



Strong ergodicity breaking in aging of mean-field spin glasses

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Out-of-equilibrium relaxation processes show aging if they become slower as time passes. Aging processes are ubiquitous and play a fundamental role in the physics of glasses and spin glasses and in other applications (e.g., in algorithms minimizing complex cost/loss functions). The theory of aging in the out-of-equilibrium dynamics of mean-field spin glass models has achieved a fundamental role, thanks to the asymptotic analytic solution found by Cugliandolo and Kurchan. However, this solution is based on assumptions (e.g., the weak ergodicity breaking hypothesis) which have never been put under a strong test until now. In the present work, we present the results of an extraordinary large set of numerical simulations of the prototypical mean-field spin glass models, namely the Sherrington–Kirkpatrick and the Viana–Bray models. Thanks to a very intensive use of graphics processing units (GPUs), we have been able to run the latter model for more than 2^{64} spin updates and thus safely extrapolate the numerical data both in the thermodynamical limit and in the large times limit. The measurements of the two-times correlation functions in isothermal aging after a quench from a random initial configuration to a temperature $T < T_c$ provides clear evidence that, at large times, such correlations do not decay to zero as expected by assuming weak ergodicity breaking. We conclude that strong ergodicity breaking takes place in mean-field spin glasses aging dynamics which, asymptotically, takes place in a confined configurational space. Theoretical models for the aging dynamics need to be revised accordingly.

spin glasses | phase transitions | off-equilibrium dynamics

Aging is a fundamental process in out-of-equilibrium relaxation dynamics. It refers to the observation that relaxation or correlation timescales grow without bound in time, so the system under study looks slower and slower as time goes by. This phenomenon has been initially discovered experimentally in structural glasses (1, 2) and spin glasses (3, 4), but, since then, it has been found to be a general feature of glassy systems (5, 6). The importance of observing and properly describing aging is due to the strong connection of such phenomena to the energy landscape where the relaxation dynamics takes place. Understanding the relaxation dynamics in more or less rough energy landscapes is a very interesting and essentially still open problem with important applications. Just to highlight a recent and very active topic: The training of artificial neural networks—that have recently proven to be so effective—is performed by minimizing the loss function, that is, performing a relaxation dynamics in a very high-dimensional space (7, 8).

The dimension of the space where the dynamics takes place is indeed a crucial aspect. While a dynamics relaxing in a low-dimensional space cannot be too surprising, when the dynamics happens in a very high-dimensional space, our intuition may easily fail in imaging the proper role of entropic and energetic barriers, and thus the description of the out-of-equilibrium dynamics becomes very challenging.

Actually, any statistical mechanics model in the thermodynamic limit does perform a dynamics in a very high-dimensional space. In few fortunate cases [e.g., ordered models undergoing coarsening dynamics (9, 10)], the out-of-equilibrium dynamics can be well described with a reduced number of parameters. However, in more general disordered models, our understanding is still limited and based either on numerical simulations or on the analytical solution of restricted classes of models: mainly trap and mean-field models.

Trap models (11) provide a very simplified description of aging dynamics in disordered systems by assuming that the dynamics proceeds by “jumps” between randomly chosen states. This strong assumption allows for an analytic solution, but it is not clear to what extent the actual microscopic dynamics in a generic disordered model does satisfy such a hypothesis.

In this work, we focus on disordered mean-field models, in particular, on the well-known Sherrington–Kirkpatrick (SK) model (12, 13) defined by the following Hamiltonian:

$$H_{\text{SK}} = - \sum_{i < j} J_{ij} s_i s_j, \quad [1]$$

Significance

Understanding the relaxation dynamics on complex energy landscapes is a key aspect not only for physical processes but also for many applications where the minimization of a complicated cost function is required. Mean-field models with disorder have been and still are the privileged playing field to try to understand such complex dynamical behavior. We report the results of extraordinary long numerical simulations (more than 2^{64} spin updates) of the prototypical mean-field model of spin glasses. We uncover that, contrary to common expectation, the off-equilibrium aging dynamics at low temperatures undergoes a strong ergodicity breaking and thus asymptotically remains trapped in a confined region of the configurational space. Our results ask for a fundamental revision of models for the aging dynamics.

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Data deposition: Code and data related to this paper are available in GitHub (<https://github.com/mbernaschi/StrongErgodicityBreaking>).

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where the N Ising spins s_i interact pairwise via quenched random Gaussian couplings $J_{ij} \approx \mathcal{N}(0, 1/N)$ having zero mean and variance $1/N$. This model is the prototype for disordered models having a continuous phase transition from a paramagnetic phase to a phase with long-range spin glass order (in the SK model, it takes place at a temperature $T_c = 1$).

Thanks to the fact that couplings become very weak in the thermodynamic limit, the off-equilibrium dynamics of mean-field models can be written in terms of two-times correlation and response functions,

$$C(t, t') = \frac{1}{N} \sum_{i=1}^N \langle s_i(t) s_i(t') \rangle, \quad R(t, t') = \frac{1}{N} \sum_{i=1}^N \frac{\partial \langle s_i(t) \rangle}{\partial h_i(t')}.$$

The angular brackets represent the average over the dynamical trajectories, and an infinitesimal time-dependent field is added to the Hamiltonian as $-\sum_i h_i s_i$ to compute responses.

In the large N limit, $C(t, t')$ and $R(t, t')$ do satisfy a set of integro-differential equations (14, 15), and the solution to these equations provides the typical decay of correlations in a very large sample of the SK model. Although their exact solution is unknown, Cugliandolo and Kurchan (CK) found an ansatz that, under some hypotheses, solves the equations in the large times limit (16). These equations have been rederived rigorously in some particular cases (17).

The CK asymptotic solution has become very popular, and its consequences have been investigated in detail (18, 19). It is also often used as the theoretical basis for the analysis of numerical data (20–22). Notwithstanding its success, the CK solution has never been put under a severe numerical test, due to the difficulties in simulating very large samples of the SK model. In particular, we are not aware of any really stringent numerical test on the assumptions made to derive it.

One of the main hypotheses underlying the CK asymptotic solution is that one-time quantities converge to their equilibrium value. This has been further used as a key assumption to derive the connection between statics and dynamics (23). However, this assumption is far from obvious, given the existence of many mean-field models showing a random first-order transition (RFOT) where it is apparent how that assumption may be not satisfied: The prototypical model with an RFOT is the spherical p -spin model where the energy relaxes to a value far from the equilibrium one if the temperature is below the dynamical transition temperature (15). At variance to models with an RFOT, in spin glass models undergoing a continuous transition, the common belief has been to assume convergence to equilibrium, but even these models have an exponential number of states at low enough temperatures (24), and it is not clear why the out-of-equilibrium dynamics should converge to equilibrium in this case. We remind the reader that the dynamics we are studying is obtained by taking the large N limit first, and thus activated processes between states are suppressed.

We want to put under a stringent test the above hypothesis, and we find convenient from the numerical point of view to test the so-called *weak ergodicity breaking* property, stating that, for any finite waiting time t_w , the correlation eventually decays to zero in the large time limit,

$$\lim_{t \rightarrow \infty} C(t_w + t, t_w) = 0 \quad \forall t_w. \quad [2]$$

The physical meaning of this hypothesis is clear: In an aging system, any configuration reached at a finite time is eventually forgotten completely, because the dynamics, although slower and slower, keeps wandering in a large part of the configurational space. A direct consequence of the weak ergodicity breaking is that an aging system does not keep memory of what it did at any finite time, and the “finite time regime” can be eventually

integrated out (under a further hypothesis called weak long-term memory: the long-time dynamics wipes out the response to any small initial external perturbation), leading to an asymptotic dynamics that is actually decoupled from the finite times regime. This is a key feature that allows getting to the CK solution.

However, looking retrospectively at the literature of the times when the CK solution was derived, one finds that the numerical and experimental evidence were far from definite. In numerical simulations, the correlation $C(t_w + t, t_w)$ was observed to decay with time t for each waiting time t_w , but the range of correlations explored was very limited: $C \gtrsim 0.05$ (25, 26). The same was true for experiments measuring the decay of the thermoremanent magnetization (4) (moreover, experiments were not probing a mean-field model).

In the present work, we report the results of an unprecedented numerical effort in the study of the out-of-equilibrium dynamics in mean-field spin glasses to check whether some of the hypotheses at the basis of currently available solutions are valid or not.

We started our study by simulating directly the SK model and found some numerical evidence contrasting with the weak ergodicity breaking assumption (these results are presented in detail in [SI Appendix](#)). However, we soon realized that such a model would prevent us from reaching sizes and times scales large enough to support any solid statement. Indeed, the simulation time for such a model scales quadratically with the system size N and rapidly makes the simulation unfeasible.

We then resorted to a spin glass model defined on a sparse graph (requiring simulation times scaling linearly with the system size), similar to the Viana–Bray (VB) model (27),

$$H_{\text{VB}} = - \sum_{(ij) \in E} J_{ij} s_i s_j, \quad [3]$$

where the edge set E defines the interaction graph, and the couplings do not need to be rescaled with the system size (as in realistic models). In order to reduce the effects of the spatial heterogeneity in the H_{VB} Hamiltonian, we simulate the model on a random regular graph (RRG) of fixed degree 4, and we use couplings of fixed modulus, that is, $J_{ij} = \pm 1$, with equal probability. We measure self-averaging quantities, and, for the very large sizes that we simulate, there is no visible dependence on the specific random graph or couplings realization.

One may question whether the out-of-equilibrium dynamics of the VB model is equivalent to the SK case. The common belief about their equivalence is based on the observation that they are both mean-field approximations of the same spin glass model. Notice also that the SK model is equivalent to the large degree limit of the VB model (once couplings or temperatures are properly rescaled). However, one may also argue, instead, that the SK and the VB models have a key difference: Only in the former, the couplings become very weak in the thermodynamic limit, and this may produce visible differences in their out-of-equilibrium dynamics. In order to better understand how similar these two mean-field models are, one should study the scaling of their free-energy barriers with system size, $\Delta F \propto N^\psi$. We have done a preliminary analysis computing the characteristic time, $\tau(N)$, of the decay of the spin correlation at equilibrium, as in ref. 28. At $T = 0.8T_c$ and in the range $64 \leq N \leq 16,384$, the divergence of $\tau(N)$ can be well fitted by $\log(\tau(N)) \propto N^\psi$, with $\psi \approx 0.22$ in the SK model and $\psi \approx 0.21$ in the VB model. So we are reasonably confident that their out-of-equilibrium dynamics are similar.

Results and Discussion

We have simulated the VB model on an RRG of degree 4 at temperature $T = 0.8T_c$, where $T_c = 1/\text{atanh}(1/\sqrt{3})$, starting from a random initial configuration. We have measured the

correlation function $C(t_w + t, t_w)$ for $t_w = 2^2, 2^4, 2^6, 2^8, 2^{10}, 2^{12}$ and very large times t . Statistical errors have been strongly reduced averaging over a huge number of samples (see *Methods* and *SI Appendix* for more details).

We show, in Fig. 1, the decay of the correlation function $C(t_w + t, t_w)$ as a function of t for $t_w = 4$ and a number of different system sizes. The reader should appreciate that we are working in the regime $C(t, t') \ll q_{EA} \simeq 0.285$, which has never been reached before in any study of the out-of-equilibrium dynamics of mean-field spin glass models (the approximate value for q_{EA} is obtained via the replica symmetric cavity method). The plot is in a double logarithmic scale, so an upward curvature is a clear indication that correlation is either decaying slower than a power law or not decaying to zero at all.

Let us start discussing equilibration effects. For small enough N , we expect the system to thermalize and the correlation function to decay to the equilibrium value $C = 0$. The thermalization time $t_{eq}(N)$ is strongly dependent on the system size N , and, in Fig. 1, it is signaled by the shoulder clearly visible in the data for $2^{12} \leq N \leq 2^{15}$ and partially in the data for $N = 2^{16}$. Wanting to study the out-of-equilibrium regime, we must impose times to be much smaller than t_{eq} . For $N \geq 2^{18}$, such a thermalization effect is absent for the times we are probing, and we can safely consider the data as representative of the out-of-equilibrium regime. However, even considering only data for $N \geq 2^{18}$, Fig. 1 shows that finite size effects become apparent when measuring very small correlations. Such finite size corrections are otherwise negligibly small at the correlation scale

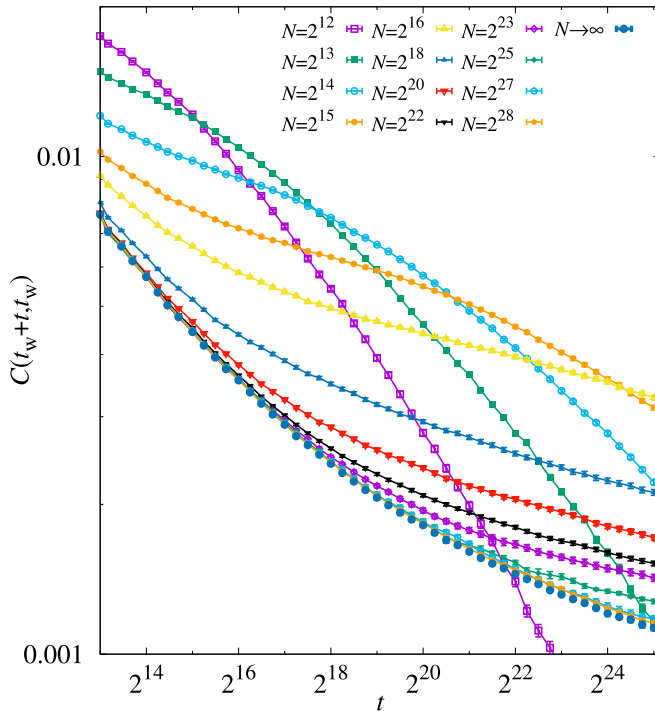


Fig. 1. Decay of the two-times correlation function $C(t_w + t, t_w)$ with $t_w = 4$ in a spin glass defined on an RRG of degree 4, initialized in a random configuration and evolved at $T = 0.87\zeta$. We show data for many different system sizes (lines are only a guide to the eye), and the extrapolation to infinite size for every value of t (represented by the series of full symbols at the bottom). It is worth noticing that we are probing a regime of very large times and extremely small correlations, $C(t_w + t, t_w) \ll q_{EA}$, never reached before in simulations of mean-field models. The upward curvature in the thermodynamic limit is a strong indication against a power law decay to zero correlation, that is, against the weak ergodicity breaking scenario.

$C \approx 0.1$ which has been the mostly probed one in the past. It is worth noticing that the clear identification of these finite size effects has been possible thanks to the very small uncertainties that we have reached by averaging over a very large number of samples (see *SI Appendix* for details on simulation parameters) and over a geometrically growing time window: In practice, the measurement at time t is the average over the time interval $[t/2^{1/4}, t]$.

In order to reach any definite conclusion in the analysis of the out-of-equilibrium dynamics, it is mandatory to take into account these finite size effects accurately. As shown in Fig. 1, off-equilibrium correlations measured in a system of smaller size decay asymptotically slower than those measured in a larger system (compare data in the range $2^{18} \leq N \leq 2^{28}$ at the largest times). Without taking properly the thermodynamic limit, one cannot argue too much about the large time limit of correlation functions. This is the reason why the data we initially got for the SK model, although pointing to the same conclusion we will draw from VB model simulations, were considered not conclusive.

We are interested in the limit of very large times taken after the thermodynamic limit. In this limit, we are probing the aging dynamics where activated processes do not play any role. For each fixed time, we extrapolate the data shown in Fig. 1 to the thermodynamic limit, following the procedure explained in *SI Appendix*, and we get the curve shown with label “ $N \rightarrow \infty$ ” in Fig. 1. It is evident that a strong upward curvature remains in the thermodynamical limit, thus suggesting that a power law decay to zero is very unlikely and would require an unnatural, very small value of the power law exponent.

Hereafter, we concentrate only on the analysis of data already extrapolated to the large N limit. We aim at understanding what is the most likely behavior of the thermodynamic correlation in the limit of large times. We are aware that, solely from numerical measurements taken at very large but finite times, we cannot make an unassailable statement, and one could always claim that, on larger times, the decay could change. Notwithstanding, we believe (and we assume in our analysis) that, for the very large times we have reached in our numerical simulations, the asymptotic behavior already set in. Thus the results of the analysis should be stable if we change the time window over which the analysis is carried out (always in a way such that $t \gg t_w$ and thus $C(t_w + t, t_w) \ll q_{EA}$).

In Fig. 2, we present the results of the asymptotic analysis that we find most stable and thus most likely, according to the above prescription. The data for the correlations $C(t_w + t, t_w)$ extrapolated in the large N limit are shown as a function of an inverse power of the time for different waiting times, ranging from $t_w = 4$ to $t_w = 4,096$. We immediately notice that all data follow a nice linear behavior in this scale; the only data departing from the linear behavior are those at very short times, that violate the condition $t_w \ll t$. The good agreement with the linear behavior—the lines are fits to data points in $t \in [2^{17}, 2^{25}]$ —implies that a fit to the form

$$C(t_w + t, t_w) = A(t_w) \exp [B(t_w) t^{-\gamma_{\text{exp}}}] \quad [4]$$

is very stable upon changing the time window (mind the log scale on the y axis). A joint fit to all t_w data with a t_w -independent exponent gives the value $\gamma_{\text{exp}} \simeq 0.17$, a weakly t_w -dependent coefficient $B(t_w)$, and definitely a nonnull extrapolation to infinite time $A(t_w) = \lim_{t \rightarrow \infty} C(t_w + t, t_w)$. For a quick reference, we may call “exponential” the fit above, although the asymptotic decay is like $A + A B t^{-\gamma_{\text{exp}}}$.

In order to test the null hypothesis, that is, the weak ergodicity breaking scenario where $\lim_{t \rightarrow \infty} C(t_w + t, t_w) = 0$ for any finite t_w , we tried to fit the data extrapolated in the large N limit to a function compatible with this limit. Given the upward curvature of the correlation function in a double logarithmic scale

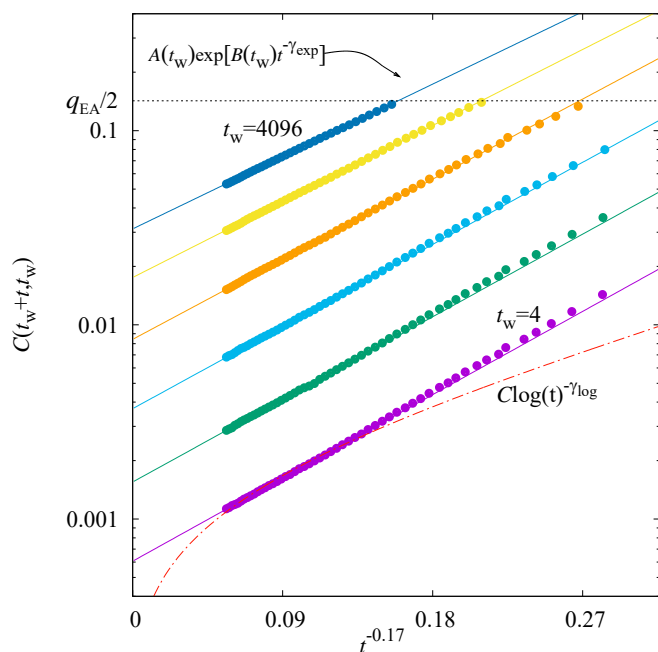


Fig. 2. The correlation functions $C(t_w + t, t_w)$ extrapolated in the thermodynamic limit as a function of time t , for all of the waiting times t_w studied. We report only data which are safely in the aging regime $C(t_w + t, t_w) < q_{EA}/2$. Straight lines through data points are best fits to functional form in Eq. 4 on the time window $t \in [2^{17}, 2^{25}]$. The dash-dotted curve is a best fit to a logarithmic decay extrapolating to zero correlation.

(Fig. 1), one may propose a very slow decay according to an inverse power of $\log(t)$. It turns out that a fit to a logarithmic decay $C(t_w + t, t_w) = C(t_w) \log(t)^{-\gamma_{\log}}$, implying a null limiting correlation, yields a value of the sum of squared residuals per degree of freedom one order of magnitude larger than the fit in Eq. 4. Moreover, the values of the fitting exponents are also very strongly dependent on the fitting window (see below and *SI Appendix*).

The most striking consequence of the analysis shown in Fig. 2 is that, for finite t_w values, in the $t \rightarrow \infty$ limit, the correlation $C(t_w + t, t_w)$ does not decay to zero. This is a very surprising result, as it implies—at variance with the widely diffused common belief—that an aging spin glass can asymptotically remember, to some extent, the configurations it reached at finite times. This positive long-term correlation confutes the weak ergodicity breaking assumptions and implies a much stronger ergodicity breaking. The present result requires rethinking the asymptotic solutions for the aging dynamics in mean-field spin glass models.

We show now some numerical evidence of why we consider the strong ergodicity breaking as the most likely scenario. In order to test the stability of the fitting procedure with respect to the choice of the time interval, we perform the analysis on intervals $t \in [2^{n-k}, 2^n]$ with fixed $k=8$ and n running on all of the time series. A fit to the function $C(t_w + t, t_w) = C(t_w) \log(t)^{-\gamma_{\log}(t_w)}$ with t_w -dependent parameters returns values of γ_{\log} that strongly depend on n , that is, on the position of the fitting window (see *SI Appendix* for details). Thus, fit results are very dependent on the time t , implying strong corrections to the asymptotic scaling. Again, we cannot completely exclude this scenario, but it is very unlikely under the hypothesis that corrections to scaling are weak at the very large times we reached at the end of our simulations.

The asymptotic scaling for the correlation decay which has been mostly used until now is an inverse power law of time. Thus

we have also fitted our data in the time window $t \in [2^{n-k}, 2^n]$ according to low-order polynomials in $t^{-\gamma}$,

$$P_M(t_w + t, t) = A_M(t_w) + \sum_{m=1}^M D_M^{(m)}(t_w) t^{-m\gamma_M(t_w)}, \quad [5]$$

with exponents and coefficients depending on both t_w and M .

Please note that, for $D_M^{(m)}(t_w) = A_M(t_w) B_M(t_w)^m$, the polynomial in Eq. 5 is nothing but the M th-order Taylor expansion of the function in Eq. 4 with $A_M(t_w) = A(t_w)$ and $B_M(t_w) = B(t_w)$. It turns out that the identification $D = A B^m$ is necessary in order to obtain numerically stable results. Moreover, assuming such a relation, we reduce the number of free parameters and improve correlations among them. At large times, where corrections to the asymptotic scaling should be negligible, the low-order polynomials should yield results in agreement with those of the analysis made with the form in Eq. 4. Any deviation would give a measure of the systematic error introduced by ignoring some corrections terms in the asymptotic behavior.

We report, in Fig. 3, the best fitting parameters according to fitting functions in Eq. 4 and in Eq. 5 with $M=2$ and assuming $D_M^{(m)} = A_M B_M^m$. Results are shown as a function of the upper limit of the time window $[t/2^8, t]$ where the fit is performed. It is clear that the resulting best fit parameters are very stable, that is, weakly dependent on the position of the fitting window, and this is a strong indication that we are probing the asymptotic regime with a functional form suffering only tiny finite time corrections. We also notice that the two estimates of the decay exponent, γ_{\exp} and $\gamma_{M=2}$, shown in Fig. 3, *Upper* and *Middle*, become compatible at large times. The asymptotic value for the correlation function $C(\infty, t_w)$, shown in Fig. 3, *Lower*, is very stable, too, and clearly different from zero.

Analogous fits to any functional form assuming weak ergodicity breaking, that is $C(\infty, t_w) = 0$, return best fitting parameters strongly dependent on the position of the fitting window, and a sum of squared residuals per degree of freedom which is typically one order of magnitude larger than the fits discussed above and illustrated in Fig. 3.*

In conclusion, the most likely scenario, which is fully supported by our data, is the one where the limiting value for the correlation function $C(\infty, t_w)$ is strictly positive for any waiting time t_w . In Fig. 4, we report the estimates of $C(\infty, t_w) = A_M(t_w)$ obtained from the last fitting window, $t \in [2^{17}, 2^{25}]$, via the polynomial in Eq. 5 with $M=1$ and $M=2$, together with the values of $A(t_w)$ obtained via the fit to Eq. 4. The three estimates are compatible within errors. In Fig. 4, *Inset*, we report the best estimates for the decay exponent obtained from the same fits. The exponent is weakly dependent on t_w , and systematic errors are more evident, indicating that we are far from a regime in which the decay to the residual correlation can be described by a single power law. Notwithstanding this, different models with different decay exponents agree both qualitatively and quantitatively with a nonnull value for the asymptotic correlation.

Computing the value of $C(\infty, t_w)$ in the large t_w limit is out of scope with present data and would require new and longer simulations with larger t_w values. However, given that we are working in the aging regime under the condition $C(t_w + t, t_w) < q_{EA}$, it is easy to get a conservative upper bound to that limiting correlation, that is, $\lim_{t_w \rightarrow \infty} C(\infty, t_w) \leq q_{EA}$. Moreover, noticing that the plot in Fig. 4 is in double logarithmic scale and that we still do not see any downward curvature, even if correlation values

*Since we are dealing with strongly correlated data, constructing a proper χ^2 estimator is a challenging task, but the sum of the squared residuals can give, anyhow, an indication of the relative goodness of different interpolating functions.

are not far from q_{EA} , we may conjecture that the upper bound is saturated, that is,

$$\lim_{t_w \rightarrow \infty} C(\infty, t_w) = q_{EA}. \quad [6]$$

The above conjecture would lead to the unexpected scenario where the out-of-equilibrium relaxation asymptotically gets trapped in an equilibrium state, randomly chosen according to the initial condition and the thermal noise at finite times.

The physical picture that emerges from the above strong ergodicity breaking scenario corresponds to a system that, while relaxing in a complex energy landscape, remains confined in regions of the configurational space becoming smaller and smaller during the evolution. Whether this is asymptotically a single state or a marginal manifold extending over a small fraction of the configurational space is not possible to deduce from our data, and further studies will be needed. Nonetheless, if this

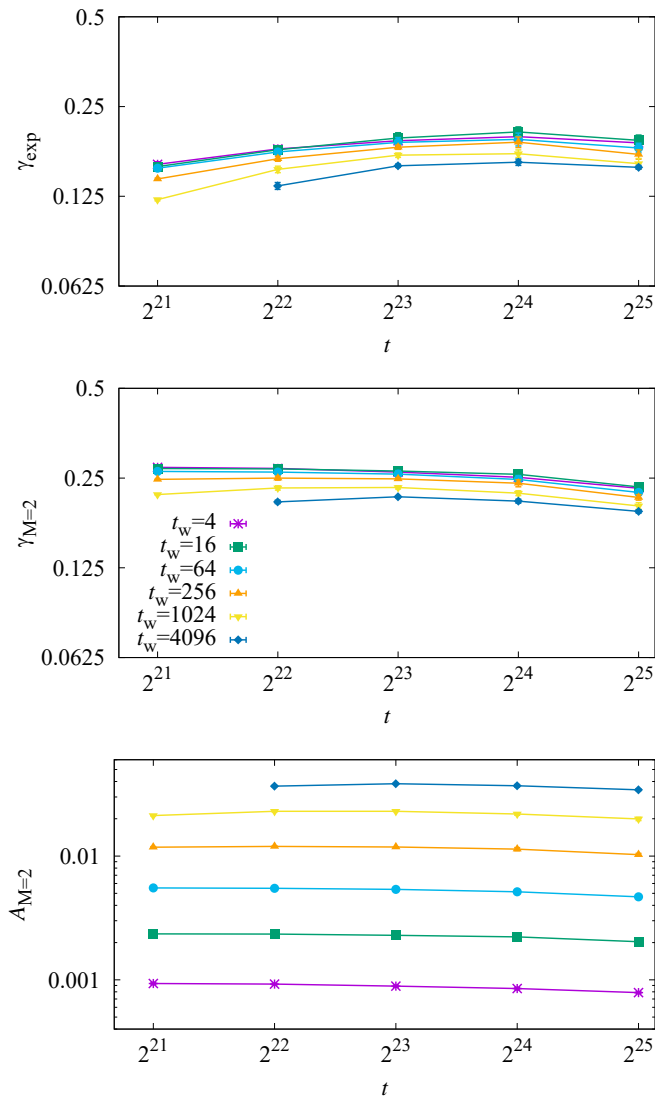


Fig. 3. The results of fitting $C(t_w + t, t_w)$ to Eq. 4 (Upper) and to Eq. 5 with $M=2$ and assuming $D_M^{(m)} = A_M B_M^n$ (Middle and Lower). In all panels, the time on the abscissa $t = 2^n$ is the upper limit of the fitting range $t \in [2^{n-8}, 2^n]$. We notice that the results of these fitting procedures are very stable, that is, vary little when moving the fitting time window, at variance to fits assuming weak ergodicity breaking.

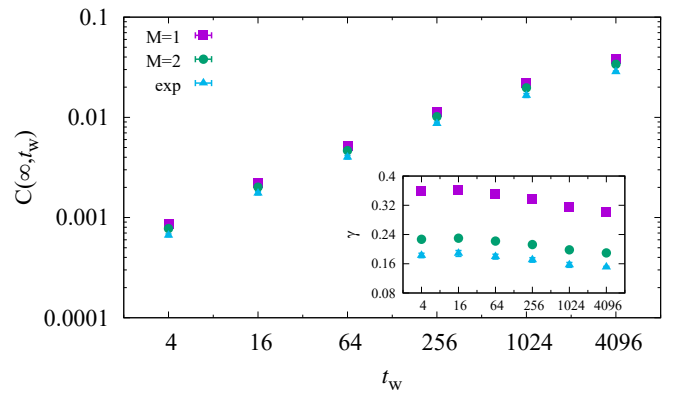


Fig. 4. The infinite time limit $C(\infty, t_w)$ estimated via the most stable fits, that is, the one in Eq. 5 with $M=1$ and $M=2$, and the one in Eq. 4. The fitting range is $t \in [2^{17}, 2^{25}]$. (Inset) The best estimate for the decay exponent.

strong ergodicity breaking scenario turns out to be the correct one, we have to abandon the physical idea of aging as a dynamical process exploring a marginal manifold extending all over the configurational space.

Recently, the study of the out-of-equilibrium dynamics in a different mean-field spin glass model, namely the spherical mixed p spin, has shown—via analytical solutions—a similar phenomenon (29): Depending on the initial condition, the asymptotic aging dynamics takes place in a restricted part of the configurational space, and memory of the initial condition is preserved. This result corroborates those presented in the present work and strongly suggests that, in mean-field spin glasses, the most general off-equilibrium relaxation is not the one we had in mind until now (a slow and unbounded wandering in the entire configurational space) but a slow evolution in a confined subspace, determined by the initial condition and the early times dynamics.

How much of the above scenario extends to finite-dimensional models is hard to say. Barriers may grow in a milder way in finite dimensions, and this may reduce the constraining effect. Unfortunately, past experiments have measured the decay of the thermoremanent magnetization on too short timescales (4). Also, the recent out-of-equilibrium simulations by the Janus collaboration (22, 30) do not reach correlations small enough to put the weak ergodicity breaking assumption under a stringent test; however, the analysis carried out in ref. 30 presented hints for a nonzero extrapolation in the large time limit. So, we believe it is time to run longer simulations on the Janus II computer to check for strong ergodicity breaking also in finite-dimensional models.

In conclusion, we have put under a severe test one of the most widely assumed hypotheses in the aging dynamics of mean-field glassy models, namely, the weak ergodicity breaking scenario. Our results are clearly in favor of a strong ergodicity breaking scenario, where the two-times correlation function does not decay to zero in the limit of large times. We have been able to achieve such unexpected results thanks to 1) the use of sparse mean-field spin glass models, 2) a careful analysis taking care of both finite size and finite time effects, and 3) an extraordinary numerical effort based on very optimized codes running on latest generation graphics processing units (GPUs).[†]

It is fun to notice that the number of spin flips we performed on the largest simulated systems is of the order of 2^{64} , the

[†]The computer code we used to run the simulations can be downloaded from <https://github.com/mbernaschi/StrongErgodicityBreaking>.

same number of wheat grains asked by Sessa, the inventor of chess, to sell his invention. Such an incredibly large number, that determined the destiny of Sessa, allows now the uncovering of unexpected physics!

Methods

We simulated many samples of the VB model with sizes in the range $2^{10} \leq N \leq 2^{28}$ for times up to 2^{25} Monte Carlo sweeps. We report all details and parameters of the simulations in [SI Appendix](#). Every simulation starts from a random initial configuration and evolves using the Metropolis algorithm at temperature $T = 0.8T_c = 0.8/\text{atanh}(1/\sqrt{3}) \simeq 0.42$ (31). This temperature is a good trade-off because it is not too low (and thus the evolution is not too slow), while being in the low temperature phase and thus having an Edwards–Anderson order parameter q_{EA} sensibly different from zero. We remind that the aging dynamics takes place in the large times limit only under the condition $C(t, t') < q_{EA}$. The thermodynamical properties of typical samples of the VB model can be computed via the cavity method: Although, for $T < T_c$, the exact solution would require breaking the replica symmetry in a continuous way, we can get a reasonable approximation to the value of q_{EA} via the replica symmetric solution providing $q_{EA}(T = 0.8T_c) \simeq 0.285$.

For simulating huge systems for long times, it is necessary to resort to parallel processing. To that purpose, we implemented the VB model on a

random regular bipartite graph (RRBG), making it possible to exploit the features of GPU accelerators despite the very irregular memory access pattern ([SI Appendix](#)). We have checked, on intermediate sizes, that results obtained on RRG and RRBG are statistically equivalent (see results in [SI Appendix](#)). The theoretical argument supporting the statistical equivalence of the VB model defined on RRG and RRBG goes as follows: RRG may have loops of any length, while RRBG only have even-length loops; since, in the VB model with symmetrically distributed couplings (we use $J_{ij} = \pm 1$), every loop can be frustrated with probability $1/2$ independently of its length, we do not expect any difference between the two graph ensembles in the thermodynamic limit, where loops become long. To speed up further the numerical simulations, we resorted to multispin coding techniques where copies evolve in parallel on the same graph, but with different couplings and different initial configurations.

Extrapolation to the thermodynamical limit is an important technical aspect of the present work (see [SI Appendix, section 2](#)). The results presented here have been obtained by fitting to a quadratic function $C(N = \infty) + AN^{-\nu} + BN^{-2\nu}$ with $\nu = 2/3$. The value of the exponent has been fixed according to well-known results in the literature (32–34).

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