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KANTOROVICH-BERNSTEIN α-FRACTAL FUNCTION IN $\mathcal{L}^P$ SPACES

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Abstract. Fractal interpolation functions are fixed points of contraction maps on suitable function spaces. In this paper, we introduce the Kantorovich-Bernstein α-fractal operator in the Lebesgue space $\mathcal{L}^p(I)$, $1 \le p \le \infty$. The main aim of this article is to study the convergence of the sequence of Kantorovich-Bernstein fractal functions towards the original functions in $\mathcal{L}^p(I)$ spaces and Lipschitz spaces without affecting the non-linearity of the fractal functions. In the first part of this paper, we introduce a new family of self-referential fractal $\mathcal{L}^p(I)$ functions from a given function in the same space. The existence of a Schauder basis consisting of self-referential functions in $\mathcal{L}^p$ spaces is proven. Further, we derive the fractal analogues of some $\mathcal{L}^p(I)$ approximation results, for example, the fractal version of the classical Müntz-Jackson theorem. The one-sided approximation by the Bernstein α-fractal function is developed.

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1. Introduction and preliminaries. First we will briefly describe the construction of a fractal interpolation function from an iterated function system (IFS) [4]. This method was first introduced by Barnsley in the reference [3]. Let $\mathbb{N}_i$ denote the first i natural numbers, and $\{(x_i, y_i), i = 1, 2, \dots, N\}$ be a set of

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data points with $x_1 < x_2 < \cdots < x_N$, $I = [x_1, x_N]$, and $I_i = [x_i, x_{i+1}]$. Let $L_i : I \to I_i, i \in \mathbb{N}_{N-1}$ be contractive homeomorphisms such that

$$L_i(x_1) = x_i,\ L_i(x_N) = x_{i+1}. \tag{1}$$

Let $K = I \times \mathbb{R}$, and $N-1$ continuous mappings $F_i : K \to \mathbb{R}$ be satisfying

$$F_i(x_1, y_1) = y_i,\ F_i(x_N, y_N) = y_{i+1},\ |F_i(x,y) - F_i(x,y')| \le c_i|y - y'|, \tag{2}$$

where $(x,y),(x,y') \in K,\ \ 0 \le c_i < 1,\ i \in \mathbb{N}_{N-1}$. Now define $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ as $w_i(x,y) = (L_i(x), F_i(x,y))\ \forall\, i \in \mathbb{N}_{N-1}$. Define

$$\mathcal{C}^*(I) = \{h \in \mathcal{C}(I) :\ h(x_1) = y_1,\ h(x_N) = y_N\}.$$

Then $\mathcal{C}^*(I)$ is a closed subspace of the Banach space $(\mathcal{C}(I), \|.\|_\infty)$. Define the Read-Bajraktarević (RB) operator $T : \mathcal{C}^*(I) \to \mathcal{C}^*(I)$ as

$$(Th)(x) = F_i(L_i^{-1}(x), h(L_i^{-1}(x))),\ x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

T is a contraction map with the contraction factor $c := \max\{c_i,\ i \in \mathbb{N}_{N-1}\}$. The Banach fixed point theorem guarantees that T has a unique fixed point g which interpolates the data $\{(x_i, y_i), i = 1, 2, \ldots, N\}$ [3] and satisfies the self-referential equation

$$g(L_i(x)) = F_i(x, g(x)),\ i \in \mathbb{N}_{N-1}.$$

The previous function is called the fractal interpolation function (FIF) corresponding to the IFS $\mathcal{I} = \{I \times K, w_i(x,y) = (L_i(x), F_i(x,y)), i \in \mathbb{N}_{N-1}\}$. A fractal function with variable scaling is obtained from the following IFS:

$$\{K; (L_i(x), F_i(x,y)),\ i \in \mathbb{N}_{N-1}\},\ \ L_i(x) = a_i x + b_i,\ \ F_i(x,y) = \alpha_i(x)y + q_i(x). \tag{3}$$

Here α_i are continuous functions on I and $\boldsymbol{\alpha} = (\alpha_1(x), \ldots, \alpha_{N-1}(x)) \in (\mathcal{L}^\infty(I))^{N-1}$ satisfying $\|\boldsymbol{\alpha}\|_\infty := \max\{\|\alpha_i\|_\infty;\ i \in \mathbb{N}_{N-1}\} < 1$, where $\|\alpha_i\|_\infty := \sup\{|\alpha_i(x)|, x \in I\}$ and $q_i(x_1) = y_i - \alpha_i(x_1)y_1,\ q_i(x_N) = y_{i+1} - \alpha_i(x_N)y_N$. For different choices of q_i, one can get different kinds of fractal functions ([15], [20]). Navascués [16] observed that a class of continuous functions can be obtained from a given continuous function on a compact set with the definition $q_i(x) = f(L_i(x)) - \alpha_i(x)b(x)$, where b is a continuous function satisfying $b(x_1) = f(x_1), b(x_N) = f(x_N)$. For this choice of q_i and a fixed partition $\Delta := \{x_1, x_2, \ldots, x_N : x_1 < x_2 < \cdots < x_N\}$, the corresponding RB operator has a fixed point $f^{\boldsymbol{\alpha}}$. This $\boldsymbol{\alpha}$-fractal function is the fixed point of

$$(T^{\boldsymbol{\alpha}}_{\Delta,b,f}g)(x) = f(x) + \alpha_i(L_i^{-1}(x))(g - b)(L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}, \tag{4}$$

and hence enjoys the self-referential equation

$$f^{\boldsymbol{\alpha}}(x) = f(x) + \alpha_i(L_i^{-1}(x))(f^{\boldsymbol{\alpha}} - b)(L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}. \tag{5}$$

The fractal dimension of $f^{\boldsymbol{\alpha}}$ depends on the scaling function $\boldsymbol{\alpha}$ [1]. The uniform error bound for the process of approximation $f^{\boldsymbol{\alpha}}$ to f can be obtained [14] from (5) as

$$\|f^{\boldsymbol{\alpha}} - f\|_\infty \le \frac{\|\boldsymbol{\alpha}\|_\infty}{1 - \|\boldsymbol{\alpha}\|_\infty}\|f - b\|_\infty. \tag{6}$$

It is clear from (6) that the convergence of $f^{\boldsymbol{\alpha}}$ to f is guaranteed whenever the scaling function is chosen as $\|\boldsymbol{\alpha}\|_\infty \to 0$ for a fixed partition Δ and a fixed base function b. Till now almost all researchers have studied the convergence of different smooth and non-smooth fractal functions to an original function with the concept of $\|\boldsymbol{\alpha}\|_\infty$ tending to 0 (see for instance [5], [17], [23]). When the original function is non-smooth, we should use a sequence of non-smooth fractal functions for convergence results. To facilitate it, Vijender [19] chose the base function $b(x)$ as the classical Bernstein function in $[x_1, x_N]$ $\forall\, x \in I, n \in \mathbb{N}$ as

$$B_n(f;x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k} (x - x_1)^k (x_N - x)^{n-k} f\left(x_1 + \frac{k(x_N - x_1)}{n}\right). \tag{7}$$

Thus there is a sequence of Bernstein polynomials corresponding to each f in $\mathcal{C}(I)$. It is also known [10] that the sequence of Bernstein polynomials of f converges uniformly to it on $[x_1, x_N]$. Though the convergence is slow, it has the shape preserving properties that help us to transmit the properties of f to $B_n(f)$. It is also clear from (7) that $B_n(f;x_1) = f(x_1)$, $B_n(f;x_N) = f(x_N)$ $\forall\, n \geq 1$ so that a Bernstein polynomial to f interpolates it at both endpoints of $[x_1, x_N]$. Thus for every $n \in \mathbb{N}$, the corresponding fractal function $f_n^{\boldsymbol{\alpha}}$ is called the n-th Bernstein $\boldsymbol{\alpha}$-fractal function of $f \in \mathcal{C}(I)$, and it is defined implicitly as

$$f_n^{\boldsymbol{\alpha}}(x) = f(x) + (f_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) - B_n(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x)) \;\forall\, x \in I_i,\ n \in \mathbb{N},\ i \in N_{N-1}. \tag{8}$$

It is noted that for a given $f \in \mathcal{C}(I)$, there exists a sequence $\{f_n^{\boldsymbol{\alpha}}(x)\}_{n=1}^{\infty}$ of Bernstein $\boldsymbol{\alpha}$-fractal functions. From (8), it is easy to deduce that

$$\|f_n^{\boldsymbol{\alpha}} - f\|_\infty \leq \frac{\|\boldsymbol{\alpha}\|_\infty}{1 - \|\boldsymbol{\alpha}\|_\infty} \|f - B_n(f)\|_\infty, \tag{9}$$

which gives the convergence of Bernstein $\boldsymbol{\alpha}$-fractal function towards f as $n \to \infty$. When we approximate an original function which is irregular in nature, the current approach is more appropriate over the existing methods. The Bernstein operator $B_n : (\mathcal{C}[x_1, x_N], \|.\|_\infty) \to (\mathcal{C}[x_1, x_N], \|.\|_\infty)$ is linear, and $\|B_n\|_{(\mathcal{C}[x_1,x_N],\|.\|_\infty)} = 1$, but $B_n : (\mathcal{L}^p, \|.\|_p) \to (\mathcal{L}^p, \|.\|_p)$ is not bounded on $(\mathcal{L}^p[x_1, x_N], \|.\|_p)$. In order to achieve the approximation in $\mathcal{L}^p$-norm, Kantorovich [12] modified the Bernstein polynomial as

$$K_n(f;x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k} (x - x_1)^k (x_N - x)^{n-k} (n+1) C_k,$$

where $C_k = \int_{x_1 + \frac{k(x_N - x_1)}{n+1}}^{x_1 + \frac{(k+1)(x_N - x_1)}{n+1}} f(t)dt$. Here K_n maps each space $\mathcal{L}^p$, $1 \leq p \leq \infty$ into itself with norm one. Also $\|f - K_n(f)\|_p \to 0$ as $n \to \infty$ for each $f \in \mathcal{L}^p$ [13]. Indeed for continuous f, we have uniform convergence $K_n(f) \to f$. Navascués, Chand and Viswanathan [22] extended the notion of $\boldsymbol{\alpha}$-fractal function to $\mathcal{L}^p$ spaces and derived some approximation properties while approaching the scaling function tending to 0. In this article, we introduce a new construction Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal function in $\mathcal{L}^p(I)$ and obtain some approximation properties without affecting the scaling functions. The rest of the paper is organized as follows: In Section 2, we construct the Lipschitz Bernstein $\boldsymbol{\alpha}$-fractal function from a given Lipschitz function. We introduce the construction of the Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal functions and deduce some properties of the corresponding multi-fractal operators in Section 3. Later we derive the fractal Müntz polynomials and prove the density theorems for the case of continuous and p-integrable functions without assuming conditions on the scaling functions in Section 4. Finally, we obtain an one-sided approximation using the Bernstein $\boldsymbol{\alpha}$-fractal functions.

2. Bernstein $\boldsymbol{\alpha}$-fractal function in Lipschitz spaces. The fractal dimension of $f^{\boldsymbol{\alpha}}$ depends on the scaling function $\boldsymbol{\alpha}$. Nasim et al.[1] computed the box dimension of the α-fractal functions by using relevant conditions on the scaling function, the original function f, and the base function b. The following proposition furnishes the details.

PROPOSITION 2.1. *Let $f \in \mathcal{C}(I)$ and $b : I \mapsto \mathbb{R}$ be the Lipschitz functions satisfying $b(x_1) = f(x_1), b(x_N) = f(x_N)$. Suppose $\Delta = \{x_1, x_2, \dots, x_N\}$ is a partition of I satisfying $x_1 < x_2 < \cdots < x_N$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}) \in (-1, 1)^{N-1}$. Also assume that the interpolation points are not colinear and $\gamma = \sum_{i=1}^{N-1} |\alpha_i|$, then the graph of f^α has the box dimension*

$$D = \begin{cases} 1 + \log_N \gamma, & \text{if } \gamma > 1, \\ 1, & \text{otherwise.} \end{cases}$$

For $0 < d \leq 1$, the Lipschitz space is defined as

$$\mathrm{Lip}d = \left\{ f : I \to \mathbb{R} : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^d} < \infty \right\}.$$

Define $\|f\|_{0,d} = \max\{\|f\|_\infty, |f|_d\}$, where

$$|f|_d = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^d}, x, y \in I, x \neq y \right\}.$$

Then $(\mathrm{Lip}d, \|\cdot\|_{0,d})$ is a complete metric space. In [7] Brown et al. showed that for a given $f \in \mathrm{Lip}d$, its Bernstein polynomial $B_n(f) \in \mathrm{Lip}d$ for each $n \geq 1$. The next theorem demonstrates that for a given Lipschitz function, we can construct a Lipschitz Bernstein $\boldsymbol{\alpha}$-fractal function. Since the fractal dimension is a quantifier of the irregularity of the approximated function, and the base function plays an important role to find the box dimension, the following theorem is useful.

THEOREM 2.2. *Let $f \in \mathrm{Lip}d$. Suppose $\Delta = \{x_1, x_2, \dots, x_N\}$ is a partition of I satisfying $x_1 < x_2 < \cdots < x_N$, $I_i := [x_i, x_{i+1}]$. Let $L_i : I \to I_i$ be given by $L_i(x) = a_i x + b_i$. Choose the base function as $b(x) = B_n(f; x)$ and scaling function*

$\alpha_i \in \mathrm{Lip}d, i \in \mathbb{N}_{N-1}$. *Define the RB-operator* $T : (\mathrm{Lip}d, \|\cdot\|_{0,d}) \to (\mathrm{Lip}d, \|\cdot\|_{0,d})$ *by*

$$Tg(x) = f(x) + \boldsymbol{\alpha}_i(L_i^{-1}(x))(g - B_n f)(L_i^{-1}(x)).$$

Further, if the scaling functions satisfy the condition $\max\left\{\frac{\|\alpha_i\|_\infty}{a_i^d} : i \in \mathbb{N}_{N-1}\right\} < 1$, *then the operator* T *has a unique fixed point* $f^{\boldsymbol{\alpha}} \in \mathrm{Lip}d$.

Proof. Let $g \in \mathrm{Lip}d$. Now for each $n \in \mathbb{N}$,

$$\begin{aligned}
|Tg|_d &= \sup_{x,y\in I, x\neq y} \frac{|Tg(x) - Tg(y)|}{|x-y|^d} \\
&= \sup_{x,y\in I_i, x\neq y} \frac{|f(x) - f(y) + \alpha_i(g - B_n(f))\circ(L_i^{-1}(x)) - \alpha_i(g - B_n(f))\circ(L_i^{-1}(y))|}{|x-y|^d} \\
&\leq \sup_{x,y\in I_i, x\neq y} \left\{\frac{|f(x)-f(y)|}{|x-y|^d}\right\} + \max_{i\in\mathbb{N}_{N-1}}(\|\alpha_i\|_\infty) \sup_{x,y\in I_i, x\neq y}\Big[\frac{|g(L_i^{-1}(x)) - g(L_i^{-1}(y))|}{|x-y|^d} \\
&\qquad + \frac{|B_n(f; L_i^{-1}(x)) - B_n(f; L_i^{-1}(y))|}{|x-y|^d}\Big] \\
&= |f|_d + \max_{i\in\mathbb{N}_{N-1}}(\|\alpha_i\|_\infty) \sup_{x,y\in I_i, x\neq y}\left[\frac{|g(L_i^{-1}(x)) - g(L_i^{-1}(y))|}{a_i^d|L_i^{-1}(x) - L_i^{-1}(y)|^d}\right. \\
&\qquad \left. + \frac{|B_n(f; L_i^{-1}(x)) - B_n(f; L_i^{-1}(y))|}{a_i^d|L_i^{-1}(x) - L_i^{-1}(y)|^d}\right] \\
&= |f|_d + \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right) \sup_{x^*,y^*\in I, x^*\neq y^*}\left[\frac{|g(x^*) - g(y^*)|}{|x^*-y^*|^d} + \frac{|B_n(f;x^*) - B_n(f;y^*)|}{|x^*-y^*|^d}\right] \\
&= |f|_d + \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right)(|g|_d + |B_n(f)|_d).
\end{aligned}$$

Since $f \in \mathrm{Lip}d$, $B_n(f) \in \mathrm{Lip}d$ [7], we obtain $|Tg|_d < \infty$ from the above inequality, and hence $Tg \in \mathrm{Lip}d$. Also for $g, g^* \in \mathrm{Lip}d$,

$$\|Tg - Tg^*\|_\infty \leq \max_{i\in\mathbb{N}_{N-1}}(\|\alpha_i\|_\infty)\|g - g^*\|_\infty.$$

Using similar steps in the estimation of $|Tg|_d$, we obtain

$$|Tg - Tg^*|_d \leq \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right)|g - g^*|_d.$$

Combining the above two inequalities, we get

$$\begin{aligned}
\|Tg - Tg^*\|_{0,d} &= \max\{\|Tg - Tg^*\|_\infty, |Tg - Tg^*|_d\} \\
&\leq \max\left\{\max_{i\in\mathbb{N}_{N-1}}(\|\alpha_i\|_\infty)\|g - g^*\|_\infty, \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right)|g - g^*|_d\right\} \\
&= \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right)\max\{\|g - g^*\|_\infty, |g - g^*|_d\} \\
&= \max_{i\in\mathbb{N}_{N-1}}\left(\frac{\|\alpha_i\|_\infty}{a_i^d}\right)\|g - g^*\|_{0,d}.
\end{aligned}$$

Thus, under the given assumption on the scaling functions, T is a contraction. $\square$

THEOREM 2.3. *Let $f \in$ Lip$d$. Choose the scaling function $\boldsymbol{\alpha}$ such that $\max\left\{\frac{\|\alpha_i\|_\infty}{a_i^d} : i \in \mathbb{N}_{N-1}\right\} < 1$. Then the sequence of Bernstein $\boldsymbol{\alpha}$-fractal functions $\{f_n^\alpha\}_{n=1}^\infty$ converges to the original function f when $n \to \infty$.*

Proof. From the definition of the Bernstein $\boldsymbol{\alpha}$-fractal function, $f_n^{\boldsymbol{\alpha}}$ obeys the following self-referential equation:

$$f_n^{\boldsymbol{\alpha}}(x) = f(x) + (f_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) - B_n(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x))\ \forall x \in I_i,\ n \in \mathbb{N},\ i \in N_{N-1}.$$

From the above equation and using the Lipschitz norm, it can be easily deduced that

$$\|f_n^{\boldsymbol{\alpha}} - f\|_{0,d} \le \frac{B}{1-B}\|f - B_n(f)\|_{0,d}, \text{ where } B = \max\left\{\frac{\|\alpha_i\|_\infty}{a_i^d} : i \in \mathbb{N}_{N-1}\right\} < 1. \tag{10}$$

It is known from [8] that

$$\|f - B_n(f)\|_{0,d} \to 0 \text{ as } n \to \infty. \tag{11}$$

Thus the proof follows using (10) and (11). □

3. Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal function.

Using the base function as $K_n(f;x)$, we define the Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal function as

$$f_n^{\boldsymbol{\alpha}}(x) = f(x) + (f_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) - K_n(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x))\ \forall\, x \in I_i,\ n \in \mathbb{N},\ i \in N_{N-1}. \tag{12}$$

THEOREM 3.1. *Let $f \in \mathcal{L}^p(I)$, $1 \le p \le \infty$. Suppose $\Delta = \{x_1, x_2, \dots, x_N\}$ is a partition of I satisfying $x_1 < x_2 < \cdots < x_N$, $I_i := [x_i, x_{i+1})$, $i \in \mathbb{N}_{N-2}$, $I_{N-1} = [x_{N-1}, x_N]$. Let $L_i(x) = a_i x + b_i$ be satisfying $L_i(x_1) = x_i$, $L_i(x_N^-) = x_{i+1}$ for $i \in \mathbb{N}_{N-1}$, and $L_{N-1}(x_1) = x_{N-1}$, $L_{N-1}(x_N) = x_N$. Choose $\alpha_i \in \mathcal{L}^\infty(I)\,\forall\, i \in \mathbb{N}_{N-1}$, and $b(x) = K_n(f;x) \in \mathcal{L}^p(I)$. Then the RB-operator given in (4) defines a self-map on $\mathcal{L}^p(I)$. Further, if the scaling function $\boldsymbol{\alpha}$ satisfies*

$$\begin{cases} \left[\displaystyle\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < 1 & \text{if } 1 \le p < \infty, \\ \|\boldsymbol{\alpha}\|_\infty < 1 & \text{if } p = \infty, \end{cases}$$

then T is a contraction on $\mathcal{L}^p(I)$. Further, the fixed point $f_n^{\boldsymbol{\alpha}} \in \mathcal{L}^p(I)$ of T satisfies the self-referential equation (12).

Proof. The proof can be obtained using arguments similar to those used in [22]. □

PROPOSITION 3.2. *Let $f \in \mathcal{L}^p(I)$. If $f_n^{\boldsymbol{\alpha}}$ is constructed according to Theorem 3.1, then we have the following error estimation:*

$$\|f_n^{\boldsymbol{\alpha}} - f\|_p < \begin{cases} \left[\sum\limits_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} \|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_p, & \text{for } 1 \le p < \infty, \\ \|\boldsymbol{\alpha}\|_\infty \|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_\infty, & \text{for } p = \infty. \end{cases}$$

Proof. From the functional equation (12), we have

$$\begin{aligned} \|f_n^{\boldsymbol{\alpha}} - f\|_p^p &= \int_I |(f_n^{\boldsymbol{\alpha}} - f)(x)|^p dx, \quad 1 \le p < \infty, \\ &= \sum_{i \in \mathbb{N}_{N-1}} \int_{I_i} \left|(f_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) - K_n(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x))\right|^p dx \\ &= \sum_{i \in \mathbb{N}_{N-1}} a_i \int_I |(f_n^{\boldsymbol{\alpha}}(x^*) - K_n(f; x^*))\alpha_i(x^*)|^p \, dx^* \\ &\le \sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \int_I |(f_n^{\boldsymbol{\alpha}}(x^*) - K_n(f; x^*))|^p \, dx^* \\ &= \sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_p^p. \end{aligned}$$

For $p = \infty$, the proof follows from straightforward computation. □

PROPOSITION 3.3. *For $f \in \mathcal{L}^p(I)$, $1 \le p \le \infty$, we have the following estimate:*

$$\|f_n^{\boldsymbol{\alpha}} - f\|_p < \begin{cases} \dfrac{\left[\sum\limits_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}}{1 - \left[\sum\limits_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}} \|f - K_n(f)\|_p, & \text{for } 1 \le p < \infty, \\ \dfrac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - K_n(f)\|_\infty, & \text{for } p = \infty. \end{cases}$$

Proof. Following the proof of Proposition 3.2, we obtain

$$\|f_n^{\boldsymbol{\alpha}} - f\|_p^p \le \sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_p^p,$$

which gives

$$\begin{aligned} \|f_n^{\boldsymbol{\alpha}} - f\|_p &\le \left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} \|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_p \\ &\le \left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} [\|f_n^{\boldsymbol{\alpha}} - f\|_p + \|f - K_n(f)\|_p]. \end{aligned}$$

The proof follows from further simplifications. □

THEOREM 3.4. *Let $n \in \mathbb{N}$ and $1 \le p \le \infty$. For each scale function $\boldsymbol{\alpha}$, the self-referential Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal operator $\mathcal{F}_n^{\boldsymbol{\alpha}} : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ defined by $\mathcal{F}_n^{\boldsymbol{\alpha}}(f) = f_n^{\boldsymbol{\alpha}}$ is a bounded linear operator, and $\mathcal{F}_n^{\boldsymbol{\alpha}}$ reduces to identity for $\boldsymbol{\alpha} = 0$.*

Proof. Let $f, g \in \mathcal{L}^p(I)$ and λ_1, λ_2 be real scalars. The functional equations for the corresponding Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal functions are given by

$$f_n^{\boldsymbol{\alpha}}(x) = \alpha_i(L_i^{-1}(x))(f_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) + f(x) - K_n(f; L_i^{-1}(x))),$$

$$g_n^{\boldsymbol{\alpha}}(x) = \alpha_i(L_i^{-1}(x))(g_n^{\boldsymbol{\alpha}}(L_i^{-1}(x)) + g(x) - K_n(g; L_i^{-1}(x)))\ \forall x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

Therefore,

$$\begin{aligned}(\lambda_1 f_n^{\boldsymbol{\alpha}} + \lambda_2 g_n^{\boldsymbol{\alpha}})(x) = (\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))& \\ \big((\lambda_1 f_n^{\boldsymbol{\alpha}} + \lambda_2 g_n^{\boldsymbol{\alpha}})(L_i^{-1}(x)) - K_n(\lambda_1 f + \lambda_2 g; L_i^{-1}(x))\big)&\end{aligned}$$

from which we obtain that $\lambda_1 f_n^{\boldsymbol{\alpha}} + \lambda_2 g_n^{\boldsymbol{\alpha}}$ is a fixed point of the operator

$$(Th)(x) = (\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))(h - K_n(\lambda_1 f + \lambda_2 g; L_i^{-1}(x)).$$

Thus, by the uniqueness of fixed point, $\mathcal{F}_n^{\boldsymbol{\alpha}}(\lambda_1 f + \lambda_2 g) = \lambda_1 f_n^{\boldsymbol{\alpha}} + \lambda_2 g_n^{\boldsymbol{\alpha}} = \lambda_1 \mathcal{F}_n^{\boldsymbol{\alpha}}(f) + \lambda_2 \mathcal{F}_n^{\boldsymbol{\alpha}}(g)$.
Consider

$$L = \begin{cases} \left[\sum\limits_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \|\boldsymbol{\alpha}\|_\infty, & \text{if } p = \infty. \end{cases} \tag{13}$$

Now

$$\begin{aligned}\|\mathcal{F}_n^{\boldsymbol{\alpha}}(f)\|_p &= \|f_n^{\boldsymbol{\alpha}}\|_p \\ &\le \|f_n^{\boldsymbol{\alpha}} - f\|_p + \|f\|_p \\ &\le \frac{L}{1-L}\|f - K_n(f)\|_p + \|f\|_p \\ &\le \frac{L}{1-L}\|Id - K_n\|\|f\|_p + \|f\|_p. \end{aligned} \tag{14}$$

The third step in the previous computation uses the Proposition 3.3. Since $\|Id - K_n\|$ is bounded in $\mathcal{L}^p$, there exists ξ such that $\|Id - K_n\| < \xi$ for all $n \in \mathbb{N}$. Then from (14) we get

$$\|\mathcal{F}_n^{\boldsymbol{\alpha}}\| \le \left(1 + \frac{L}{1-L}\xi\right).$$

Consequently, $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is a bounded operator for each $n \in \mathbb{N}$. □

Let us introduce the following terminologies that are required hereafter.

DEFINITION 3.5. A continuous map $f : X \to Y$ between two topological spaces X and Y is said to be a homeomorphism (topological isomorphism) if it has a continuous inverse.

DEFINITION 3.6. If $f : X \to X$ is a continuous map on the topological space X having a continuous inverse, then f is said to be a topological automorphism.

THEOREM 3.7. *Let $n \in \mathbb{N}$. Suppose the scaling function $\boldsymbol{\alpha}$ satisfies*

$$\left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < \min\{1, \|K_n\|^{-1}\}, \text{ if } 1 \le p < \infty,$$

$$\|\boldsymbol{\alpha}\|_\infty < \min\{1, \|K_n\|^{-1}\}, \text{ if } p = \infty.$$

Then the corresponding fractal operator is bounded below. In particular, $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is injective and has a closed range. Also $\mathcal{F}_n^{\boldsymbol{\alpha}} : \mathcal{L}^p(I) \to \mathcal{F}_n^{\alpha}(\mathcal{L}^p(I))$ is a topological isomorphism for each $n \in \mathbb{N}$.

Proof. For $1 \le p \le \infty$, from the reverse triangle inequality and the Proposition 3.2, we obtain

$$\begin{aligned}
\|f\|_p - \|f_n^{\boldsymbol{\alpha}}\|_p &\le \|f - f_n^{\boldsymbol{\alpha}}\|_p \\
&\le L\|f_n^{\boldsymbol{\alpha}} - K_n(f)\|_p \\
&\le L\|f_n^{\boldsymbol{\alpha}}\|_p + L\|K_n\|\|f\|_p, \\
\Rightarrow \|f\|_p &\le \frac{1+L}{1 - L\|K_n\|}\|f_n^{\alpha}\|_p. \qquad (15)
\end{aligned}$$

Since $L < \|K_n\|^{-1}$, the operator $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is bounded below and so injective. Now to prove that $\mathcal{F}_n^{\boldsymbol{\alpha}}$ has a closed range, let $f_{n,m}^{\boldsymbol{\alpha}}$ be a sequence in $\mathcal{F}_n^{\boldsymbol{\alpha}}(\mathcal{L}^p(I))$ such that $f_{n,m}^{\boldsymbol{\alpha}} \to h$, i.e., $f_{n,m}^{\boldsymbol{\alpha}}$ is a Cauchy sequence in $\mathcal{F}_n^{\boldsymbol{\alpha}}(\mathcal{L}^p(I))$. Now

$$\|f_m - f_r\|_p \le \frac{1+L}{1 - L\|K_n\|}\|f_{m,n}^{\boldsymbol{\alpha}} - f_{r,n}^{\boldsymbol{\alpha}}\|_p$$

which shows that $\{f_m\}$ is a Cauchy sequence in $\mathcal{L}^p(I)$. Since $\mathcal{L}^p(I)$ is a complete metric space, there exists $f \in \mathcal{L}^p(I)$ such that $f_m \to f$. Using the continuity of $\mathcal{F}_n^{\boldsymbol{\alpha}}$, we have $h = \mathcal{F}_n^{\boldsymbol{\alpha}}(f) = f_n^{\boldsymbol{\alpha}}$. Thus using the bounded inverse theorem, we found that the inverse of the map $\mathcal{F}_n^{\boldsymbol{\alpha}} : \mathcal{L}^p(I) \to \mathcal{F}_n^{\boldsymbol{\alpha}}(\mathcal{L}^p(I))$ is a bounded linear operator for each $n \in \mathbb{N}$. □

THEOREM 3.8. *The fractal operator $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is a topological automorphism on $\mathcal{L}^p(I)$ if the variable scaling function $\boldsymbol{\alpha}$ obeys*

$$\left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < (1 + \|Id - K_n\|)^{-1}, \text{ if } 1 \le p < \infty,$$

$$\|\boldsymbol{\alpha}\|_\infty < (1 + \|Id - K_n\|)^{-1}, \text{ if } p = \infty.$$

Also

$$\frac{1 - L\|K_n\|}{1+L}\|f\|_p \le \|\mathcal{F}_n^{\boldsymbol{\alpha}}\| \le 1 + \frac{L}{1-L}\|Id - K_n\|\|f\|_p,$$

where L is defined in (13).

Proof. From (14), we obtain the inequality $\|Id - \mathcal{F}_n^{\boldsymbol{\alpha}}\| \leq \frac{L}{1-L}\|Id - K_n\|$. Since $L(1+\|Id - K_n\|) < 1$, we get $\|Id - \mathcal{F}_n^{\boldsymbol{\alpha}}\| < 1$. Thus $\sum\limits_{j=0}^{\infty}(Id - \mathcal{F}_n^{\boldsymbol{\alpha}})^j$ is convergent in the operator norm, and $\mathcal{F}_n^{\boldsymbol{\alpha}} = Id - (Id - \mathcal{F}_n^{\boldsymbol{\alpha}})$ is invertible. The bounds follow from the Proposition 3.3. □

The existence of a Schauder basis of $\mathcal{L}^p(I)$ is useful for finding the best approximation of an element in $\mathcal{L}^p(I)$ from a finite dimensional subspace of $\mathcal{L}^p(I)$. Also the Schauder basis of $\mathcal{L}^p(I)$ is helpful to approximate the solution of the first order non-linear mixed Fredholm-Volterra integro-differential equations [6]. In some applications, it is required to maintain the global structure involved in a given problem, and self-referentiality may be beneficial. Hence, it is worth to find a Schauder basis of $\mathcal{L}^p(I)$ consisting of fractal functions. First we recall the definition of a Schauder basis of a Banach space:

Definition 3.9. A sequence $\{x_n\}$ of a Banach space X is a Schauder basis if for every $x \in X$, there exists a unique representation of x as $x = \sum\limits_{m=1}^{\infty} c_m x_m$, where $\{c_m\}$ is a sequence of scalars.

Example 3.10. The Haar system is a Schauder basis of $\mathcal{L}^p(I)$ for $1 \leq p < \infty$.

The following theorem guarantees the existence of a Schauder basis of fractal functions in $\mathcal{L}^p(I)$. Without loss of generality assume that $I = [0, 1]$.

Theorem 3.11. *For $1 \leq p \leq \infty$, the space $\mathcal{L}^p(I)$ admits a Schauder basis consisting of self-referential functions.*

Proof. Let $\{f_m\}$ be a Schauder basis of $\mathcal{L}^p(I)$ with the associated coefficient functionals $\{\lambda_m\}$. Suppose that the scaling function $\boldsymbol{\alpha}$ is chosen according to the Theorem 3.8. Then, $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is a topological isomorphism for each $n \in \mathbb{N}$. Let $f \in \mathcal{L}^p(I)$. Then $(\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f) \in \mathcal{L}^p(I)$ and

$$(\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f) = \sum_{m=1}^{\infty} \lambda_m \left((\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f)\right) f_m.$$

Since $\mathcal{F}_n^{\boldsymbol{\alpha}}$ is a linear and continuous map,

$$f = \sum_{m=1}^{\infty} \lambda_m \left((\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f)\right) f_{m,n}^{\boldsymbol{\alpha}}.$$

To prove the uniqueness of the representation, let us assume another representation of f as

$$f = \sum_{m=1}^{\infty} \gamma_m f_{m,n}^{\boldsymbol{\alpha}}.$$

Using the continuity of $(\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}$, $(\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f) = \sum_{m=1}^{\infty} \gamma_m f_m$ and hence $\gamma_m = \lambda_m\left((\mathcal{F}_n^{\boldsymbol{\alpha}})^{-1}(f)\right)$, $m \in \mathbb{N}$. Thus, for fixed n, $\{f_{m,n}^{\boldsymbol{\alpha}}\}$ is a Schauder basis of $\mathcal{L}^p(I)$ consisting of self-referential functions. $\square$

4. Fractal version of the Müntz-Jackson theorem. Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct, non-negative real numbers with $\lambda_0 = 0$. The non-negative valued functions x^{λ_i} are well defined on $[0, \infty]$. The collection $\Lambda_m = \{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_m}\}$ is called a finite Müntz system. The linear space $M_m(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_m}\}$ is called a Müntz space. Let $I = [0,1]$ and $\Delta := \{x_1, \dots, x_N\}$ be a partition of I satisfying $0 = x_1 < \cdots < x_N = 1$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{N-1}) \in (\mathcal{L}^\infty(I))^{N-1}$. As mentioned earlier $K_n : \mathcal{L}^p(I) \to \mathcal{L}^p(I)$ is a bounded linear map. Also, the Müntz monomials $x^{\lambda_i} \in \mathcal{L}^p(I)$ even if $\lambda_i > -\frac{1}{p}$. Using the construction described in the introductory section, we can define the fractal analogue as $(x_n^{\boldsymbol{\lambda}_i})^{\boldsymbol{\alpha}} = \mathcal{F}_n^{\boldsymbol{\alpha}}(x^{\lambda_i})$. In this case, a Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal Müntz polynomial is a linear combination of the function $(x_n^{\lambda_i})^{\boldsymbol{\alpha}}, n \in \mathbb{N}$, where $\lambda_i \in \Lambda$.

THEOREM 4.1. *Let $1 \le p \le \infty$ and $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers such that $\lambda_i > -\frac{1}{p}$ for each i. Suppose $\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{(\lambda_i + \frac{1}{p})^2 + 1} = \infty$. Then $A = \operatorname{span}\{(x_n^{\lambda_i})^{\boldsymbol{\alpha}} : i, n \in \mathbb{N}\}$ is dense in $\mathcal{L}^p(I)$.*

Proof. Let $f \in \mathcal{L}^p(I)$ and $\epsilon > 0$ be arbitrary. Under the stated conditions on λ_i, it follows from the full-Müntz theorem [11] that there exists a Müntz polynomial $q_m \in M_m(\Lambda)$ and a natural number N_1 such that

$$\|f - q_m\|_p < \frac{\epsilon}{2}\, \forall\, m \ge N_1. \tag{16}$$

With the aid of q_m, let $q_{m,n}^{\boldsymbol{\alpha}}$ be the Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal Müntz polynomial determined by the IFS $\{[0,1] \times \mathbb{R}; (L_i(x), F_{m,n,i}(x,y)),\ i \in \mathbb{N}_{N-1}\}$, where $F_{m,n,i}(x,y) = \alpha_i(x)y + q_m(L_i(x)) - \alpha_i(x)K_n(q_m; x)$, $i \in \mathbb{N}_{N-1}$. Now, $q_n^{\boldsymbol{\alpha}}$ satisfies

$$q_{m,n}^{\boldsymbol{\alpha}}(x) = \alpha_i(x) q_{m,n}^{\boldsymbol{\alpha}}(L_i^{-1}(x)) + q_m(x) - \alpha_i(x) K_n(q_m; L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}, n \in \mathbb{N}. \tag{17}$$

Using Proposition 3.3, it is easy to verify that $q_{n,m}^{\boldsymbol{\alpha}}(x)$ satisfies the following inequality

$$\|q_{m,n}^{\boldsymbol{\alpha}} - q_m\|_p \le \frac{L}{1-L} \|q_m - K_n(q_m)\|_p. \tag{18}$$

Choose $C = \frac{L}{1-L} > 0$. From the convergence result of the Kantorovich-Bernstein polynomials [13], it follows that for each $m \in \mathbb{N}$, there exists a sequence $\{K_n(q_m; x)\}_{n=1}^{\infty}$ of Kantorovich-Bernstein polynomials of q_m that converges to it with respect to the p-norm. Therefore, for a given $\epsilon > 0$, there exists a natural number N_2 such that

$$\|q_m - K_n(q_m)\|_p < \frac{\epsilon}{2C}\, \forall\, n \ge N_2. \tag{19}$$

Thus, using (19) in (18), we obtain

$$\|q^{\boldsymbol{\alpha}}_{m,n} - q_m\|_p \leq \frac{\epsilon}{2} \, \forall \, n \geq N_2. \tag{20}$$

Choose $N = \max\{N_1, N_2\}$. Using (16) and (20) for $n \geq N$, we obtain

$$\|f - q^{\boldsymbol{\alpha}}_{m,n}\|_p \leq \|f - q_m\|_p + \|q_m - q^{\boldsymbol{\alpha}}_{m,n}\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the density theorem. □

Next, we will give the Jackson type rate of convergence. Here we want to find the possible degree of approximation by fractal polynomials in an arbitrary space $\Lambda^{\boldsymbol{\alpha}}_{m,n} = \mathcal{F}^{\boldsymbol{\alpha}}_n(\Lambda_m) = \{1, (x_n^{\lambda_1})^{\boldsymbol{\alpha}}, \dots, (x_n^{\lambda_m})^{\boldsymbol{\alpha}}\}$. For our results, we need the $\mathcal{L}^p$-modulus of continuity:

DEFINITION 4.2. Let $f \in \mathcal{L}^p([0,1])$, $1 \leq p \leq \infty$. The $\mathcal{L}^p$-modulus of continuity is defined by

$$\omega_p(f, \delta) = \sup_{|t| \leq \delta} \|f(x+t) - f(x)\|_p.$$

THEOREM 4.3. *Suppose $\{\lambda_i\}_{i=1}^n$ satisfies the growth condition $\lambda_k \geq Sk,\ S > 2$. Then for all $f \in \mathcal{C}([0,1])$, there exists $Q^{\boldsymbol{\alpha}}_n(x) \in \Lambda^{\boldsymbol{\alpha}}_{m,n}$ such that*

$$\|f - Q^{\boldsymbol{\alpha}}_n\|_\infty \leq \frac{A\omega_\infty(f;\epsilon)}{S-2} + \frac{\|\boldsymbol{\alpha}\|_\infty}{1 - \|\boldsymbol{\alpha}\|_\infty}\|Q - B_n(Q)\|_\infty,$$

where $\epsilon = \exp(-2 \sum\limits_{i=1}^{n} \frac{1}{\lambda_i})$.

Proof. Under the stated hypotheses, the reference [2] asserts the existence of a polynomial $Q(x) \in \Lambda_m$ such that

$$\|f - Q\|_\infty \leq \frac{A\omega_\infty(f;\epsilon)}{S-2}. \tag{21}$$

Let us consider the Bernstein $\boldsymbol{\alpha}$-fractal function corresponding to this Müntz polynomial $Q(x)$ as $Q^{\boldsymbol{\alpha}}_n = \mathcal{F}^{\boldsymbol{\alpha}}_n(Q)$. Using (9), we obtain

$$\|Q^{\boldsymbol{\alpha}}_n - Q\|_\infty \leq \frac{\|\boldsymbol{\alpha}\|_\infty}{1 - \|\boldsymbol{\alpha}\|_\infty}\|Q - B_n(Q)\|_\infty. \tag{22}$$

Finally, combining (21) and (22), we have

$$\|f - Q^{\boldsymbol{\alpha}}_n\|_\infty \leq \frac{A\omega_\infty(f;\epsilon)}{S-2} + \frac{\|\boldsymbol{\alpha}\|_\infty}{1 - \|\boldsymbol{\alpha}\|_\infty}\|Q - B_n(Q)\|_\infty.$$

This completes the proof. □

The $\mathcal{L}^p$ analogue of the Müntz-Jackson theorem for the Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal Müntz polynomial is given in the following result:

THEOREM 4.4. *Let $2 \le p \le \infty$ and $\Lambda^{\boldsymbol{\alpha}}_{m,n}$ be defined as above; $\lambda_k \ge 2k$. Then for every $f \in \mathcal{L}^p([0,1]), n \in \mathbb{N}$, there exists a Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal Müntz polynomial $Q^{\boldsymbol{\alpha}}_n \in \Lambda^{\boldsymbol{\alpha}}_{m,n}$ such that*

$$\|f - Q^{\boldsymbol{\alpha}}_n\|_p \le A\omega_p(f,\epsilon) + \frac{L}{1-L}\|Q - K_n(Q)\|_p,$$

where L is defined in (13).

Proof. Under the stated hypothesis, it is known from [2] that there exists a Müntz polynomial $Q \in \Lambda_m$ such that

$$\|f - Q\|_p \le A\omega_p(f,\epsilon). \tag{23}$$

With the help of this Müntz polynomial Q and for a given partition of $[0,1]$, construct the Kantorovich-Bernstein $\boldsymbol{\alpha}$-fractal Müntz polynomial $Q^{\boldsymbol{\alpha}}_n$ using (12). Now $Q^{\boldsymbol{\alpha}}_n \in \Lambda^{\boldsymbol{\alpha}}_{m,n}$ and from the Proposition 3.3,

$$\|Q^{\boldsymbol{\alpha}}_n - Q\|_p \le \frac{L}{1-L}\|K_n(Q) - Q\|_p. \tag{24}$$

The proof follows immediately using (23), (24) in

$$\|f - Q^{\boldsymbol{\alpha}}_n\|_p \le \|f - Q_n\|_p + \|Q_n - Q^{\boldsymbol{\alpha}}_n\|_p. \qquad \square$$

5. Application. In this section, we will prove the existence of a fractal one-sided best approximation. The properties of the Bernstein $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}}_n$ depend on the scaling function and the base function. Several shape preserving properties of the fractal functions have been studied in ([9], [21]) for the choice of arbitrary b and $\boldsymbol{\alpha}$ satisfying certain conditions. In our case, considering $b(x) = B_n(f;x)$ and using the properties of $B_n(f)$, we get the following result.

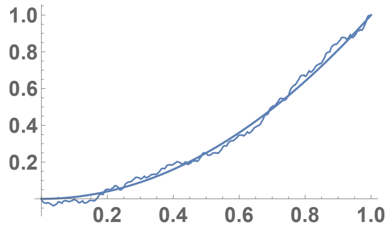
THEOREM 5.1. *Let $f \in \mathcal{C}(I)$ be convex on $[0,1]$ and $\Delta := \{0 = x_1 < x_2 < \cdots < x_N = 1\}$. Consider the IFS as described in (3), where $q_{i,n}(x) = f(L_i(x)) - \alpha_i(x)B_n(f;x)$ and $\alpha_i(x) \in \mathcal{C}(I)$. The corresponding Bernstein $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}}_n$ satisfies $f^{\boldsymbol{\alpha}}_n(x) \le f(x)\ \forall\, x \in I, n \in \mathbb{N}$, and the equality holds at the knot points provided that $\alpha_i(x) \ge 0\ \forall\, x \in I$.*

Proof. The Bernstein $\boldsymbol{\alpha}$-fractal function $f^{\boldsymbol{\alpha}}_n$ satisfies

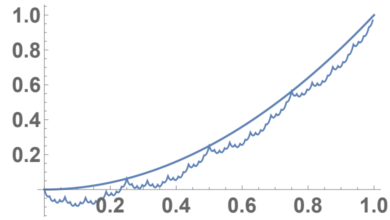
$$f^{\boldsymbol{\alpha}}_n(L_i(x)) = f(L_i(x)) + (f^{\boldsymbol{\alpha}}_n(x) - B_n(f;x))\alpha_i(x)\ \forall x \in I_i,\ n \in \mathbb{N},\ i \in N_{N-1}. \tag{25}$$

Clearly from the construction of the fractal function, we can observe that the next generation values of $(f^{\boldsymbol{\alpha}}_n)(L_i(x_j))$ depend on the current values $f^{\boldsymbol{\alpha}}_n(x_j)$ for $j = 1, 2\ldots, N$ at the grid points. Since $(f^{\boldsymbol{\alpha}}_n)(x_m) = f(x_m)$, that is, $(f^{\boldsymbol{\alpha}}_n - f)(x_m) \le 0\ \forall\, m = 1, 2, \ldots, N$, to prove the proposed condition it is enough to check that $(f^{\boldsymbol{\alpha}}_n - f)(L_i(x)) \le 0\ \forall\, i \in \{1, 2, \ldots, N\}$. Thus, from (25), we have

$$\begin{aligned}(f^{\boldsymbol{\alpha}}_n - f)(L_i(x)) &= \alpha_i(x)(f^{\boldsymbol{\alpha}}_n(x) - B_n(f;x)) \\ &= \alpha_i(x)(f^{\boldsymbol{\alpha}}_n(x) - f(x)) + \alpha_i(x)(f(x) - B_n(f;x)).\end{aligned}$$



(a) Fractal function $f^{\boldsymbol{\alpha}}$ not satisfying $f^{\boldsymbol{\alpha}} \leq f$



(b) Fractal function $f^{\boldsymbol{\alpha}}$ satisfying $f^{\boldsymbol{\alpha}} \leq f$

Figure 1: Fractal functions for $f(x) = x^2$.

Since $B_n(f;x) \geq f(x)$ for a convex function f [18], $(f_n^{\boldsymbol{\alpha}} - f)(L_i(x)) \leq 0$ if $\alpha_i(x) \geq 0 \ \forall \ x \in I$ as $(f_n^{\boldsymbol{\alpha}} - f)(x) \leq 0$. The theorem follows from the iterative nature of a fractal function. $\square$

EXAMPLE 5.2. Consider $I = [0,1]$ with a uniform partition Δ of step size $h = \frac{1}{4}$. Suppose the original convex function is $f(x) = x^2$. The fractal function $f^{\boldsymbol{\alpha}}$ is constructed with a uniform partition Δ, scaling vector $\boldsymbol{\alpha} = (0.5, -0.45, 0.5, -0.65)$, and base function $b(x) = xe^{x-1}$. The corresponding graph is depicted in Fig. 1(a). Note that $b(x) = xe^{x-1}$ does not satisfy $b(x) \geq f(x)$ and α is not a positive vector in this case, and hence the fractal function $f^{\boldsymbol{\alpha}}$ does not lie completely below f (see Fig. 1(a)). With a choice of $\boldsymbol{\alpha}$ satisfying the conditions prescribed in Theorem 5.1, namely, $\boldsymbol{\alpha} = (0.2, 0.1, 0.3, 0.25)$, and $b(x) = B_2(f;x) = \frac{x^2+x}{2} \geq f(x)$, we obtain a fractal function $f^{\boldsymbol{\alpha}}$ that lies completely below f (see Fig. 1(b)).

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