

FURTHER GENERALIZATIONS OF THE PARALLELOGRAM LAW

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ABSTRACT. In recent work [2], a generalization of the parallelogram law in any dimension $N \geq 2$ was given by considering the ratio of the quadratic mean of the measures of the $N - 1$ -dimensional diagonals to the quadratic mean of the measures of the faces of a parallelotope. In this paper, we provide a further generalization considering not only $(N - 1)$ -dimensional diagonals and faces, but the k -dimensional ones for every $1 \leq k \leq N - 1$.

1. INTRODUCTION

If we consider the usual Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, the well-known identity

$$(1) \quad \|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2)$$

is called the *parallelogram law*.

This identity can be extended to higher dimensions in several ways. For example, it is straightforward to see that

$$(2) \quad \|a + b + c\|^2 + \|a + b - c\|^2 + \|a - b + c\|^2 + \|a - b - c\|^2 = 4(\|a\|^2 + \|b\|^2 + \|c\|^2)$$

with subsequent analogue identities arising inductively. There are, in fact, many works devoted to provide generalizations of (1) in many different contexts [1, 3, 4]

Note that if we rewrite (1) as

$$(3) \quad \frac{\|a + b\|^2 + \|a - b\|^2}{2} = 2 \frac{(\|a\|^2 + \|b\|^2 + \|a\|^2 + \|b\|^2)}{4}$$

it just means that in any parallelogram, the ratio of the quadratic mean of the lengths of its diagonals to the quadratic mean of the lengths of its sides equals $\sqrt{2}$. With this interpretation in mind, Alessandro Fonda [2] has recently proved the following interesting generalization.

Theorem 1. *Given linearly independent vectors $a_1, \dots, a_N \in \mathbb{R}^n$, it holds that*

$$\begin{aligned} & \sum_{i < j} \left(\left\| (a_i + a_j) \wedge \bigwedge_{k \neq i, j} a_k \right\|^2 + \left\| (a_i - a_j) \wedge \bigwedge_{k \neq i, j} a_k \right\|^2 \right) = \\ & = (N - 1) \sum_{k=1}^N 2 \|a_1 \wedge \dots \wedge \widehat{a_k} \wedge \dots \wedge a_N\|^2. \end{aligned}$$

In other words, for any N -dimensional parallelotope, the ratio of the quadratic mean of the $(N - 1)$ -dimensional measures of its diagonals to the quadratic mean of the $(N - 1)$ -dimensional measures of its faces is equal to $\sqrt{2}$.

In this work we extend this result considering the faces of dimension k for every $1 \leq k \leq N-1$ and a suitable definition of k -dimensional diagonal of a parallelotope. Then, Theorem 1 will just be a particular case of our result for $k = N-1$. Indeed, our result can be stated as follows.

Theorem 2. *Let us consider an N -dimensional parallelotope and let $1 \leq k \leq N-1$. The ratio of the quadratic mean of the k -dimensional measures of its k -dimensional diagonals to the quadratic mean of the k -dimensional measures of its k -dimensional faces is equal to $\sqrt{N-k+1}$.*

In fact, our generalization goes in the line of the work [3] but considering the definition of diagonal face given in [2].

2. NOTATION AND PRELIMINARIES

In this section we are going to introduce some notation and to present some basic facts that will be useful in the sequel. Let us consider a parallelotope \mathcal{P} generated by a family of linearly independent vectors $\mathcal{B} = \{a_1, a_2, \dots, a_N\} \subseteq \mathbb{R}^n$. This means that

$$\mathcal{P} = \left\{ \sum_{i=1}^N \alpha_i a_i : \alpha_i \in [0, 1] \right\}.$$

Let us fix $1 \leq k \leq N-1$. Then, given k different vectors $\mathcal{S} = \{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$, we can consider the face generated by them:

$$\mathcal{F}(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v v : \alpha_v \in [0, 1] \right\}.$$

This face can now be translated by one or more of the remaining vectors thus obtaining a face

$$\mathcal{F}^I(\mathcal{S}) = \left\{ \sum_{v \in \mathcal{S}} \alpha_v a_v + \sum_{w \in \mathcal{B} \setminus \mathcal{S}} \alpha_w w \in \mathcal{P} : \alpha_w \in \{0, 1\} \right\},$$

where $I = (\alpha_v)_{v \notin \mathcal{S}} \in \{0, 1\}^{N-k}$. Since each choice of a set $\mathcal{S} \subseteq \mathcal{B}$ and a vector $I \in \{0, 1\}^{N-k}$ leads to a different face and every face can be obtained in this way, it follows the well-known result that \mathcal{P} has exactly $2^{N-k} \binom{N}{k}$ k -dimensional faces. Moreover, it is clear that all the 2^{N-k} different faces $\mathcal{F}^I(\mathcal{S})$ are congruent to the set generated by \mathcal{S} , $\mathcal{F}(\mathcal{S})$.

Now, we focus on the k -dimensional diagonals which will be defined following the ideas in [2]. Let us consider $N-k+1$ different vectors $\mathcal{T} = \{a_{i_1}, \dots, a_{i_{N-k+1}}\} \subseteq \mathcal{B}$ and let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ be any partition. Then, the following set

$$\mathcal{D}(\mathcal{T}_1, \mathcal{T}_2) = \left\{ \alpha \sum_{v \in \mathcal{T}_1} v + (1-\alpha) \sum_{v \in \mathcal{T}_2} v + \sum_{w \in \mathcal{B} \setminus \mathcal{T}} \alpha_w w : \alpha, \alpha_w \in [0, 1] \right\}.$$

is called the k -dimensional diagonal associated to $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2)$. Clearly each choice of a set $\mathcal{T} \subseteq \mathcal{B}$ and a partition of \mathcal{T} leads to a different diagonal. Thus, it readily follows that \mathcal{P} has exactly $2^{N-k} \binom{N}{N-k+1}$ different k -dimensional diagonals. Moreover, if we define the vector

$$V(\mathcal{T}_1, \mathcal{T}_2) = \sum_{v \in \mathcal{T}_1} v - \sum_{v \in \mathcal{T}_2} v,$$

we have that

$$\mathcal{D}(\mathcal{T}_1, \mathcal{T}_2) = \left\{ \alpha V(\mathcal{T}_1, \mathcal{T}_2) + \sum_{v \in \mathcal{T}_2} v + \sum_{w \in \mathcal{B} \setminus \mathcal{T}} \alpha_w w : \alpha, \alpha_w \in [0, 1] \right\}$$

and, consequently, it is clear that the diagonal $\mathcal{D}(\mathcal{T}_1, \mathcal{T}_2)$ is just a translation of the set generated by $\{V(\mathcal{T}_1, \mathcal{T}_2), w : w \in \mathcal{B} \setminus \mathcal{T}\}$ and, hence, it is congruent to it.

3. PROOF OF THEOREM 2

After introducing the notation and the main objects involved in this work, we are now in the condition to prove the main result of the paper.

Let \mathcal{P} be a parallelotope generated by $\mathcal{B} = \{a_1, a_2, \dots, a_N\} \subseteq \mathbb{R}^n$. We first compute the quadratic mean of the k -dimensional measures of its k -dimensional faces. To do so, we first note that, for every $\mathcal{S} = \{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$, the k -dimensional measure of the face $\mathcal{F}(\mathcal{S})$ is $\|a_{i_1} \wedge \dots \wedge a_{i_k}\|$. In the previous section we have seen that \mathcal{P} has exactly $2^{N-k} \binom{N}{k}$ k -dimensional faces and, moreover, that there are exactly 2^{N-k} copies of each face $\mathcal{F}(\mathcal{S})$. Consequently, the quadratic mean of the k -dimensional measures of the k -dimensional faces of \mathcal{P} is:

$$(4) \quad \frac{2^{N-k} \sum \|a_{i_1} \wedge \dots \wedge a_{i_k}\|^2}{2^{N-k} \binom{N}{k}}.$$

Now we have to compute the quadratic mean of the k -dimensional measures of the k -dimensional diagonals of \mathcal{P} . First of all, recall that \mathcal{P} has exactly $2^{N-k} \binom{N}{N-k+1}$ different k -dimensional diagonals. Each of them is the translation of the set generated by $\{V(\mathcal{T}_1, \mathcal{T}_2), w : w \in \mathcal{B} \setminus \mathcal{T}\}$ for exactly one choice of $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2)$. The k -dimensional measure of this latter set is $\left\| V(\mathcal{T}_1, \mathcal{T}_2) \wedge \bigwedge_{w \in \mathcal{B} \setminus \mathcal{T}} w \right\|$. Consequently, the quadratic mean of the k -dimensional measures of the k -dimensional diagonals of \mathcal{P} is:

$$(5) \quad \frac{\sum_{\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2} \left\| V(\mathcal{T}_1, \mathcal{T}_2) \wedge \bigwedge_{w \in \mathcal{B} \setminus \mathcal{T}} w \right\|^2}{2^{N-k} \binom{N}{N-k+1}}.$$

Now, using the bilinearity of the scalar product and taking into account the definition of $V(\mathcal{T}_1, \mathcal{T}_2)$, it can be easily seen that when we vary $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2)$, we get the term $\|a_{i_1} \wedge \dots \wedge a_{i_k}\|^2$ exactly $2^{N-k} k$ times for every possible choice of $\{a_{i_1}, \dots, a_{i_k}\} \subseteq \mathcal{B}$. This implies that the quadratic mean of the k -dimensional measures of the k -dimensional diagonals of \mathcal{P} (5) can in fact be written as:

$$(6) \quad \frac{2^{N-k} k \sum \|a_{i_1} \wedge \dots \wedge a_{i_k}\|^2}{2^{N-k} \binom{N}{N-k+1}}.$$

Finally, in order to obtain Theorem 2 it is enough to divide (6) by (4):

$$\frac{(6)}{(4)} = \frac{k \binom{N}{k}}{\binom{N}{N-k+1}} = N - k + 1.$$

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