(1+\frac{n-1}{p}\right)\Gamma\left(1+\frac{n+3}{p}\right)\ge\Gamma\left(1+\frac{n+1}{p}\right)^2$. Also observe that, by Stirling's formula, $\frac{\mathbb{E}S^{p-1}\mathbb{E}S^{p+3}}{(\mathbb{E}S^{p+1})^2}$ is bounded from above by an absolute constant. Proceeding as in the proof of Proposition 4.2, by direct computation and using Lemma 2.2, $\mathbb{E}h^2(g_1,g_2,g_3,g_4,\bar g_1,\bar g_2)\le C$. In the sequel we shall use the same letter $c, C...$ to denote possibly different values of an absolute constant $c, C... > 0$.

According to Lemma 4.4 and Lemma 2.2, the second summand has absolute value bounded by

$$\mathbb{E}|\overline{Z}|(\mathbb{E}h^2)^{1/2}(\mathbb{E}|\overline{Y_2}|^2)^{1/2} \leq \frac{C}{\sqrt{p}}\mathbb{E}|\overline{Z}|$$

and in the same way, the third summand has absolute value bounded by

$$\mathbb{E}|Z|(\mathbb{E}h^2)^{1/2}(\mathbb{E}|Y_4|^2)^{1/2} \leq \frac{C}{\sqrt{p}}\mathbb{E}|Z|$$

Similarly, the forth summand has absolute value bounded by $\frac{C}{p}$ and finally, Lemma 4.5 implies $c\frac{\sqrt{n}}{\sqrt{p}} \leq \mathbb{E}\,|Z|\,, \mathbb{E}\,|\overline{Z}| \leq C\frac{\sqrt{n}}{\sqrt{p}}$ whenever $p \leq n$.

We put all estimates together and conclude that for $1 \leq p \leq n$ and some absolute constants:

$$\frac{np\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}{\mathbb{E}S^{p-1}}f(\overline{\xi}_1, \overline{\xi}_2) \leq -C_1 n + C_2\sqrt{n} + C_3\sqrt{n} + C_4.$$

The result now easily follows. □

In the following lemma we compute the value of $f(\xi_1, \xi_2)$.

Lemma 4.9. *Let X be a random vector uniformly distributed on $P_{\theta_0^\perp}B_p^n$, $p \geq 1$, and let $f : S_{\theta_0^\perp} \times S_{\theta_0^\perp} \to \mathbb{R}$ be given by*

$$f(\eta_1, \eta_2) = \mathbb{E}\langle X, \eta_1\rangle^2\langle X, \eta_2\rangle^2 - \mathbb{E}\langle X, \eta_1\rangle^2\mathbb{E}\langle X, \eta_2\rangle^2.$$

Write $\xi_1 = \frac{e_1 - e_2 + e_3 - e_4}{2}$, $\xi_2 = \frac{e_1 - e_2 - e_3 + e_4}{2}$. Then

$$f(\xi_1, \xi_2) = \frac{\mathbb{E}S^{p-1}\mathbb{E}h(g_1, g_2, g_3, g_4, \overline{g}_1, \overline{g}_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}},$$

where $h : \mathbb{R}^6 \to \mathbb{R}$ is defined by

$$\begin{aligned} h(x_1, x_2, x_3, x_4, x_5, x_6) \quad &= \quad \frac{1}{4}x_1^4 - x_1^3x_2 - \frac{3}{4}x_1^2x_2^2 + x_1^2x_2x_3 - \frac{1}{2}x_1x_2x_3x_4 + x_1x_2^2 \\ &- \quad \frac{\mathbb{E}S^{p-1}\mathbb{E}S^{p+3}}{(\mathbb{E}S^{p+1})^2}(x_1^2x_5^2 - 2x_1x_2x_5^2 + x_1x_2x_5x_6). \end{aligned}$$

Proof. We have seen in the proof of Proposition 4.1 that

$$\begin{aligned} \mathbb{E}(g_1^4 - 4g_1^3g_2 - 3g_1^2g_2^2 \quad &+ \quad 12g_1^2g_2g_3 - 6g_1g_2g_3g_4)\psi_{\theta_0} = \\ &= \quad 4\mathbb{E}\langle G, \xi_1\rangle^2\langle G, \xi_2\rangle^2\psi_{\theta_0} - 4\mathbb{E}\langle G, \overline{\xi}_1\rangle^2\langle G, \overline{\xi}_2\rangle^2\psi_{\theta_0} \end{aligned}$$

and then, taking into account that

$$\mathbb{E}\langle G, \overline{\xi}_1\rangle^2\langle G, \overline{\xi}_2\rangle^2\psi_{\theta_0} = \mathbb{E}(g_1g_2^2 - 2g_1^2g_2g_3 + g_1g_2g_3g_4)\psi_{\theta_0}$$

we obtain that

$$\mathbb{E}\langle X, \xi_1\rangle^2\langle X, \xi_2\rangle^2 = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}}\mathbb{E}\langle G, \xi_1\rangle^2\langle G, \xi_2\rangle^2\psi_{\theta_0}$$

$$= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}\Big(\mathbb{E}(\frac{1}{4}g_1^4 - g_1^3g_2 - \frac{3}{4}g_1^2g_2^2 + g_1^2g_2g_3 - \frac{1}{2}g_1g_2g_3g_4 + g_1g_2^2)\psi_{\theta_0}\overline{\psi}_{\theta_0}\Big)$$

Since, by Lemma 4.1

$$\mathbb{E}\langle X,\xi_1\rangle^2\mathbb{E}\langle X,\xi_2\rangle^2=\frac{(\mathbb{E}S^{p-1})^2\mathbb{E}(g_1^2-g_1g_2)(\overline{g}_1^2-\overline{g}_1\overline{g}_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}}{(\mathbb{E}S^{p+1})^2\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}$$

and $\mathbb{E}g_1g_2\overline{g}_1^2\psi_{\theta_0}\overline{\psi}_{\theta_0}=\mathbb{E}\overline{g}_1\overline{g}_2g_1^2\psi_{\theta_0}\overline{\psi}_{\theta_0}$, we obtain the result. □

Lemma 4.10. *Let* $\xi_1=\frac{e_1-e_2+e_3-e_4}{2}$, $\xi_2=\frac{e_1-e_2-e_3+e_4}{2}\in S_{\theta_0^\perp}$. *There exists* $n_0\in\mathbb{N}$ *such that for every* $n\geq n_0$ *and* $1\leq p\leq 2$

$$f(\xi_1,\xi_2)<0$$

Proof. By Lemma 4.9 we must compute the sign of $\mathbb{E}h(g_1,g_2,g_3,g_4,\overline{g}_1,\overline{g}_2)\psi_{\theta_0}\overline{\psi}_{\theta_0}$. For that matter we apply Lemma 4.7 with the same choice of random variables and h as above. The behaviour of the sign is determined by the sign of

$$\begin{aligned}\mathbb{E}h &= \frac{\Gamma\left(\frac{3}{p}\right)^2}{4\Gamma\left(\frac{1}{p}\right)^2}\left(F\left(\frac{1}{p}\right)-3-\frac{4\Gamma\left(1+\frac{n-1}{p}\right)\Gamma\left(1+\frac{n+3}{p}\right)}{\Gamma\left(1+\frac{n+1}{p}\right)^2}\right)\\ &\leq \frac{\Gamma\left(\frac{3}{p}\right)^2}{4\Gamma\left(\frac{1}{p}\right)^2}(6-3-4)=-\frac{\Gamma\left(\frac{3}{p}\right)^2}{4\Gamma\left(\frac{1}{p}\right)^2}.\end{aligned}$$

As in Proposition 4.3, we bound the terms in Lemma 4.7 in the same manner and conclude as before that for some absolute constants,

$$\frac{np\mathbb{E}S^{p+3}\mathbb{E}\psi_{\theta_0}\overline{\psi}_{\theta_0}}{\mathbb{E}S^{p-1}}f(\xi_1,\xi_2)\leq -C_1n+C_2\sqrt{n}+C_3\sqrt{n}+C_4. \qquad \square$$

Proof of Corollary 1.2. It follows from Lemmas 4.8 and 4.10 and Theorem 1.2. □

Remark. By a closer study it is possible to state the results of this section letting p grow with n. Using the estimate $\frac{c'\sqrt{n}}{\sqrt{p}}\leq\mathbb{E}\psi_{\theta_0}\leq\frac{C'\sqrt{n}}{\sqrt{p}}$, $p\geq n$, proven in [AB3], our method works for at least $p\leq cn^2$. However, by viewing the situation at $p=\infty$ mentioned in the introduction, we believe Corollary 1.3 should hold for C independent of p.

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