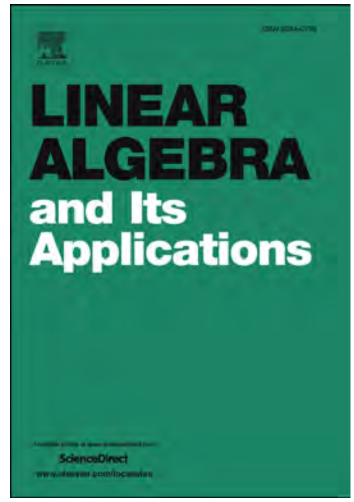


# Accurate inverses of Nekrasov Z-matrices

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# Accurate inverses of Nekrasov Z-matrices

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**Abstract**

We present a parametrization of a Nekrasov $Z$-matrix that allows us to compute its inverse with high relative accuracy. Numerical examples illustrating the accuracy of the method are included.

*MSC:* 15B48, 15A09, 15B35, 65F05

*Key words:* Accuracy, Inverse, $H$-matrices, Strictly diagonally dominant matrices, Nekrasov matrices

## 1 Introduction

Recent research in Numerical Linear Algebra has shown that, for some classes of structured matrices, some algebraic computations can be performed to high relative accuracy (HRA), independently of the size of the classical condition number. These classes of matrices are defined by special sign or other structure. It is well–known (cf. p. 52 of [7]) that, if an algorithm is subtraction–free, its output can be computed to HRA. For these classes of matrices, knowing an adequate parametrization has been a crucial start point for the construction of the corresponding accurate algorithms, being many of them subtraction–free. In contrast to these classes of matrices, for other structured classes of matrices it is not possible to construct such HRA algorithms (cf. [6]).

In this paper, we present a parametrization for Nekrasov $Z$-matrices, which allows us to construct a subtraction–free (and so, HRA) efficient algorithm to compute their inverses.

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Let us now recall some basic definitions on classes of matrices used in this paper. A real matrix $A$ is a *$Z$-matrix* if all its off–diagonal entries are nonpositive. A $Z$-matrix $A$ is a nonsingular *$M$-matrix* if its inverse is nonnegative. Given a complex matrix $A = (a_{ij})_{1\le i,j\le n}$, its *comparison matrix* $\mathcal{M}(A) = (\tilde{a}_{ij})_{1\le i,j\le n}$ has entries $\tilde{a}_{ii} := |a_{ii}|$ and $\tilde{a}_{ij} := -|a_{ij}|$ for all $j \ne i$ and $i, j = 1, \dots, n$. We say that a complex matrix is a nonsingular *$H$-matrix* if its comparison matrix is a nonsingular $M$-matrix. This concept corresponds with the concept of $H$-matrix of invertible class given in [4]. A matrix $A = (a_{ij})_{1\le i,j\le n}$ is *SDD* (strictly diagonally dominant by rows) if $|a_{ii}| > \sum_{j\ne i} |a_{ij}|$ for all $i = 1, \dots, n$, and $A$ is *DD* (diagonally dominant by rows) if $|a_{ii}| \ge \sum_{j\ne i} |a_{ij}|$ for all $i = 1, \dots, n$. It is well–known that an SDD matrix is nonsingular and that a square matrix $A$ is a nonsingular $H$-matrix if and only if there exists a diagonal matrix $W$ with positive diagonal entries such that $AW$ is SDD. Nekrasov matrices (see [14]) are defined in Section 2 and form another subclass of $H$-matrices that includes SDD matrices. Some recent applications of Nekrasov matrices can be seen in [5], [10], [11] or [12].

Let us present the layout of the paper. Section 2 presents the parametrization of Nekrasov $Z$-matrices, some auxiliary results and the construction of the subtraction–free algorithms for the inverse of a Nekrasov $Z$-matrix in a particular case. The algorithm for a general Nekrasov $Z$-matrix $A$ is constructed in Section 3. Section 4 includes some algorithms used in our method and presents numerical examples showing its accuracy. Our method also allows us to compute the solution of a linear system $Ax = b$ with $b \ge 0$ to HRA. The numerical examples also show great accuracy of our method even when $b$ does not satisfy this requirement.

The following notations will be also used in this paper. A matrix $A = (a_{ij})_{1\le i,j\le n}$ (resp., a vector $v = (v_1, \dots, v_n)^T$) is nonnegative if $a_{ij} \ge 0$ for all $i, j$ (resp., $v_i \ge 0$ for all $i$), and we write $A \ge 0$ (resp., $v \ge 0$).

## 2 Parametrization of Nekrasov matrices and HRA

Let us start by defining the concept of a Nekrasov matrix (see [5,14]). For this purpose, let us define recursively for a complex matrix $A = (a_{ij})_{1\le i,j\le n}$ with $a_{ii} \ne 0$, for all $i = 1, \dots, n$,

$$h_1(A) := \sum_{j\ne 1} |a_{1j}|, \quad h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, \dots, n. \tag{1}$$

Let $N := \{1, \dots, n\}$. We say that $A$ is a *Nekrasov matrix* if $|a_{ii}| > h_i(A)$ for all $i \in N$. A Nekrasov matrix is a nonsingular $H$-matrix [14]. Therefore, a Nekrasov $Z$-matrix with positive diagonal entries is a nonsingular $M$-matrix.

**Remark 2.1** *Let us recall that DD $M$-matrices admit some algebraic computations with high relative accuracy (HRA). A key tool is the use of an adequate parametrization of these matrices, which was provided by the off–diagonal entries and the row sums (cf. [1],[8], [13], [2]). We shall call these $n^2$ parameters for an $n \times n$ DD $M$-matrix $A$ as DD-parameters. If these DD-parameters are known with HRA, then some algebraic computations of $A$ can be performed with HRA as it is shown in the previous references.*

In this paper we also study computations with HRA for the class of Nekrasov $Z$-matrices. Here, a good choice of parameters will also be crucial. The parameters that we shall use for an $n \times n$ Nekrasov $Z$-matrix $A = (a_{ij})_{1\leq i,j\leq n}$ with positive diagonal are the following $n^2$ parameters, which will be called *N-parameters*:

$$\begin{cases} a_{ij}, & i \neq j \\ \Delta_j(A) := a_{jj} - h_j(A), & j \in N \end{cases} \tag{2}$$

We can characterize an $n \times n$ Nekrasov $Z$-matrix with positive diagonal through the $n^2$ signs of the parameters given in (2). In fact, $A$ is a Nekrasov $Z$-matrix with positive diagonal if and only if the first $n^2 - n$ parameters (corresponding to the off–diagonal entries, $a_{ij}$ with $i \neq j$) are nonpositive and the last $n$ parameters ($\Delta_j(A)$ for all $j \in N$) are positive.

Since a Nekrasov matrix is a nonsingular $H$-matrix, there exists a positive diagonal matrix $W$ such that $AW$ is *SDD*. The following lemma shows that the very simple diagonal matrix

$$S = \begin{pmatrix} \frac{h_1(A)}{a_{11}} & & & \\ & \frac{h_2(A)}{a_{22}} & & \\ & & \ddots & \\ & & & \frac{h_n(A)}{a_{nn}} \end{pmatrix} \tag{3}$$

holds that $AS$ satisfies the weaker property of being *DD*.

**Lemma 2.2** *Let $A$ be a Nekrasov $Z$-matrix with positive diagonal and let $S$ be the matrix given by* (3)*. Then the matrix $AS$ is a DD $Z$-matrix.*

**PROOF.** Observe that $\frac{h_i(A)}{a_{ii}} \geq 0$ for $i \in N$, and so, $S \geq 0$. Then $B := AS$ preserves the signs of $A$, and the elements of $B = (B_{ij})_{1\leq i,j\leq n}$ are:

$$B_{ij} = \begin{cases} a_{ij}\frac{h_j(A)}{a_{jj}}, \text{ if } i \neq j, \\ h_i(A), \text{ if } i = j. \end{cases}$$

Since $A$ is a $Z$-matrix, $B$ is also a $Z$-matrix. It remains to prove that $B$ is also DD. Since $A$ is a Nekrasov matrix, $h_j(A) < a_{jj}$ for all $j \in N$. For each $i \in N$,

$$h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{a_{jj}} + \sum_{j=i+1}^{n} |a_{ij}| \geq \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{a_{jj}} + \sum_{j=i+1}^{n} |a_{ij}| \frac{h_j(A)}{a_{jj}}$$

and so $B$ is DD.

For a Nekrasov $Z$-matrix $A$ and the diagonal matrix $S$ given by (3), the following result shows that if we know the $n^2$ N-parameters in (2) of $A$, then we can compute the $n^2$ DD-parameters of the DD $M$-matrix $AS$ with HRA.This fact will allow us to take advantage of properties of DD $M$-matrices to obtain algorithms with HRA for Nekrasov $Z$-matrices.

**Theorem 2.3** *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a Nekrasov $Z$-matrix with positive diagonal entries and let $S$ be the matrix given by (3). Given the $n^2$ N-parameters (2), we can compute the row sums and the off–diagonal entries of $AS$ (its DD-parameters) by a subtraction–free algorithm (and so, with HRA), with at most $\frac{3n(n-1)}{2}$ additions, $2n(n-1)$ multiplications and $2n-1$ quotients.*

**PROOF.** Observe that by (2),

$$a_{jj} = \Delta_j(A) + h_j(A), \qquad j \in N. \tag{4}$$

Let us start by computing $h_1(A)$, $a_{11}$, $h_2(A)$, $a_{22}$, $\ldots$, $h_n(A)$, $a_{nn}$ using the formulas (4) and (1). We carry out $n$ sums computing the diagonal entries by (4), $n$ quotients in order to obtain $\frac{h_j(A)}{a_{jj}}$ when needed (and we store them) and $\frac{(n-1)n}{2}$ products and $n(n-2)$ sums to calculate $h_j(A)$ for all $j \in N$ using (1). Then we obtain the off–diagonal entries of $AS$, $a_{ij}\frac{h_j(A)}{a_{jj}}$, which requires $n(n-1)$ products. Finally, we compute the row sums of $AS$. The row sum of the $i$th row is:

$$\sum_{j=1}^{i-1} a_{ij} \frac{h_j(A)}{a_{jj}} + h_i(A) + \sum_{j=i+1}^{n} a_{ij} \frac{h_j(A)}{a_{jj}},$$

which can be expressed in the following form using (1), (2) and the sign pattern of a $Z$-matrix:

$$\sum_{j=i+1}^{n} (-a_{ij}) \left(1 - \frac{h_j(A)}{a_{jj}}\right) = \sum_{j=i+1}^{n} |a_{ij}| \frac{a_{jj} - h_j(A)}{a_{jj}} = \sum_{j=i+1}^{n} |a_{ij}| \frac{\Delta_j(A)}{a_{jj}}. \tag{5}$$

Computing the row sums requires $n-1$ quotients of the form $\frac{\Delta_j(A)}{a_{jj}}$ for $j = 2, \ldots, n$, $\frac{n(n-1)}{2}$ sums and $\frac{n(n-1)}{2}$ products. The total number of required operations is at most $\frac{3n(n-1)}{2}$ additions, $2n(n-1)$ multiplications and $2n-1$

quotients. We do not perform any subtraction in this procedure and so it is subtraction–free.

Let us introduce some basic notations related with Gaussian and Gauss–Jordan elimination. Gaussian elimination without pivoting for a nonsingular $n \times n$ matrix $A$ consists of a procedure of at most $n-1$ steps resulting in the following sequence of matrices:

$$A =: A^{(1)} \longrightarrow A^{(2)} \longrightarrow \cdots \longrightarrow A^{(n)}, \tag{6}$$

where $A^{(t)}$ has zeros below its main diagonal in the first $(t-1)$ columns and $A^{(n)}$ is an upper triangular matrix. To obtain $A^{(t+1)}$ from $A^{(t)}$ we produce zeros in column $t$ below the pivot element $a_{tt}^{(t)}$ by subtracting adequate multiples of row $t$ from the rows beneath it. The same transformation can be performed with the matrix $(A \mid B^{(1)})$, where $B^{(1)} := I$ is the identity matrix,

$$(A \mid I) =: \left(A^{(1)} \mid B^{(1)}\right) \longrightarrow \left(A^{(2)} \mid B^{(2)}\right) \longrightarrow \cdots \longrightarrow \left(A^{(n)} \mid B^{(n)}\right). \tag{7}$$

Now we proceed analogously, starting from the last row and producing zeros above the main diagonal of $A^{(k)}$ $(n \leq k \leq 2n-1)$ to obtain the sequence:

$$\left(A^{(n)} \mid B^{(n)}\right) \longrightarrow \cdots \longrightarrow \left(A^{(2n-1)} \mid B^{(2n-1)}\right) \longrightarrow \left(A^{(2n)} \mid B^{(2n)}\right) =: (I \mid A^{-1}). \tag{8}$$

In this case, $A^{(t)} = (a_{ij}^{(t)})_{1\leq i,j\leq n}$, $t = n+1, \ldots, 2n-1$, has zeros above its main diagonal in the last $(t-n)$ columns. To obtain $A^{(t+1)}$ from $A^{(t)}$, $t = n, \ldots, 2n-1$, we produce zeros in column $2n-t$ above the pivot element $a_{2n-t,2n-t}^{(t)}$ by subtracting multiples of row $2n-t$ from the rows above it. Finally, $A^{(2n)} = I$ is obtained from $A^{(2n-1)}$ by dividing each row of $A^{(2n-1)}$ by its diagonal entries. This well–known method is called Gauss–Jordan elimination.

Let $Q_{k,n}$ be the set of increasing sequence of $k$ positive integers in $N$. Given $\alpha, \beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$. If $\alpha = \beta$, then we have the principal submatrix $A[\alpha] := A[\alpha|\alpha]$. The complement $\alpha^C$ is the increasingly rearranged $N \setminus \alpha$.

For DD $M$-matrices, algorithms with HRA starting from their DD-parametrization were presented in [8] and [13]. In both papers, Gaussian elimination is used, but with a different pivoting strategy in each of them. In order to obtain the inverse with HRA, a pivoting strategy is not necessary, as the following result shows.

**Proposition 2.4** *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a DD nonsingular $Z$-matrix with positive diagonal entries. If we know the row sums and the off–diagonal entries* *of A (i.e., its DD-parameters), then we can compute $A^{-1}$ with a subtraction–free algorithm (and so, with HRA) performing $\mathcal{O}(n^3)$ elementary operations.*

**PROOF.** By the hypotheses, $A$ is nonsingular and $A + D$ is SDD (and so, nonsingular) for any positive diagonal matrix $D$. Then, by using property $(C_{10})$ of Theorem 2.2. of chapter 6 of [3] we deduce that $A$ is a nonsingular $M$-matrix. In order to obtain $A^{-1}$ with HRA, we are going to use Gauss–Jordan elimination without pivoting. We form the augmented matrix $\tilde{M} := (A|I|s)$, which coincides with the first matrix in (7) but with a last column with the vector $s$ formed by the row sums of $A$: $s = (s_1, \ldots, s_n)^T$ and, for $i = 1, \ldots, n$, $s_i := \sum_{j=1}^n a_{ij}$. Then, we apply the elementary operations of the Gaussian elimination of $A$ to the whole matrix $\tilde{M}$. We start by computing the first pivot, $a_{11}$, by adding $s_1 (\geq 0)$ and the sum of the absolute values of the first row off–diagonal entries: $a_{11} = s_1 + \sum_{j \neq 1} |a_{1j}|$. Then we produce zeros in the first column of $A$ by adding positive multiples of the first row and, with the exception of the diagonal entries of $A^{(2)}[2, ..., n]$, every entry of $\tilde{M}^{(2)} = (A^{(2)} \mid B^{(2)} \mid s^{(2)})$ is computed with HRA. Nevertheless, we can obtain analogously the first diagonal entries of $A^{(2)}[2, ..., n], \ldots, A^{(n-1)}[n-1, n]$ with HRA when they are needed as pivots at the corresponding steps of the Gaussian elimination of $A$, and $a_{nn}^{(n)}$ after finishing the elimination procedure. In order to start the second iteration, it only remains to obtain $a_{22}^{(2)}$ with HRA.

Since $A^{(2)}[2, ..., n]$ is the Schur complement of an $M$-matrix it is also an $M$-matrix (see [9]). The vector of row sums is obtained as $Ae = s$, where $e = (1, \ldots, 1)^T$. Observe that $s = s^{(1)} \geq 0$ and the way of constructing $\tilde{M}^{(2)}$ from $\tilde{M}$ imply that $s^{(2)} \geq 0$. Besides, $e$ will be also the solution of the linear system $A^{(2)} x = s^{(2)}$, which implies by the sign pattern of $A^{(2)}$ that the components of $s^{(2)}$ coincide again with the row sums of $A^{(2)}$. So, $a_{22}^{(2)}$ can be computed with HRA by adding $s_2^{(2)} (\geq 0)$ and the absolute values of the off–diagonal entries of the second row of $A^{(2)}$. Now we continue the Gaussian elimination and make zeros in the second column below $a_{22}^{(2)}$. We repeat this procedure until when we obtain the upper triangular matrix $U := A^{(n)}$ with HRA. Then $A^{(n)}$ preserves the $Z$-matrix sign pattern. In this process, the identity matrix becomes the lower triangular matrix $B^{(n)}$, with ones on the diagonal and nonnegative entries below it.

Now, we continue the elimination procedure of $A^{(n)}$ starting with the last row and producing zeros above the main diagonal of $A^{(k)}$ $(n \leq k \leq 2n - 1)$, as described in (8), and we apply it to the whole matrix $\left(A^{(n)} \mid B^{(n)}\right)$. The sign pattern of $\left(A^{(n)} \mid B^{(n)}\right)$ allows us to carry out this elimination process without subtractions, and so, with HRA.

The computational cost is given by the cost of Gauss–Jordan elimination (and so of $\mathcal{O}(n^3)$ elementary operations) in addition to the elementary operations to compute the pivots $a_{11}, a_{22}^{(2)}, \ldots, a_{nn}^{(n)}$ and to update the vectors $s, s^{(2)}, \ldots, s^{(n)}$ (of $\mathcal{O}(n^2)$ elementary operations in both cases).

**Remark 2.5** *By the characterization ($I_{28}$) of Theorem (2.3) of chapter 6 of [3], a Z-matrix $A$ is a nonsingular M-matrix if and only if there exists a vector $z$ with positive entries such that $s := Az$ has positive entries. Then the same proof of Proposition 2.4 can be used to prove that, if we know the $n^2 + n$ parameters of $A$ given by its $n^2 - n$ off–diagonal entries, the $n$ entries of $z := (z_1, \ldots, z_n)^T$ and the $n$ entries of $Az = s(= (s_1, \ldots, s_n)^T)$, then we can compute $A^{-1}$ with HRA. The analogous proof to that of Proposition 2.4 will use now the augmented matrix $\tilde{M} := (A|I|s)$, where $s = Az$, $z$ will play the role of $(1, \ldots, 1)^T$ and the expression of $a_{11}$ will be now $a_{11} = \left(s_1 + \sum_{j \neq 1} |a_{1j}| z_j\right)/z_1$. Besides, $z$ will be again the solution of the linear systems $A^{(k)}x = s^{(k)}$ for $k = 2, \ldots, n$. The result can be stated as follows: "If $A = (a_{ij})$ is a nonsingular M-matrix and we know its off–diagonal entries as well as $z > 0$ such that $s := Az > 0$, then we can compute $A^{-1}$ with a SF algorithm (and so with HRA) performing $\mathcal{O}(n^3)$ elementary operations".*

The following result is a consequence of Theorem 2.3 and Proposition 2.4 and guarantees the construction of the inverse of Nekrasov $Z$-matrices $A$ in the particular case that $h_i(A) \neq 0$ for all $i$.

**Corollary 2.6** *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a Nekrasov $Z$-matrix with positive diagonal entries such that $h_i(A) \neq 0$ for $i = 1, \ldots, n$ (see (1)). If we know its $n^2$ N-parameters (2) then we can compute $A^{-1}$ with a subtraction–free algorithm (and so, with HRA) performing $\mathcal{O}(n^3)$ elementary operations.*

**PROOF.** Let $S$ be the matrix given by (3), which can be obtained with HRA and $\mathcal{O}(n^2)$ elementary operations, without performing any subtraction. Then $B := AS$ is a nonsingular diagonally dominant $M$-matrix and by Theorem 2.3 we can compute its DD-parameters (i.e., off–diagonal entries and row sums) with HRA. With these DD-parameters we can compute $B^{-1}$ with HRA by the procedure described in Proposition 2.4.

Since $B = AS$, we conclude that $A^{-1} = SB^{-1}$ and so each entry of the inverse of $A$ can be computed by multiplying the corresponding entry of $B^{-1}$ by the corresponding diagonal entry of $S$. This step can be computed with $n^2$ elementary operations, without performing any subtraction.

**Remark 2.7** *The accurate inverse $A^{-1}$ obtained in Corollary 2.6 (and also for a general Nekrasov $Z$-matrix with positive diagonal entries, obtained in the following section) can be used to compute with HRA the solution of a linear system $Ax = b$ with $b \geq 0$ by the direct computation $x = A^{-1}b$, since the constructed matrix with HRA $A^{-1} \geq 0$ and so subtractions are not performed.*

*In Section 4, our numerical experiments also show that the solution of the linear system $Ax = b$ for any $b$, computed by this procedure, is also accurately computed.*

## 3 Accurate inverses in the general case

We show in this section that the condition $h_i(A) \neq 0$ for $i = 1, \ldots, n$ can be suppressed in Corollary 2.6. In order to prove this fact, it is crucial to study first the distribution of the zero entries of a Nekrasov matrix that satisfies $h_i(A) = 0$ for some $i \in N$.

**Lemma 3.1** *Let $A = (a_{ij})_{1\le i,j\le n}$ be a Nekrasov matrix, and let $J = \{i_1, \ldots, i_k\} \subseteq N$ $(i_1 \le i_2 \le \cdots \le i_k)$ be the ordered set of indices such that $h_{i_j}(A) = 0$. Then at least $n-j$ off–diagonal elements of the row $i_j$ are zero for each $j = 1, \ldots, k$.*

**PROOF.** Assuming that $J \neq \emptyset$, we start by considering the row $i_1$:

$$h_{i_1}(A) = \sum_{k=1}^{i_1-1} |a_{i_1k}| \frac{h_k(A)}{|a_{kk}|} + \sum_{k=i_1+1}^{n} |a_{i_1k}| = 0. \tag{9}$$

Since $h_k(A) \neq 0$ for $k < i_1$, we deduce from (9) that $a_{i_1k} = 0$ when $k \neq i_1$, that is, all the off–diagonal entries of the $i$ th row are zero. Now we consider the row $i_j \in J$ with $j > 1$:

$$\sum_{k=1}^{i_j-1} |a_{i_jk}| \frac{h_k(A)}{|a_{kk}|} + \sum_{k=i_j+1}^{n} |a_{i_jk}| = \sum_{k=1,\ k\notin J}^{i_j-1} |a_{i_jk}| \frac{h_k(A)}{|a_{kk}|} + \sum_{k=i_j+1}^{n} |a_{i_jk}| = 0.$$

In this case, we have that $a_{i_jk} = 0$ whenever $k \notin \{i_1, \ldots, i_j\}$. So there are at least $n - j$ zero entries corresponding to the columns with index $k \notin \{i_1, \ldots, i_j\}$.

By the previous result, observe that the first row of a Nekrasov matrix $A = (a_{ij})_{1\le i,j\le n}$ that satisfies $h_i(A) = 0$ has exactly $n-1$ zero entries.

**Theorem 3.2** *Let $A = (a_{ij})_{1\le i,j\le n}$ be a Nekrasov $Z$-matrix with positive diagonal entries. If we know its $n^2$ N-parameters (2), then we can compute $A^{-1}$ with HRA performing a subtraction–free algorithm of $\mathcal{O}(n^3)$ elementary operations.*

**PROOF.** We start by computing $h_1(A), a_{11}, \ldots, h_n(A), a_{nn}$ by (1) and (2) from the N-parameters of $A$ without subtractions. This computation requires

$\mathcal{O}(n^2)$ elementary operations. Let us define the ordered set $I \subseteq N$ given by the increasing sequence of indices such that $h_i(A) \neq 0$. If $I = N$ we can apply Theorem 2.6. So, from now on we consider the case $I \neq N$.

Let $S$ be the diagonal matrix given by (3). We define the submatrices $\hat{A} := A[I]$ and $B := (AS)[I]$. Observe that $B$ is DD because it is a principal submatrix of $AS$. It is possible to compute its inverse without performing subtractions and with $\mathcal{O}(n^3)$ elementary operations. In order to prove it, we first obtain an adecuate parametrization of $B$ with a subtraction–free algorithm. In this case, the required parameters are its off–diagonal elements, $a_{ij}\frac{h_j(A)}{a_{jj}}$, and its row sums (i.e., its DD-parameters), which can be written by the choice of $I$, formulae (1), (2) and the sign pattern of a $Z$-matrix in the following form, as in (5):

$$\sum_{j\in I, j\neq i} a_{ij}\frac{h_j(A)}{a_{jj}}+h_i(A) = \sum_{j=1}^{i-1} a_{ij}\frac{h_j(A)}{a_{jj}}+h_i(A)+\sum_{j=i+1}^{n} a_{ij}\frac{h_j(A)}{a_{jj}} = \sum_{j=i+1}^{n} |a_{ij}|\frac{\Delta_j(A)}{a_{jj}}.$$

So, the mentioned DD-parametrization of $B$ can be obtained from (2) by a subtraction–free procedure and $\mathcal{O}(n^2)$ elementary operations. With these parameters we can apply Theorem 2.6 in order to obtain the inverse of the diagonally dominant $M$-matrix $B = (AS)[I]$ with a subtraction–free algorithm and $\mathcal{O}(|I|^3)$ elementary operations. Then it is straightforward to compute accurately (and with $\mathcal{O}(|I|^2)$ elementary operations) $\hat{A}^{-1} = S[I]B^{-1}$.

The $|I| \times |I|$ matrices $\hat{A}$ and $\hat{A}^{-1}$ allow us to define the following procedure, key to obtain $A^{-1}$ with HRA. It consists of $n - |I|$ major steps resulting in a sequence of matrices as follows:

$$\hat{A} := \hat{A}^{(1)} \to \hat{A}^{(2)} \to \ldots \to \hat{A}^{(n-|I|+1)} = A, \tag{10}$$

For each $p \in \{2, \ldots, n - |I| + 1\}$, we obtain the matrix $\hat{A}^{(p)}$ by adding to $\hat{A}^{(p-1)}$ the row and column of $A$ corresponding to the biggest index $i \in N$ that was not already involved in it. We form the new matrix keeping the row/column ordering of $A$, and then we construct the inverse $\left(\hat{A}^{(p)}\right)^{-1}$ using the information provided by $\left(\hat{A}^{(p-1)}\right)^{-1}$. The last step will give us $A^{-1}$. To carry out the first step we start by choosing the biggest $k \in I^c$ . Then we form the $(n - |I| + 1) \times (n - |I| + 1)$ matrix $\hat{A}^{(2)}$ adding the corresponding entries of the $k$th row and column of $A$ to $\hat{A}$ in the corresponding place. In order to obtain the inverse of this new matrix from $C = \hat{A}^{-1}$ we use Lemma 3.1, which states that the new row added to $\hat{A}$ has at least $|I|$ zeros that will appear as off–diagonal elements. The new row has only one element in $\hat{A}^{(2)}$ different from zero, $a_{kk}$. In this case the entries of $C^{(2)} := (\hat{A}^{(2)})^{-1}$ are the following:

$$c_{ij}^{(2)} = \begin{cases} c_{ij}, & i,j \in I, \\ \frac{1}{a_{kk}}, & i = j = k, \\ 0, & i = k,\ j \in I, \\ c_{ik}^{(2)}, & i \in I,\ j = k. \end{cases}$$

We need to check this fact and find the expression of the entries $c_{ik}^{(2)}$. We consider the product $\hat{A}^{(2)}C^{(2)}$, which has to be the identity matrix of order $|I|+1$. Let us start with the case when both $i, j \in I$. Since the inverse of $\hat{A}$ is $C$, the performed operation to obtain the element $(i,j)$ of the product is:

$$\sum_{s\in I} a_{is}c_{sj} + a_{ik}\cdot 0 = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Now, if $i = k$, $j \in I$, we have

$$\sum_{s\in I} a_{ks}c_{sj} + a_{kk}c_{kj} = \sum_{s\in I} 0\cdot c_{sj} + a_{kk}\cdot 0 = 0$$

If $i = j = k$, we obtain

$$\sum_{s\in I} a_{ks}c_{sk} + a_{kk}c_{kk} = \sum_{s\in I} 0\cdot c_{sk} + \frac{a_{kk}}{a_{kk}} = 1$$

It remains the case $i \in I$, $j = k$, which determines the missing entries of $C^{(2)}$:

$$\sum_{s\in I} a_{is}c_{sk}^{(2)} + \frac{a_{ik}}{a_{kk}} = 0, \qquad i \in I.$$

Let us define $c := \left(c_{ik}^{(2)}\right)_{i\in I}$, the vector composed by the missing entries. Then, we can express the system of equations in terms of the matrix $\hat{A}$:

$$\hat{A}c = -\left(a_{ik}\right)_{i\in I}\left(\frac{1}{a_{kk}}\right).$$

We have already computed $\hat{A}^{-1}$ with HRA, and the right hand side is nonnegative, so we obtain $c$ with HRA (see Theorem 2.6) by performing the product:

$$c = C\left(a_{ik}\right)_{i\in I}\left(\frac{-1}{a_{kk}}\right) = \hat{A}^{-1}(a_{ik})_{i\in I}\left(\frac{-1}{a_{kk}}\right).$$

So we obtain $C^{(2)}$. We can continue analogously. In general, after performing $p-1$ major steps we may obtain $A^{-1}$ and finish the procedure, or we may have to continue it adding the row and column of index $k$, where $k \in I^c$ is the biggest index such that the $k$th row was not involved in $\hat{A}^{(p-1)}$. The added row had at least $|I|+p-1$ zeros in the original matrix, $A$. Now these zeros are the off–diagonal elements of the added row. We define $I^{(p)}$, the ordered set of indices of the rows from $A$ used in $\hat{A}^{(p-1)}$. Then we perform the product $c = C^{(p-1)} \left(a_{ik}\right)_{i\in I^{(p-1)}} \left(\frac{-1}{a_{kk}}\right)$ in order to obtain the missing entries of the matrix $C^{(p)} = (\hat{A}^{(p)})^{-1}$. After computing $c$, we build $C^{(p)}$:

$$c_{ij}^{(p)} = \begin{cases} c_{ij}^{(p-1)}, & i,j \in I^{(p)}, \\ \frac{1}{a_{kk}}, & i=j=k, \\ 0, & i=k,\ j\in I^{(p)}, \\ c, & i \in I^{(p)},\ j=k. \end{cases}$$

Clearly, we can perform these calculations with HRA and with $\mathcal{O}(n^3)$ elementary operations.

## 4 Algorithms and numerical tests

In the previous section we have presented a procedure that allows us to compute the inverse of a Nekrasov $Z$-matrix accurately if we know its N-parameters (2) with HRA. In this section, we are going to present the algorithms to compute such inverses following Theorem 3.2 and we are going to test them with some numerical examples.

The first algorithm introduced, Algorithm 1, starts with the N-parameters of the Nekrasov $Z$-matrix and performs the required preparation to compute its inverse depending on the distribution of the zero entries of the matrix.

If $h_i(A) \neq 0$ for $i = 1, \ldots, n$ the procedure corresponds to Theorem 2.3, and it calculates the DD-parameters of $AS$. Otherwise, the algorithm works with the adequate submatrix as described in Theorem 3.2. The output consists of the matrix $A$, where the parameters of $(AS)[I]$ are stored in the submatrix $A[I]$ (the case $I = N$ corresponds to Theorem 2.3), the ordered set of indices $I$ and, if the cardinal $|I| > 1$, the diagonal matrix $S$.

Once we obtain the DD-parameters of the DD $M$-matrix $AS$ (or $(AS)[I]$), our goal is to compute its inverse with HRA. We can compute it using the subtraction–free implementation of Gauss–Jordan elimination without pivoting described in the proof of Proposition 2.4. For brevity, we do not include this algorithm, which can be easily derived. The inverse can be stored in $A$ using again the submatrix $A[I]$.

**Algorithm 1** nektoDD

**Input:** $A = (a_{ij})(i \neq j)$, $\Delta$ ▷ The N-parameters (2)
**for** $i = 1 : n$ **do**
  $h_i = \sum_{j=1}^{i-1} a_{ij}k_j + \sum_{j=i+1}^{n} a_{ij}$
  $a_{ii} = \Delta_i + h_i$
  $k_i = h_i / a_{ii}$
**end for**
Build I, the set of indices such that $h_i(A) \neq 0$.
**if** $|I| > 1$ **then**
  **for** $i = I$ **do**
    $a_{ii} = \sum_{j=i+1}^{n} a_{ij}\Delta_j / a_{jj}$
    **for** $j = I \backslash \{i\}$ **do**
      $a_{ij} = a_{ij}k_j$
    **end for**
  **end for**
  Build $S$, the $|I| \times |I|$ diagonal matrix whose diagonal entries are $k_j$, $j \in I$.
**else if** $|I| = 1$ **then**
  $a_{II} = 1/a_{II}$
**else**
  $a_{nn} = 1/a_{nn}$
  $I = [n]$
**end if**

If we have the case that $h_i(A) \neq 0$ for $i = 1, \ldots, n$ (analogously, $|I| = n$), it only remains to perform the product $S(AS)^{-1}$, since we obtained $(AS)^{-1}$ applying Gauss–Jordan elimination. Otherwise, we need to build the inverse of $AS$ starting with $((AS)[I])^{-1}$. Algorithm 2 performs this computation. Its input is the matrix $A$ obtained after running Algorithm 1 and the set of indices $I$ (we just need to perform the direct product $S[I]((AS)[I])^{-1}$ before, as done in Algorithm 3).

**Algorithm 2** buildnekinv

**Input:** $A, I$ ▷ $A[I]$ contains $A[I]^{-1}$
Build the set of ordered indices $J := I^c = \{j_1, \ldots, j_k\}$ such that $j_1 > j_2 > \ldots > j_k$.
**for** $i = J$ **do**
  $a_{ii} = 1/a_{ii}$
  $A[I|i] = -A[I](A[I|i]. * a_{ii})$ ▷ .∗ means component–wise multiplication
  $I = I \cup \{i\}$ (ordered)
**end for**

With Algorithm 1, Gauss–Jordan elimination adapted according to Proposition 2.4 and Algorithm 2, we can give a general method to compute the inverse of a Nekrasov $Z$-matrix with positive diagonal with HRA starting with its N-parameters. Algorithm 3 performs all the process.

**Algorithm 3** Computation of the inverse

**Input:** $A = (a_{ij})(i \neq j)$, $\Delta$ ▷ N-parameters
$[A, I, S] = nektoDD(A = (a_{ij})(i \neq j), \Delta)$
**if** $|I| > 1$ **then**
  Compute $B = A[I]^{-1}$ using the adapted Gauss–Jordan elimination
  $A[I] = S * B$
**end if**
$A^{-1} = buildnekinv(A)$

| Condition number | MATLAB | HRA |
|---|---|---|
| 6.7161e+03 | 2.5585e-13 | 4.4536e-15 |
| 7.4296e+04 | 5.4000e-12 | 8.9743e-15 |
| 2.1634e+06 | 3.7380e-11 | 4.4752e-15 |
| 1.2159e+05 | 4.5739e-12 | 2.9456e-15 |
| 6.4136e+03 | 5.1254e-13 | 1.6247e-15 |
| 1.6378e+05 | 1.9921e-11 | 2.2964e-15 |
| 1.9344e+06 | 1.3436e-13 | 3.7407e-14 |
| 2.0715e+05 | 3.0038e-11 | 2.2757e-15 |
| 2.9297e+05 | 7.1062e-12 | 1.5991e-15 |
| 1.7608e+03 | 4.9903e-14 | 1.6191e-15 |

Table 1
Maximum relative errors when computing $A^{-1}$

The numerical experiments have been carried out computing the inverses with Algorithm 3. The errors were estimated comparing the computed approximations with the exact arithmetic solutions obtained with the Symbolic Math Toolbox of MATLAB. In order to illustrate the accuracy of the method presented in this paper, the same problems are also solved using the usual MATLAB commands. In Table 1 we show the maximum relative errors obtained computing the inverse of ten $20 \times 20$ Nekrasov $Z$-matrices generated randomly. The column labeled MATLAB shows the error when the inverse is computed using the MATLAB command *inv*, and the column HRA shows the error when the inverse is obtained from the N-parameters using the procedure with HRA. We observe better results with our method, but the obtained difference is not large since the generated examples are not ill–conditioned.

Besides, since all off–diagonal entries are generated randomly, these first examples do not include any matrix satisfying $h_i(A) = 0$ for some $i = 1, \dots, n$. One way to obtain examples with a greater condition number consists precisely in generating matrices using this additional condition. If we impose $h_j(A) = 0$ whenever $j \in J \subseteq N$, the entries $a_{ij}$ with $j \in J$ and $i > j$ may be arbitrar-

| Condition number | MATLAB | HRA |
|---|---|---|
| 4.8463e+11 | 1.4620e-04 | 8.7273e-16 |
| 1.8512e+11 | 1.5955e-12 | 9.0349e-16 |
| 1.1334e+11 | 9.1933e+14 | 6.3183e-16 |
| 3.6138e+11 | 2.4059e-05 | 8.5640e-16 |
| 8.4356e+10 | 3.1290e+05 | 9.1776e-16 |
| 1.0278e+11 | 3.7958e-02 | 8.7454e-16 |
| 1.0960e+11 | 1.0160e-12 | 7.6520e-16 |
| 1.1049e+12 | 2.2165e-04 | 3.8750e-15 |
| 2.0787e+11 | 2.2370e-05 | 1.5643e-15 |
| 1.8109e+11 | 5.8134e-06 | 1.1298e-15 |

Table 2
Maximum relative errors when computing $A^{-1}$, with the condition $h_i(A) = 0$ for some $i$

ily large. By generating these entries significantly larger than the others, we obtain Nekrasov matrices that are far from being diagonally dominant. For such matrices, the MATLAB command *inv* gives unaccurate inverses and the procedure with HRA introduced in Theorem 3.2 performs as expected. The results can be seen in Table 2, which contains ten examples of $20 \times 20$ Nekrasov $Z$-matrices.

As we mentioned earlier in Remark 2.7, computing the inverse of a Nekrasov $Z$-matrix $A$ with HRA also allows us to solve with HRA the linear system $Ax = b$ with $b \geq 0$ by performing the computation $x = A^{-1}b$. In Table 3, we show the maximum relative error obtained computing the solution in ten cases considering $b = e = (1, \ldots, 1)^T$. The involved matrices are $20 \times 20$ Nekrasov $Z$-matrices with positive diagonal, generated as in the previous case. We show the results obtained computing the solution with the MATLAB command \ and the method with HRA, which computes the inverse from the N-parameters and performs the direct computation $x = A^{-1}b$. We observe the great accuracy of our method, in contrast to MATLAB.

In order to assure the HRA, we required $b \geq 0$. However, we may obtain accurate solutions even without this requirement. For this purpose, we generated ten $20 \times 20$ Nekrasov $Z$-matrices with positive diagonal entries and we solved the system $Ax = b$ with $b = (b_i)_{1 \leq i \leq n}$, $b_i = (-1)^{i+1}$. Table 4 shows the results obtained with the MATLAB command \ and with the procedure with HRA.

| Condition number | MATLAB | HRA |
|---|---|---|
| 1.5337e+11 | 5.5757e-05 | 9.7469e-16 |
| 5.2794e+10 | 9.0848e-07 | 3.4470e-16 |
| 8.7214e+10 | 1.5188e-05 | 6.6034e-16 |
| 1.2596e+11 | 2.3053e-14 | 1.3981e-15 |
| 6.3378e+10 | 2.6565e-14 | 5.2600e-16 |
| 7.1578e+10 | 4.8790e-05 | 3.6567e-16 |
| 4.6526e+10 | 1.3704e-14 | 5.5542e-16 |
| 7.7622e+10 | 5.8318e-06 | 5.1943e-16 |
| 4.1758e+10 | 4.5051e-15 | 5.8087e-16 |
| 7.2351e+10 | 1.2952e-13 | 5.6605e-16 |

Table 3
Maximum relative errors when solving $Ax = b$ with $b = e$

| Condition number | MATLAB | HRA |
|---|---|---|
| 1.8080e+11 | 2.0521e-13 | 3.9922e-16 |
| 1.6297e+12 | 8.7030e-14 | 3.2741e-14 |
| 1.6561e+12 | 6.1643e-04 | 1.5069e-15 |
| 6.7289e+10 | 1.9126e-14 | 2.6656e-15 |
| 1.1951e+11 | 3.0361e-14 | 8.7663e-16 |
| 2.7320e+11 | 7.5251e-15 | 1.1204e-15 |
| 4.6654e+10 | 9.9933e-06 | 7.0215e-16 |
| 1.1328e+11 | 2.3099e-06 | 3.1267e-16 |
| 5.7753e+11 | 1.7024e-13 | 3.5437e-15 |
| 7.4226e+10 | 1.9813e-06 | 8.1572e-16 |

Table 4
Maximum relative errors when solving $Ax = b$ with $b_i = (-1)^{i+1}$

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