Set-theoretic methods in infinite abelian group theory

Fernando Barrera

Universidad Zaragoza

1542

Máster en Modeliazación e Investigación Matemática, Estadística y Computación

Supervisors: **Fernando Montaner Frutos - Joan Bagaria i Pigrau**

September 2020

Contents

Preface

In the present work we see how advanced set-theoretic methods apply to the study of infinite abelian groups. It has been written under the supervision of Prof. Dr. Fernando Montaner Frutos, from the University of Zaragoza, and Prof. Dr. Joan Bagaria i Pigrau, from the University of Barcelona, and it has been partially supported by the IUMA (Instituto Universitario de Matemáticas y Aplicaciones de la Universidad de Zaragoza) under grant PEX-17-007.

The first chapter is a brief collection of preliminary results needed for the remaining chapters. Most of the proofs are omitted but can be easily found in any standard textbook in the topic. In the second chapter, we see a few examples in which some instances of large cardinals such as measurable, strongly compact and δ-strongly compact cardinals naturally arise when dealing with infinte abelian groups. In particular, we see Eda's Theorem and some results regarding the Dugas-Göbel cardinal. The third and last chapter focuses on the Whitehead's Problem, which asks whether every Whitehead group is free. Although its restriction to groups of countable cardinality has a positive solution in ZFC, the general problem is undecidable. Indeed, both a positive and a negative answer for groups of size $\aleph_1$ are consistent with ZFC. An Appendix at the end intends to be a short and intuitive introduction to the technique of forcing, including the iteration of forcing used by Martin and Solovay to prove the consistency of Martin's Axiom.

Introduction

Most of mathematics can be done within ordinary set theory (that is, ZFC) or even smaller fragments of it. In fact, the working mathematician rarely needs to explicitly mention the axioms that he or she uses, with the possible exception of the Axiom of Choice. However, it might happen that a given proposition neither can be proved nor disproved in ZFC. Indeed, after Gödel's Incompleteness Theorems we know that every recursive axiomatic system powerful enough to formalize arithmetic, and ZFC is one of them, is either consistent or complete. Of course, mathematicians believe that ZFC is consistent. But assuming that ZFC is consistent implies accepting its incompleteness, that is, the existence of mathematical assertions which neither can be proved nor disproved in ZFC. This motivates a rich debate which is still on going on the necessity of new axioms for mathematics (see [FFMS]), being the most popular position among set-theorists that new axioms are needed. For instance, although ZFC is not able to decide the Continuum Hypothesis, mathematicians could eventually agree on the new axioms to be added to ZFC so that ZFC plus *those new axioms* is still consistent and powerful enough to decide whether the Continuum Hypothesis does or doesn't hold. The purpose of this work, however, is not to convince the reader to embrace this position. Less ambitious, we will just focus on how different set-theoretic methods like large cardinal axioms, forcing and ultrapowers apply to the study of infinite abelian groups.

It usually happens that if we restrict ourselves to the study of mathematical objects of countable size, things are provable in ZFC. This does not mean that one cannot deal with objects of uncountable size in ZFC. Indeed, as we shall see, Specker's Lemma, which shows that the additive group $\mathbb{Z}$ is slender, can be generalized to the product of uncountable-many copies of $\mathbb{Z}$. More precisely, for any uncountable cardinal κ and any homomorphism $h : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$, then $h(e_\alpha) = 0$ for all but finitely many α, where e_α is the function $e_n : \kappa \longrightarrow \{0,1\}$ such that $e_\alpha(\beta) = 1$ if and only if $\alpha = \beta$. However, although it is a theorem of ZFC that $Hom(\mathbb{Z}^\omega/\mathbb{Z}^{<\omega}, \mathbb{Z})$, where the classes in $\mathbb{Z}^\omega/\mathbb{Z}^{<\omega}$ consists of the vectors in $\mathbb{Z}^\omega$ differing in just finitely many coordinates, is the trivial group, it is independent of ZFC whether the same holds for the group $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z})$ with κ uncountable. Indeed, Eda's Theorem shows that $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z})$ is not the trivial group if and only if there exists a measurable cardinal. As we shall see, the existence of measurable cardinals cannot be proved in ZFC (assuming ZFC is consistent). Therefore, the question of whether $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z})$ with κ uncountable is the trivial group is independent of ZFC.

Throughout the second chapter we shall see some other examples of independent statements regarding infinite abelian groups apart from Eda's result. For instance, the so-called Dugas-Göbel cardinal of a strongly cotorsion-free group is, if it exists, greater than or equal to the first measurable cardinal. Some other large cardinal notions such as strongly compact and δ-strongly compact will appear. As it happens for measurable cardinals, strongly compact and δ-strongly compact cardinals can be defined in terms of complete filters which can be extended to complete ultrafilters. To provide a more practical characterization of the weaker δ-strongly compact cardinals dealing with ultrapowers and Łoś' results is necessary. This characterization of δ-strongly compact cardinals will be useful to prove Dugas-Eda-Abe's Theorem, which states that if κ is a δ-strongly compact cardinal, then $R_X = R_X^\kappa$ for every group of size less than δ, from which it follows that $R_\mathbb{Z} = R_\mathbb{Z}^\kappa$ if and only if κ is ω_1-strongly compact. Again, since the existence of ω_1-strongly compact cardinals cannot be proved in ZFC (provided ZFC is consistent), knowing whether $\mathbb{Z}$ satisfies the cardinal condition for some cardinal κ is another example of a mathematical proposition that cannot be decided in ZFC for it follows from the existence of a large cardinal.

In the third chapter we will just focus on the Whitehead's problem. The Whitehead's problem asks whether every W-group (an abelian group is said to be a W-group if every homomorphism onto A whose kernel is isomorphic to $\mathbb{Z}$ splits) is free. Although in 1951 Stein answered this question in the positive for groups of countable cardinality, the question remained open for groups of arbitrary cardinality until Saharon Shelah proved in 1974 that, restricted to groups of size $\aleph_1$, both a negative and a positive answer are consistent with ZFC. To be precise, Shelah showed that in the constructible universe L every W-group of size $\aleph_1$ is free and that under the assumption of Martin's Axiom one can always find a W-group of cardinality $\aleph_1$ which is not free. Since both the Axiom of Constructibility and Martin's Axiom are consistent with ZFC, the undecidability of the Whitehead's problem follows. Gödel proved the consistency of the Axiom of Constructibility in the late 30's by constructing the universe L, an inner model of ZF where he showed that the Axiom of Choice and the Continuum Hypothesis hold. To prove the consistency of Martin's Axiom, Martin and Solovay, based on Solovay and Tennenbaum's work on the consistency of the Suslin's Hypothesis, built a model of ZFC in which MA holds. To follow the proof some background in forcing is required. In the Appendix the reader will find a short exposition of forcing, Solovay-Tennenbaum's iterated forcing and Solovay-Martin's proof.

Chapter 1

Preliminaries

In this chapter we review the set-theoretic and algebraic background which is needed for the next chapters. We assume that the reader is familiar with the basic notions of first-order logic, the logical machinery we use to build statements and conditions of sets. If any doubt, we suggest the reader to consult [Men97]. In order to avoid this chapter to be unnecessary long, we omit most of the proofs. Nevertheless, everything here can be easily found in any standard book on the topic. For questions related to the set-theoretic part, we refer to [Kun13], specifically I.7-11 and I.13; and [Jech03], Chapters 2 and 3; and for those related to the algebraic part, we refer to [Lan02], I, III.2, and XX; [EM02], II; and [Kap69].

1.1 Set-theoretic background

The language of set theory consists of the following symbols:

(1) the *variables*, which run exclusively over sets;

(2) the *logical symbols*:

   (i) the *logical connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, to be read as *not, and, or, if and if and only if*, respectively;

   (ii) the *quantifiers* $\forall, \exists$, to be read as *for all* and *there exists*;

   (iii) the *identity symbol* $=$;

(3) A relation symbol $\in$ which is the *membership relation.*

Moreover, we will use some *auxiliary symbols* like parentheses, square brackets, etc.

The *atomic formulas* are strings of symbols of the form $x \in y$ or $x = y$ for any pair of variables x, y. The remaining formulas are recursively built from the atomic formulas applying the following rules:

(1) If φ is a formula, then $\neg\varphi$ is a formula.

(2) If φ and ψ are formulas, then $\varphi * \psi$ is a formula as well, where $*$ might be $\wedge, \vee, \rightarrow$ or $\leftrightarrow$.

(3) If φ is a formula and x is a variable, so are $\forall x\varphi$ and $\exists x\varphi$.

The occurrences of variables under the scope of a quantifier are said to be *bound*. Otherwise, they are said to be *free*. A formula with no free variables is called *sentence*. A formula φ in which one variable occurs free expresses a *property*. If a is a set and x occurs free in $\varphi(x)$, we say that a satisfies the property φ if $\varphi(a)$ holds.

1.1.1 The axioms of ZFC

We will work in the Zermelo-Fraenkel with the Axiom of Choice axiom system, simply denoted by ZFC. We assume that ZFC is consistent with no further comment. The theorems of ZFC are the sentences which logically follow from the axioms according to any logical calculi for first-order logic with equality. The axioms of ZFC are the following:

Existence. There is at least one set:

$$\exists x(x = x).$$

Extensionality. Sets with exactly the same elements are equal:

$$\forall a \forall b[\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b].$$

Pair. If a, b are sets, there exists a set containing both a and b as elements.

$$\forall a \forall b[\exists x(a \in x \wedge b \in x)].$$

Union. If a is a set, there exists a set whose elements are the elements of the elements of a.

$$\forall a[\exists x \forall y \in a \forall z \in y(z \in a)].$$

Power Set. If a is a set, there exists a set whose elements are the subsets of a, that is, the sets whose elements are elements of a.

$$\forall a[\exists x \forall y(\forall z \in y(z \in a) \rightarrow y \in x)].$$

We will denote by $\mathcal{P}(a)$ the power set of a.

Infinity. There exists an inductive set.

$$\exists x(\exists y(y \in x) \wedge \forall y \in x \exists z \in x(y \in z)).$$

Foundation. Every non-empty set contains and $\in$-minimal element.

$$\forall a[\exists y(y \in a) \rightarrow \exists y \in a \forall z \in a(z \notin y)].$$

Separation Schema. For every set a and every property φ there is a set whose elements are exactly the elements in a satisfying the property φ.

$$\forall a[\exists x \exists y(y \in x \leftrightarrow y \in a \wedge \varphi(y))],$$

for every $\varphi(y)$ where x does not occur. Note that since this happens for every formula φ, this is a list of infinite-many axioms.

Replacement Schema For every *definable function* on a set a, there is a set whose elements are the values of this function.

$$\forall a[\forall x \in a \exists! y \varphi(x, y) \rightarrow \exists z \forall x \in a \exists y \in z \forall(x, y)].$$

A function f is said to be *definable* in a if there exists a formula $\varphi(x, y)$ such that for every $x \in a$ there exists a unique y ($\exists! y$) such that $\varphi(x, y)$ holds. Again, this is a list of infinite-many axioms, one for each definable function.

Axiom of Choice (AC) For every set a of pairwise disjoint non-empty sets, there is a set that contains exactly one element from each set in a.

In ZF, AC is equivalent to *Zermelo's Well-ordering Principle*: for every set a there exists a well-ordering in a, that is, a linear ordering in which every non-empty set has a least element. AC is equivalent to Zorn's Lemma, modulo ZF, too.

Remark. The objects of ZFC are sets. However, we shall consider collections of objects that are not sets. We call *proper class* to any collection of sets which is not a set. For instance, the collection V of all sets, determined by the formula $x = x$, is a proper class. It cannot be a set because it doesn't satisfy the axiom of Foundation. A *class* is a collection of sets and it can be either a proper class or a set.

1.1.2 Ordinals and cardinals

Definition 1.1. A class a is said to be *transitive* if for every $y \in x \in a$, then $y \in a$. An *ordinal number* is a transitive set well ordered with respect to the memership relation $\in$.

This definition can be expressed as the following proposition shows:

Proposition 1.2. *A set a is an ordinal if and only if a is transitive, $\forall x, y \in a(x \in y \vee x = y \vee y \in x)$, and $\forall x \subseteq a(x \neq \emptyset \to \exists y \in x(y \cap x = \emptyset))$.*

We denote by OR the class of ordinals.

Proposition 1.3.

(1) *Any transitive set of ordinals is an ordinal and any element of an ordinal is an ordinal.*

(2) *If X is a set of ordinals, $\bigcup X$ is an ordinal. If α is an ordinal, $\bigcup a = a$.*

Consequently, OR is a proper class. Moreover, the empty set is an ordinal and if α is an ordinal, so is $\alpha \cup \{\alpha\}$. We usually write 0 and $\alpha + 1$ instead of $\emptyset$ and $\alpha \cup \{\alpha\}$, respectively. Also, if a is a set of ordinals, we might write $\bigcup a = \sup a$, that is, the least ordinal which is greater than or equal to any ordinal in a. Analogously, we might write $\bigcap a = \min a$.

Definition 1.4. If α is an ordinal, $\alpha + 1$ is the *successor* of α. If α isn't a successor ordinal, it is said to be a *limit ordinal*.

The class of ordinals can be well ordered by $<$, where $a < b$ if and only if $a \in b$. Indeed:

Proposition 1.5. *For any ordinals α, β:*

(1) *$\alpha \in \beta$ if and only if $\alpha \subsetneq \beta$.*

(2) *Either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha \in \beta$.*

An ordinal is said to be *countable* if it is finite or bijectable with ω. Otherwise, it is *uncountable*. The first uncountable ordinal, denoted by ω_1 is the set of countable ordinals; ω_2 is the next ordinal which is not bijectable with any of its predecessors. Analogously, we define the ordinals ω_α. At the limit stages we let ω_α to be the union of its predecessors.

The Axiom of Choice is equivalent, modulo ZF, to *Zermelo's Well-Ordering Principle*, which states that every set can be well-ordered.

Theorem 1.6 (The Enumeration Theorem)**.** *Every well-ordered set is isomorphic with an ordinal.*

Therefore we might define for every well-ordered set a its *order type*, denoted $ot(a)$, which is the unique ordinal with which a is isomorphic.

Theorem 1.7 (Transfinite Recursion)**.** *If G is a class function on V, then there is a unique class function F on the ordinals such that for each ordinal α, $F(\alpha) = G(F \upharpoonright \alpha)$. The function F is defined as $F(\alpha) = x$ if and only if there is a function f with domain α such that for every $\beta < \alpha$, $f(\beta) = G(f \upharpoonright \beta)$ and $x = G(f \upharpoonright \alpha)$.*

In OR we can define the operations of *addition*:

$$\alpha + 0 = \alpha,$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1,$$

$$\alpha + \beta = \sup\{\alpha + \xi : \xi < \beta\} \text{ if } \beta \text{ is a limit;}$$

multiplication:

$$\alpha \cdot 0 = 0,$$

$$\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + 1,$$

$$\alpha \cdot \beta = \sup\{\alpha \cdot \xi : \xi < \beta\} \text{ if } \beta \text{ is a limit;}$$

and *exponentiation*:

$$\alpha^0 = \alpha,$$

$$\alpha^{(\beta+1)} = (\alpha^\beta) \cdot \alpha,$$

$$\alpha^\beta = \sup\{\alpha^\xi : \xi < \beta\} \text{ if } \beta \text{ is a limit,}$$

for all ordinals α, β.

Remark. Ordinal addition and ordinal multiplication are not commutative. For instance, $1 + \omega = \omega \neq \omega + 1$ and $2 \cdot \omega = \omega \neq \omega \cdot 2$.

Definition 1.8. A *cardinal* number is an ordinal which is not bijectable with any predecessor.

Therefore, if κ and λ are cardinals and there exists a bijection between them, $\kappa = \lambda$. All finite ordinals are cardinals and so are the ordinals of the form $\omega, \omega_1, \dots, \omega_\omega, \dots$. Note that every cardinal is a limit ordinal.

Proposition 1.9 (AC)**.** *Every set is bijectable with a unique cardinal.*

We denote by $|a|$ the *cardinality* of a, that is, the unique cardinal which a is bijectable with.

Remark. The assertion in Proposition 1.9. is actually equivalent to the Axiom of Choice.

The least cardinal greater than a cardinal κ is the set of ordinals bijectable to a cardinal smaller or equal to κ. We denote it by κ^+.

Definition 1.10. We say that κ^+ is the *successor cardinal* of κ. If κ is not a successor cardinal, it is said to be a *limit cardinal.*

Proposition 1.11.

(1) *If κ is a limit cardinal, then for all $\lambda < \kappa$, $\lambda^+ < \kappa$.*

(2) *For every cardinal κ there is a limit cardinal λ such that $\kappa < \lambda$.*

(3) *If X is a set of cardinals, so is $\bigcup X$. If κ is a cardinal, $\bigcup \kappa = \kappa$.*

It follows from (3) that the class of cardinals, which we will denote by CARD, is proper. We will denote the transfinite sequence of cardinals as follows: $\aleph_0, \aleph_1, \ldots, \aleph_\omega, \ldots, \aleph_\alpha, \ldots$. We note that $\aleph_n = \omega_n$ for every $n < \omega$. Indeed:

Definition 1.12.

(1) $\aleph_0 = \omega$,

(2) $\aleph_\alpha = \aleph_\alpha^+$,

(3) $\aleph_\alpha = \sup\{\aleph_\beta : \beta < \alpha\}$ if α is a limit ordinal.

Of course, for every α, $\alpha \leq \aleph_\alpha$ and, if $\beta < \alpha$ then $\aleph_\beta < \aleph_\alpha$.

For every pair of cardinals κ, λ we define $\kappa + \lambda$ as $|(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$; the product is defined as $\kappa \cdot \lambda = |\kappa \times \lambda|$ and the exponentiation as $\kappa^\lambda = |\Pi_{\alpha<\lambda}\kappa|$.

Proposition 1.13. *Let κ and λ be cardinals. The union of every family of at most λ sets of cardinality at most κ has cardinality at most $\lambda \cdot \kappa$.*

Proposition 1.14. *If κ and λ are infinite cardinals, $\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}$.*

In particular, the addition and multiplication of cardinals is commutative. Although the sum and multiplication of infinite cardinals are easy to work with, exponentiation isn't trivial at all. For instance, $2^{\aleph_0}$ is undecidable in ZFC.

Definition 1.15. The *Continuum Hypothesis* CH is the assertion that $2^{\aleph_0} = \aleph_1$. The *Generalized Continuum Hypothesis* asserts that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\alpha \in OR$.

CH is independent of ZFC, thus so is GCH.

1.1.3 Cofinality

Let α be a limit ordinal. A subset b of α is said to be *unbounded* in α if and only if $sup b = \alpha$, or, equivalently, if for all $\beta < \alpha$ there is some $\gamma \in b$ such that $\beta < \gamma$. We say that an ordinal β is *cofinal* in α if and only if there i a strictly increasing function $f : \beta \longrightarrow \alpha$ whose range is unbounded in α.

Definition 1.16. The *cofinality* of α is the least ordinal which is cofinal in α. We denote it by $cf(\alpha)$.

Proposition 1.17.

(1) *If α and β are limit ordinals and α is cofinal in β, $cf(\alpha)$ is cofinal in $cf(\beta)$.*

(2) *If α is a limit ordinal, $cf(\alpha) = cf(\aleph_\alpha)$.*

Note that for every α, $cf(\alpha) \leq \alpha$.

Definition 1.18. A limit ordinal α is *regular* if $cf(\alpha) = \alpha$. Otherwise, we say that α is *singular*.

Proposition 1.19.

(1) *If α is a limit ordinal, $cf(\alpha)$ is a regular cardinal.*

(2) *If κ is an infinite cardinal, $cf(\kappa)$ is the least cardinal λ such that κ is the union of a family of λ-many sets of cardinality less than κ.*

(3) *Every infinite successor cardinal is regular.*

(4) *In general, for every infinite cardinal κ, $\kappa < \kappa^{cf(\kappa)}$. If $2 \leq \lambda$, $\kappa < cf(\lambda^\kappa)$.*

Theorem 1.20. *An infinite cardinal κ is regular if and only if the union of every family of less than κ-many seats each of cardinality less than κ is a set of cardinality less than κ.*

Proposition 1.21. *If GCH holds, for all infinite cardinals κ and λ:*

$$\kappa^\lambda = \begin{cases} \lambda^+ \text{ if } \lambda \geq \kappa \\ \kappa^+ \text{ if } cf(\kappa) \leq \lambda \leq \kappa \\ \kappa \text{ if } \lambda < cf(\kappa) \end{cases}$$

1.1.4 Models of Set Theory

We assume that ZFC is consistent. Then, there exist models of ZFC. A *model of ZFC* is a pair $\langle M, R\rangle$ where M is a non-empty class and R is a binary relation on M such that $\langle M, R\rangle$ satisfies the axioms of ZFC. We can define analogously what a *model of a fragment of ZFC* is. The relation R on a class M is said to be *well-founded* if it is set-like in the sense that for every $x \in M$ the class $\{y \in M : yRx\}$ is a set and there is no infinite descending R-chain. We say that a subset $x \in M$ is R-transitive if for every $zRyRx$, then zRx. The transfinite recursion on well-founded relations holds. We say that a model $\langle M, R\rangle$ is standard if R is the membership relation on M, that is, if $R = \in \cap (M \times M)$. A *submodel* $\langle N, \in_N\rangle$ of $\langle M, \in\rangle$ is a model such that $N \subseteq M$ and

$\in_N = \in \cap (N \times N)$. If $N \subseteq M$ is a submodel of M of ZFC and $N \vDash ZFC$, N is said to be an *inner model* of M. Whenever the context is clear, we will denote any model $\langle M, \in \rangle$ simply by M.

Theorem 1.22 (Downward Löwenheim-Skolem-Tarski Theorem)**.** *(ZFC minus Power Set) Let M be an $\mathcal{L}$-structure and let κ be a cardinal such that $max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |M|$ and fix $S \subseteq B$ with $|S| \leq \kappa$. Then, there is an elementary substructure N of M, that is, an substructure satisfying the same $\mathcal{L}$-sentences, such that $S \subseteq N$ and $|N| = \kappa$.*

The Downward Löwenheim-Skolem-Tarski theorem tell us that for every model of set theory there is always a countable elementary submodel.

Theorem 1.23 (Mostowski Collapse)**.** *If $\langle M, R \rangle$ is a well-founded model of the axiom of Extensionality, then there is a unique transitive model $\langle N, \in \rangle$ and a unique isomorphism $\pi : \langle M, R \rangle \longrightarrow \langle N, \in \rangle$.*

We call $\langle N, \in \rangle$ the *Mostowski* or *transitive collapse* of $\langle M, R \rangle$; the isomorphism π is called the *collapse mapping.* It is clear that if $\langle M, \in \rangle$ and $\langle N, \in \rangle$ are isomorphic, then $M = N$.

In ZFC the universe of all sets forms a cumulative hierarchy. Then, $x \in V$ if and only if there exists some ordinal α such that $x \in V_\alpha$, where the V_α are defined as follows:

Definition 1.24.

$V_0 = \emptyset$,

$V_{\alpha+1} = \mathcal{P}(V_\alpha)$,

$V_\alpha = \bigcup_{\beta<\alpha} V_\beta$, if α is a limit.

Then, $V = \bigcup_{\alpha \in OR} V_\alpha$.

$V \vDash ZFC$. For every set $a \in V$, the *rank* of a, denoted by $rk(a)$, is the least ordinal α such that $a \in V_\alpha$. V_α is transitive for every $\alpha \in OR$. If $\alpha < \beta$, then $V_\alpha \in V_\beta$. For every ordinal α, $\alpha \subseteq V_\alpha$ but $\alpha \notin V_\alpha$.

Lemma 1.25. *Every transitive class satisfies Extensionality and Foundation.*

Proof. Let a be a transitive class and let $x, y \in a$ which are seen in a to be equal. Since a is transitive, it contains all elements of x and y, so they have the same elements so, by Extensionality, they are equal. To see Foundation, let a be a class and suppose that $b \in a$ is non-empty. Then $a \cap b$ is non-empty and, by Foundation, $a \cap b$ contains an $\in$-minimal element. ■

Proposition 1.26.

(1) V_α *satisfies Foundation, Extensionality, Union, Separation for every ordinal* α.

(2) *If* α *is a limit ordinal, then* V_α *satisfies Pairing, Power Set and AC.*

(3) *If* $\alpha > \omega$, *then* V_α *satisfies Infinity.*

(4) *If* $\alpha = \omega$, *then* V_α *satisfies Replacement.*

Proof. Let $\alpha \in ORD$. By the previous lemma, since V_α is transitive, it satisfies Extensionality and Foundation. For Union, we proceed by transfinite induction. The limit case being clear, let $\alpha = \beta + 1$. If $a \in V_\alpha$, then $a \subseteq V_\beta$. Since $\bigcup a = \{b \in V_\beta : \beta \in a\}$, then $\bigcup a \subseteq V_\beta$, hence $\bigcup a \in V_\alpha$. For Separation we go analogously. Let φ be an arbitrary formula in the language of set theory and let $a \in V_\alpha$. Then $a \subseteq V_\beta$, so $\{b \in a : \varphi(b)\} \subseteq V_\beta$. Therefore, $\{b \in a : \varphi(b)\} \in V_\alpha$. This proves (1). To see (2), let $a, b \in V_\alpha$ with α a limit. Since $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$, let β be the least ordinal such that $a, b \in V_\beta$. Then $\{a, b\} \subseteq V_\beta$, so $\{a, b\} \in V_{\beta+1}$. Since $\beta + 1 < \alpha$, $\{a, b\} \in V_\alpha$, which proves Pairing. For Power Set let $a \in V_\alpha$, with α a limit and let $\beta < \alpha$ be the least ordinal such that $a \in V_\beta$, then $\mathcal{P}(a) \in V_{\beta+2}$. Since $\beta + 2 < \alpha$, $\mathcal{P}(a) \in V_\alpha$. Now, let $a \in V_\alpha$ and let f be a choice function for a. Then $f(b) \in b$ for every non-empty $b \in a$, so $f = \{\langle b, f(b)\rangle : b \in a, b \neq \emptyset\}$. That is, f is a set of ordered pairs of the form $\{\{b\}, \{b, f(b)\}\}$. But $b \in a$ and $f(b) \in b$ so if β is an ordinal below α such that $a \in V_{\beta+2}$, then $a \subseteq V_{\beta+1}$, hence $b \in V_{\beta+1}$, so $b \subseteq V_\beta$ and $f(b) \in V_\beta$. Therefore, $b, f(b) \in V_\alpha$. Since V_α satisfies Pairing, $\{\{b\}, \{b, f(b)\}\} \in_\alpha$ for every $b \in a$, so $f \in V_\alpha$ and V_α satisfies AC. To see that (3) holds just note that $\omega \in V_\alpha$ for every uncountable α. Since ω is an inductive set, if α is uncountable, then V_α satisfies Infinity. For (4), let $f \in V_\omega$ be a definable function with domain a, i.e., a function for which there is a formula $\varphi(x, y)$ such that $f(b) = c$ if and only if $\varphi(b, c)$ and $\varphi(x, y) \wedge \varphi(z, y) \to x = z$. Let $rg(f)$ be the range of f. Since a is a finite set, $rg(f)$ is finite as well. Note that for each $b \in rg(f)$, $rk(b) < \omega$, so let $\gamma = \sup\{rk(b) : b \in rg(f)\}$. Then, $rg(f) \subseteq V_\gamma$, hence $rg(f) \in V_{\gamma+1}$, so in $rg(f) \in V_\omega$, which finishes the proof. ∎

The following is a very useful result. The proof, which we omit, goes by induction on the complexity of the formula. See [Jech03], Theorem 12.14, p. 168.

Theorem 1.27 (Reflection Theorem. Levy, 1960)**.** *Let* $\varphi(x_1, \ldots, x_n)$ *be a formula of the language of set theory. Let* $\langle A_\alpha : \alpha \in OR\rangle$ *be a cumulative hierarchy and let* $A = \bigcup_{\alpha\in OR} A_\alpha$. *Then there is a closed proper class* C *of ordinals* β *such that for all* $a_1, \ldots, a_n \in A_\beta$,

$$A \vDash \varphi(a_1, \ldots, a_n) \text{ if and only if } A_\beta \vDash \varphi(a_1, \ldots, a_n).$$

For any infinite cardinal κ we denote by H_κ the set of all sets whose transitive closure has cardinality less than κ. The *transitive closure* of a set a, denoted by $tc(a)$ is the smallest transitive set containing a. For every κ, $H_\kappa \subseteq V_\kappa$ and $H_\omega = V_\omega$. The H_κ form a cumulative hierarchy: if $\lambda < \kappa$ then $H_\lambda \subseteq H_\kappa$, and if κ is a limit cardinal then $H_\kappa = \bigcup_{\lambda<\kappa} H_\lambda$. Then, $V = \bigcup_{\kappa \in CARD} H_\kappa$. There is a closed proper class of cardinals C such that $V_\kappa = H_\kappa$ for every $\kappa \in C$. Actually, for every uncountable cardinal κ, $H_\kappa = V_\kappa$ if and only if $\beth_\kappa = \kappa$, where $\beth$ is the beth function (see [Kun13], Definition I.13.24). H_κ in general satisfies every axiom but Infinity, Replacement and Power Set. It satisfies Infinity if and only if $\kappa > \aleph_0$, it satisfies Replacement if and only if κ is regular, and it satisfies Power Set if and only if κ is a strong limit. Therefore only if κ is inaccessible, $H_\kappa \models$ ZFC.

1.1.5 Inaccessible cardinals

Definition 1.28. A cardinal is *weakly inaccessible* if is a regular, uncountable limit cardinal. Equivalently, a cardinal κ is weakly inaccessible if and only if κ is regular and $\aleph_\kappa = \kappa$. A *strong limit cardinal* is an infinite cardinal κ such that $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. An *inaccessible* cardinal is a regular uncountable strong limit cardinal.

Theorem 1.29. *If κ is inaccessible, then $V_\kappa \models ZFC$.*

Proof. Since κ is an uncountable cardinal, so a limit ordinal, V_κ satisfies ZFC minus possibly Replacement. So let us show that if κ is inaccessible, then V_κ satisfies Replacement. Let f be a definable function with domain a with $a \in V_\kappa$. It is a well-known fact that if κ is inaccessible, then $|V_\kappa| = \kappa$ so, since $|a| < \kappa$, then $|rg(f)| < \kappa$, so there is some $\alpha < \kappa$ such that $rg(f) \subseteq V_\alpha$, hence $rg(f) \in V_\kappa$. ■

As a consequence, since ZFC is a recursive axiom system in which arithmetic can be formalized, Gödel's Second Incompleteness Theorem applies. Therefore, if the existence of inaccessibles was provable from ZFC, ZFC would prove its own consistency, which is not possible. In general, we call *large cardinals* to those cardinals whose existence must be taken as an axiom for it cannot be proven inside ZFC. Inaccessible cardinals are an example of those. As seen in Theorem 1.26, for every uncountable limit ordinal α, V_α satisfies ZFC except possibly Replacement. From the assumption of the existence of an inaccessible cardinal κ it can be proved the existence of many cardinals $\lambda < \kappa$ such that all axioms of ZFC hold in V_λ.

1.2 Algebraic background

We assume some familiarity with the basic concepts of the theory of abelian groups and category theory. If any doubt, we refer the reader to [Lan02], I.11. Some of the concepts in this section can be studied in the more general context of homological algebra. Might the reader be interested, see [Lan02], XX. The material introduced in this section will be needed to show important features of W-groups in Chapter 2.

1.2.1 Free abelian groups

Throughout the text, although otherwise specified, group will mean abelian group. Most of the notions introduced here will be used in Chapter 3. Recall that if $\{A_i : i \in I\}$ is a family of abelian groups, their direct sum $A = \bigoplus_{i \in I} A_i$ is the subgroup of the product $\prod_{i \in A}$ consisting of the families $(a_i)_{i \in I}$ where $a_i \in A_i$ for every $i \in I$ and $a_i = 0$ for all but finitely many $i \in I$. We say that a family $\{x_i : i \in I\}$ of elements of A is a *basis* for A if it is non-empty and if every element a of A has a unique expression as a linear combination $a = \sum_{i \in I} r_i x_i$ with $r_i \in \mathbb{Z}$ and almost all $r_i = 0$.

Definition 1.30. A group A is said to be *free* if it has a basis.

Equivalently, a free group is a direct sum of infinite cyclic groups. It is then clear that free groups are isomorphic to the direct sum of copies of $\mathbb{Z}$. Therefore, the additive group $\mathbb{Z}$ is a trivial example of a free group while the additive group $\mathbb{R}$ is an example of a non-free group.

Theorem 1.31 (See [Lan02], Theorem 7.3, p. 41)**.** *A subgroup of a free group is free.*

Since abelian groups are $\mathbb{Z}$-modules, most of the notions and results in this section can be generalized to the more general theory of modules. For an approximation to the topic from this perspective, use [Kap69]. Theorem 1.31 can be also found in [Kap69], Lemma 15.

Definition 1.32. A surjective homomorphism of groups $\pi : B \longrightarrow A$ *splits* if there is a homomorphism $\rho : A \longrightarrow B$ such that $\pi \circ \rho = id_A$, where id_A denotes the identity on A. The mapping ρ is sometimes called the *splitting function* for π.

Free groups can be characterized in terms of splitting homomorphisms.

Theorem 1.33. *A group A is free if and only if every homomorphism onto A splits.*

Proof. Suppose first that A is free and let $\pi : B \longrightarrow A$ be a epimorphism, with B arbitrary. Let $S = \{s_i : i \in I\}$ be a basis of A and let $b_i \in B$ such that $\pi(s_i) = b_i$ for each $i \in I$. Since S is a basis of A, there exists a unique homomorphism $\rho : A \longrightarrow B$

such that $\rho(s_i) = b_i$ for each $i \in I$. It is easy to see that ρ is a splitting function for π. Conversely, let F be the free group generated by $S = \{s_a : a \in A\}$ and let $\pi : F \longrightarrow A$ be the unique homomorphism such that $\pi(s_a) = a$ for all $a \in A$. Since π is surjective, by assumption there is a splitting homomorphism $\rho : A \longrightarrow F$ for π. But ρ is injective, so A is isomorphic to a subgroup of F. Then, by Theorem 1.31, A is free. ■

Corollary 1.34. *If B be a subgroup of A such that B and A/B are both free, then A is free. Moreover, any basis of B extends to a basis of A.*

Proof. We sketch the proof. Let $\pi : A \longrightarrow A/B$ be the canonical projection. By assumption, since A/B is free there is a splitting homomorphism $\rho : A/B \longrightarrow A$ for π. The unique presentation of any $a \in A$ as a sum of element of $\rho(A/B)$ and B is $a = \rho(\pi(a)) + (a - \rho(\pi(a))$, so $A = \rho(A/B) \oplus B$. Since ρ is injective, if S is a basis of A/B, then $\rho(Y)$ is a basis of $\rho(A/B)$. Therefore, if R is a basis of B then $\rho(S) \cup R$ is a basis of A and we are done. ■

An ascending chain of sets $A_0 \subseteq A_1 \subseteq \ldots A_\mu \subseteq \ldots$ with $\mu < \alpha$ is called a *smooth chain* if for every limit ordinal $\lambda < \alpha$, $A_\lambda = \bigcup_{\mu<\lambda} A_\mu$. It is said to be *strictly increasing* if for every $\mu < \alpha$; $A_\mu \neq A_{\mu+1}$. A family $\{A_\mu : \mu < \alpha\}$ is a chain of groups if for every $\mu < \alpha$, A_μ is a group which is a subgroup of $A_{\mu+1}$.

Theorem 1.35. *If $\{A_\mu : \mu < \alpha\}$ is a smooth chain of groups such that A_0 is free and $A_{\mu+1}/A_\mu$ is free for every $\mu < \alpha$, then $A = \bigcup_{\mu<\alpha} A_\mu$ is free. Moreover, for every $\mu < \alpha$, A/A_μ is free.*

Proof. A_0 is assumed to be free, so let S_0 be a basis of it. By transfinite induction we construct a smooth chain of sets $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_\mu \subseteq \ldots$ for $\mu < \alpha$ such that each S_μ is a basis of A_μ. Suppose we have already defined the chain up to some ordinal $\beta < \alpha$. If β is a limit, let $S_\beta = \bigcup_{\mu<\beta} S_\beta$. Then, S_β is a basis of $\bigcup_{\mu<\beta} A_\mu = A_\mu$. So suppose β is a successor ordinal, say $\beta = \delta + 1$. $A_{\delta+1}/A_\delta$ is free by hypothesis, so by Corollary 1.34, S_δ extends to a basis $S_{\delta+1}$ to $A_{\delta+1}$. Then, $S = \bigcup_{\mu<\alpha} S_\mu$ is a basis of A and $\{s + A_\mu : s \in X \setminus X_\mu\}$ is a basis of A/A_μ. ■

Recall that a group A is said to be *torsion* if all its elements are of finite order and is said to be *torsion-free* if all its elements are of infinite order.

Theorem 1.36 (See [Lan02], Theorem 8.4, p. 45)**.** *Every finitely-generated torsion-free group is free.*

Consequently, since every subgroup of a torsion-free group is torsion-free, every finitely-generated subgroup of a torsion-free group is free.

1.2.2 Basics of homological algebra

An *open complex* of groups is a sequence of groups and homomorphisms $\{(A^i, f^i)\}$

$$\to A^i \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \to$$

where i ranges over all integers, f^i maps A^i into A^{i+1} and $f^i \circ f^{i-1} = 0$ for all i. We say that the open complex is *exact* whenever $Ker(f^i) = Im(f^{i-1})$ for every i. One can consider finite sequences of homomorphism

$$A^1 \to \ldots \to A^n$$

but this can be made into a complex sequence by inserting the trivial groups at each end with the corresponding zero homomorphisms. These kind of complexes are called *short* or *finite sequences.*

$Hom(A, B)$ denotes the set of homomorphisms $A \to B$ for any two groups A, B. Together with the addition defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, the set $Hom(A, B)$ has a group structure. For every group C, $Hom(\cdot, C)$ is a contravariant functor (that is, a reversing arrow functor) from the category of abelian groups to the category of group homomorphisms into C, as every group homomorphism $f : A \to B$ induces a homomorphism $f' : Hom(B, C) \to Hom(A, C)$ given by $f'(g) = g \circ f$ for every $g \in Hom(B, C)$. Actually, from every sequence $A' \to A \to A''$ we get the induced sequence $Hom(A'', B) \to Hom(A, B) \to Hom(A', B)$. Moreover, the sequence $A' \to A \to A'' \to 0$ is exact if and only if $0 \to Hom(A'', B) \to Hom(A, B) \to Hom(A', B)$ is exact (equivalently, $0 \to A' \to A \to A''$ is exact if and only if $Hom(A'', B) \to Hom(A, B) \to Hom(A', B) \to 0$ is exact; see [Lan02], Propositions 2.1 and 2.2, pp. 122, 123). Fixing the other coordinate, note that if $0 \to B' \to B \to B''$ is a exact sequence, so is the sequence $0 \to Hom(A, B') \to Hom(A, B) \to Hom(A, B'')$.

Definition 1.37. A short exact sequence $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$ *splits* if the surjective homomorphism g splits.

It is easy to see that if $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$ splits, A is the direct sum of $Ker(f)$ and $Im(\rho)$, where ρ stands for the splitting function $\rho : A'' \to A$. Conversely, if $Ker(f)$ is a direct summand of M, the short exact sequence splits.

Proposition 1.38 (See [Lan02], Proposition 3.2, p. 132)**.** *Let* $0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$ *be an exact sequence, the following are equivalent:*

(1) *There exists a homomorphism* $\phi : A'' \longrightarrow A$ *such that* $g \circ \phi = id_A$*.*

(2) *There exists a homomorphism* $\psi : A \longrightarrow A'$ *such that* $\psi \circ f = id_{A'}$*.*

The following fact would also work if we took R-modules for any ring R. The satisfaction of these equivalent properties defines what a projective module is. We will simply talk about *projective groups*.

Proposition 1.39 (See [Lan02], p. 137)**.** *Let A be a group. The following are equivalent.*

(1) *Given a homomorphism $f : A \longrightarrow M''$ and a surjective homomorphism $g : M \longrightarrow M''$, there exists a homomorphism $h : A \longrightarrow M$ such that $g \circ h = f$.*

(2) *Every exact sequence $0 \to M' \to M'' \to A \to 0$ splits.*

(3) *A is a direct summand of a free group.*

Definition 1.40. A short exact sequence $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$, where F_1 is free is called a *free resolution*.

Note that since a free resolution is exact, f_0 is injective because $Ker(f_1) = Im(0) = 0$. Therefore, F_0 is a subgroup of F_1 so, by Theorem 1.31, is a free group.

Definition 1.41. Let $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$ be a free resolution. For any group C we define

$$Ext(A, C) = Hom(F_0, C)/Im(f_0'),$$

where f_0' denotes the induced homomorphism $f_0' : Hom(F_1, C) \longrightarrow Hom(F_0, C)$.

In Chapter 3, Ext will be shown to be very a useful tool to define Whitehead groups. We finish with the following result.

Theorem 1.42 (See [Lan02], Lemma 8.3, p. 809)**.** *Let $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$ be a free resolution (it is enough if it is exact) and let C be an arbitrary group. Then, there is an exact sequence*

$$0 \to Hom(A, C) \xrightarrow{f_1'} Hom(F_1, C) \xrightarrow{f_0'} Hom(F_0, C) \to$$
$$\to Ext(A, C) \longrightarrow Ext(F_1, C) \longrightarrow Ext(F_0, C) \longrightarrow 0$$

where f_1' and f_2' denote the induced homomorphisms by f_1 and f_2, respectively.

Chapter 2

Large cardinals and infinite abelian groups

Large cardinals axioms arise naturally in several contexts of the theory of infinite abelian groups. In this chapter we will see a characterization of the existence of measurable cardinals in terms of the group of homomorphisms from $\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}$ into $\mathbb{Z}$ discovered by Katsuya Eda in 1982. More precisely, Eda's theorem shows that there exists a cardinal κ such that $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z})$ is not the trivial group if and only if there exists a measurable cardinal. Since the existence of measurable cardinals cannot be proved in ZFC, the question of whether $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z})$ is the trivial group or not is independent of ZFC. In further sections we shall see how the so-called Dugas-Göbel cardinal relate to some large cardinals like measurables, strongly compact and δ-strongly compact cardinals. Throughout this chapter we will make use of highly useful techniques and results such as ultrapowers and the Wald-Łoś's Lemma.

2.1 Filters, ultrafilters and measurable cardinals

Definition 2.1. Let A be a non-empty set. The set $\mathcal{F}$ is a *filter* on A if $\mathcal{F} \subseteq \mathcal{P}(A)$ and

(1) $A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,

(2) $X \cap Y \in \mathcal{F}$ whenever $X, Y \in \mathcal{F}$,

(3) For every $X \in \mathcal{F}$ and $Y \in \mathcal{P}(A)$, if $X \subseteq Y$, then $Y \in \mathcal{F}$.

Note that filters on a set A formalize the idea of being a "big" subset of A. Indeed, (1) confirms the intuition that A is a big subset of A and excludes the empty set; (2) tells that two subsets of A are big only when their intersection is also big; (3) tells that if a subset X of A is big, so must be every set containing X.

Example. Clearly, $\{A\}$ is a filter on A, which we call the *trivial filter.* If $a \in \mathcal{P}(A)$ is a non-empty set, then $\mathcal{F} = \{X \in \mathcal{P} : a \subseteq X\}$ is a filter as well. If κ is an infinite cardinal, the *Fréchet filter* on κ is $\mathcal{F} = \{X \subseteq \kappa : |\kappa - X| < \kappa\}$.

Definition 2.2. A filter $\mathcal{F}$ on A is said to be *principal* if there exists some non-empty $Y \subseteq A$ such that $X \in \mathcal{F}$ if and only if $Y \subseteq X$.

It is easy to see that every filter on a finite set A is principal. Indeed, if A is finite, so is $\mathcal{F}$, hence $\bigcap \mathcal{F} \in \mathcal{F}$. Then, if $\mathcal{F}$ was non-principal, $\bigcap \mathcal{F} = \emptyset$. The Fréchet filter on κ is non-principal for every κ.

Definition 2.3. We say that a filter $\mathcal{F}$ on A is *maximal* if there is no proper filter on A containing $\mathcal{F}$. A filter $\mathcal{F}$ on a set A is an *ultrafilter* if it is maximal.

Note that if $\mathcal{F}$ is an ultrafilter and X is an arbitrary subset of A, then either X or its complement in A is a member of $\mathcal{F}$. Indeed, if there exists a subset X of A such that neither X nor $A \setminus X$ were in $\mathcal{F}$, then $X \cap Y \neq \emptyset$ for every $Y \in \mathcal{F}$. Otherwise, if $X \cap Y = \emptyset$, $Y \subseteq X \setminus A$, so $X \setminus A \in \mathcal{F}$, which contradicts our assumption. Then, since $X \cap Y \neq \emptyset$ for every $Y \in \mathcal{F}$, the intersection of any finite collection of members of $\mathcal{F} \cup \{X\}$ is non-empty. And it is easy to see that every collection of sets with this property can be extended to a filter. So let $\mathcal{G}$ be that filter. Of course, $\mathcal{F} \subseteq \mathcal{G}$, which contradicts the maximality of $\mathcal{F}$. Conversely, if for every $X \in \mathcal{P}(A)$ either $X \in \mathcal{F}$ or $A \setminus X \in \mathcal{F}$, $\mathcal{F}$ is clearly maximal, hence an ultrafilter. This proves the following:

Proposition 2.4. *A filter $\mathcal{F}$ is an ultrafilter on A if and only if for every $X \in \mathcal{P}(A)$, either $X \in \mathcal{F}$ or $A - X \in \mathcal{F}$.*

Theorem 2.5 (Tarksi)**.** *Every filter can be extended to an ultrafilter.*

Proof. Let $\mathcal{F}$ be an arbitrary filter on A. We can partially order, with respect to the inclusion relation, the set P of filters on A containing $\mathcal{F}$. Take C to be an arbitrary chain in P. Then $\bigcup C$ is a filter on A and an upper bound of C. By Zorn's Lemma, P has a maximal element, that is, an ultrafilter. ■

As next lemma shows, principal ultrafilters are easily characterizable.

Lemma 2.6. *An ultrafilter $\mathcal{F}$ on a is principal if and only if there exists some $a \in A$ such that $\mathcal{F} = \{X \subseteq A : a \in X\}$.*

Proof. The implication to the right is straightforward. Let $\mathcal{F}$ be a principal ultrafilter on a and let Y be a non-empty set such that $Y \subseteq X$ for all $X \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, if $a \in Y$, then $A - \{a\} \notin \mathcal{F}$ because $Y \nsubseteq A - \{a\}$, so $\{a\} \in \mathcal{F}$. Then $Y = \{a\}$, so $\mathcal{F} = \{X \subseteq A : a \in X\}$. ■

Clearly, if $\mathcal{F}$ is principal, the intersection of any arbitrary large collection of elements in $\mathcal{F}$ belongs to $\mathcal{F}$. This motivates the following definition.

Definition 2.7. Let κ be an infinite cardinal. A filter $\mathcal{F}$ on A is said to be *κ-complete* if the intersection of any family of less than κ-many members from $\mathcal{F}$ remains in $\mathcal{F}$.

Lemma 2.8. *If κ is singular, every κ-complete filter on κ is κ^+-complete.*

Proof. Let κ be singular and let $\mathcal{F}$ be a κ-complete filter on κ. It is enough to prove that the intersection of κ-many elements of $\mathcal{F}$ belongs $\mathcal{F}$. Let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{F}$ and let $S = \{\mu : \mu < cf(\kappa)\}$ be a cofinal sequence on κ. For every μ, let $Y_\mu = \bigcap_{\alpha<\mu} X_\alpha$. Since $\mathcal{F}$ is κ-complete and $\mu < cf(\kappa) < \kappa$, $Y_\mu \in \mathcal{F}$ for every $\mu < cf(\kappa)$. Also, since S is cofinal, for every $\alpha < \kappa$ there is some $\mu \in S$ such that $\alpha < \mu$. Therefore, $\bigcap_{\mu<cf(\kappa)} Y_\mu = \bigcap_{\alpha<\kappa} X_\alpha$. Again by the κ-completeness of $\mathcal{F}$, $\bigcap_{\mu<cf(\kappa)} Y_\mu \in \mathcal{F}$. Thus $\bigcap_{\alpha<\kappa} X_\alpha \in \mathcal{F}$ and $\mathcal{F}$ is κ^+-complete. ■

We can now introduce measurable cardinals, which will play an important role throughout this chapter.

Definition 2.9. A cardinal κ is *measurable* if it is uncountable and it has a κ-complete non-principal ultrafilter.

Measurable cardinals are large cardinals. Indeed, their existence implies the existence of inaccessible cardinals which, as we have already seen, cannot be proved in ZFC.

Theorem 2.10. *Measurable cardinals are inaccessible.*

Proof. From Definition 1.21, an inaccessible cardinal is a regular uncountable strong limit cardinal. Let κ be measurable. We already have that it is uncountable. Let us see that it is regular. By Proposition 1.19 (2), $\kappa = cf(\kappa)$ if and only if κ cannot be partitioned into less than κ-many sets of cardinality less than κ. Since κ is measurable, there exists a κ-complete non-principal ultrafilter $\mathcal{F}$ on κ.

Lemma 2.11. *Let κ, λ be infinite cardinals with $\lambda \leq \kappa$. If an ultrafilter $\mathcal{F}$ on κ is λ-complete, then for every partition $\{X_\alpha : \alpha < \mu\}$ of κ with $\mu < \lambda$ there exists an α such that $X_\alpha \in \mathcal{F}$.*

Proof of the lemma. For the sake of contradiction, suppose that there is no $X_\alpha \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $\kappa - X_\alpha \in \mathcal{F}$ for every $\alpha < \mu$. By λ-completeness, $\emptyset = \bigcap_{\alpha<\mu} \kappa - X_\alpha \in \mathcal{F}$. ■

Lemma 2.12. *Every set of a κ-complete non-principal ultrafilter on κ is of cardinality κ.*

Proof of the lemma: Assume on the contrary that $X \in \mathcal{F}$ is of cardinality less than κ. $\mathcal{F}$ is non-principal, so for every $\alpha \in X$ there is some $X_\alpha \in \mathcal{F}$ such that $\alpha \notin X_\alpha$. Otherwise, $\alpha \in \bigcap \mathcal{F}$. Since $|X| < \kappa$ and $\mathcal{F}$ is κ-complete, $\bigcap\{X_\alpha : \alpha < |X|\} \in \mathcal{F}$. But $X \cap \bigcap\{X_\alpha : \alpha < |X|\} = \emptyset$, which is a contradiction. ■

If follows from Lemma 2.11 that every partition of κ has an element X in $\mathcal{F}$. By Lemma 2.12, X has cardinality κ. Therefore, there is no partition of κ in less than κ-many sets of cardinality less than κ, hence κ is regular.

Let us see that κ is a strong limit. Suppose the contrary, that is, assume that there exists some $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$. Then there is a set $\{f_\alpha : \alpha < \kappa\}$ of κ-many functions $f_\alpha : \lambda \longrightarrow \{0,1\}$. Let $\mathcal{F}$ be a κ-complete non-principal ultrafilter. Then, for each $\beta < \lambda$ and each $\alpha < \lambda$, either $\{\alpha : f_\alpha(\beta) = 0\} \in \mathcal{F}$ or $\{\alpha : f_\alpha(\beta) = 1\} \in \mathcal{F}$ so let X_β be the one in $\mathcal{F}$ and let $\epsilon_\beta = 0$ if $\{\alpha : f_\alpha(\beta) = 0\} \in \mathcal{F}$, $\epsilon_\beta = 1$ otherwise. By κ-completeness, $\bigcap_{\beta<\lambda} X_\beta \in \mathcal{F}$. Note that the only element in $\bigcap_{\beta<\lambda} X_\beta$ is actually the ordinal α such that $f_\alpha(\beta) = \epsilon_\beta$. By Lemma 2.6, $\mathcal{F}$ is principal, which is a contradiction. We conclude that κ is a strong limit, hence inaccessible. ■

Remark. The converse of Lemma 2.11 also holds. We can prove it by induction on λ. By definition, every filter is $\aleph_0$-complete, so assume that $\mathcal{F}$ is λ-complete. We show that $\mathcal{F}$ is λ^+-complete. Let $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{F}$ and define the sequence $\{Y_\alpha : \alpha < \lambda\}$ as follows:

$$Y_0 = X_0,$$

$$Y_{\alpha+1} = Y_\alpha \cap X_{\alpha+1}$$

$$Y_\alpha = \textstyle\bigcap_{\beta<\alpha} Y_\beta \text{ if } \beta \text{ is a limit.}$$

Note that $\bigcap_{\alpha<\lambda} Y_\alpha = \bigcap_{\alpha<\lambda} X_\alpha$. Since $\mathcal{F}$ is λ-complete, $Y_\alpha \in \mathcal{F}$ for every $\alpha < \lambda$. Let $Z_\alpha = Y_\alpha - Y_{\alpha+1}$. It follows that $\{\kappa - X_0\} \cup \{Z_\alpha : \alpha < \lambda\} \cup \{\bigcap_{\alpha<\lambda} Y_\alpha\}$ is a partition of κ. Now, since $\mathcal{F}$ is an ultrafilter and $X_0 \in \mathcal{F}$, $\kappa - X_0 \notin \mathcal{F}$. Also, since $\kappa - Z_\alpha = \kappa - (Y_\alpha - Y_{\alpha+1}) = (\kappa - Y_\alpha) \cup Y_{\alpha+1} \in \mathcal{F}$, then $Z_\alpha \notin \mathcal{F}$ for any $\alpha < \lambda$. Therefore, since for every partition of κ in less than λ-many sets there must be at least one set in $\mathcal{F}$, it must be $\bigcap_{\alpha<\lambda} Y_\alpha \in \mathcal{F}$, that is, $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{F}$. Thus, $\mathcal{F}$ is λ^+-complete. For λ a limit cardinal the result follows trivially.

Remark. If $\mathcal{F}$ is an ultrafilter on κ all whose elements have cardinality κ, $\mathcal{F}$ is said to be *uniform*. Lemma 2.12 then says that every κ-complete non-principal ultrafilter on κ is uniform.

The following proposition will be useful in the next section.

Proposition 2.13. *If κ is the least cardinal for which there exists a non-principal ω_1-complete ultrafilter $\mathcal{F}$ on κ, then $\mathcal{F}$ is κ-complete.*

Proof. Let $\mathcal{F}$ be a ω_1-complete ultrafilter on κ. Assume that $\mathcal{F}$ is not κ-complete. By the Remark, there exists a partition $\{X_\alpha : \alpha < \lambda\}$ of κ with $\lambda < \kappa$ and $X_\alpha \notin \mathcal{F}$ for all $\alpha < \lambda$. Let $f : \kappa \longrightarrow \lambda$ be a surjective mapping given by $f(\alpha) = \beta$ if and only if $\alpha \in X_\beta$ for each $\alpha \in \kappa$. Then, the set $\mathcal{D} = \{X \in \mathcal{P}(\lambda) : f^{-1}(X) \in \mathcal{F}\}$ is a non-principal ultrafilter on λ. Indeed, since $f^{-1}(\lambda) = \kappa$, which is in $\mathcal{F}$, $\lambda \in \mathcal{D}$. Also, $\emptyset = f^{-1}(\emptyset)$ is not in in $\mathcal{D}$. If $X, Y \in \mathcal{D}$, then $f^{-1}(X), f^{-1}(Y) \in \mathcal{F}$, hence $f^{-1}(X) \cap f^{-1}(Y) \in \mathcal{F}$. But $f^{-1}(X) \cap f^{-1}(Y) = f^{-1}(X \cap Y)$, so $X \cap Y \in U$. And, if $f^{-1}(X) \in \mathcal{F}$ and $X \subseteq Y \subseteq \lambda$, then $f^{-1}(X) \subseteq f^{-1}(Y)$, so $Y \in \mathcal{D}$. This proves that $\mathcal{D}$ is a filter. To see that it is maximal just take an arbitrary $X \subseteq \lambda$. $\mathcal{F}$ is assumed to be an ultrafilter, so either $f^{-1}(X) \in \mathcal{F}$ or $\kappa - f^{-1}(X) \in \mathcal{F}$. If the former, $X \in \mathcal{D}$; if the latter, since $\kappa - f^{-1}(X) = f^{-1}(\lambda) - f^{-1}(X) = f^{-1}(\lambda - X)$, then $\lambda - X \in \mathcal{D}$. By Lemma 2.6, to see that $\mathcal{D}$ is non-principal it is just enough to see that there is no $\beta \in \lambda$ such that $\beta \in X$ for all $\{\beta\} \in \mathcal{D}$. Suppose the opposite. If $\{\beta\} \in \mathcal{D}$, then $f^{-1}(\{\beta\}) \in \mathcal{F}$. But $f^{-1}(\beta) = \{\alpha : f(\alpha) = \beta\} = \{\alpha : \alpha \in X_\beta\} = X_\beta$ which, by assumption, isn't in $\mathcal{F}$, so we have a contradiction. It remains to show that $\mathcal{D}$ is ω_1-complete. For this, let $\{Y_n : n < \omega\}$ be a partition of λ. If $Y_n \notin \mathcal{D}$ for any n, then $f^{-1}(Y_n) \notin \mathcal{F}$ for any $n < \omega$. But $\bigcup_{n<\omega} f^{-1}(Y_n) = \kappa$, which contradicts that $\mathcal{F}$ is ω_1-complete. We conclude that in λ there is a ω_1-complete non-principal ultrafilter, contradicting that κ was the least cardinal with that property. This is a contradiction. We therefore have that every partition of κ in less than κ-many parts has an element in $\mathcal{F}$, that is, $\mathcal{F}$ is κ-complete. ■

2.2 Eda's Theorem

For each $n < \omega$, e_n will denote the function $e_n : \omega \longrightarrow \{0, 1\}$ such that $e_n(m) = 1$ if and only if $m = 1$. Recall that for every two sets A, B, the set of functions $f : B \longrightarrow A$ is denoted by A^B. The *Baer-Specker group* is $\mathbb{Z}^\omega$ together with addition defined componentwise.

Lemma 2.14 (Specker, 1949). *If $h : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}$ is a homomorphism, then $h(e_n) = 0$ for all but finitely many n.*

Proof. We prove it by contradiction. Suppose that $h(e_n) \neq 0$ for infinitely many n. We may assume that $h(e_n) \neq 0$ for all n. Define a sequence $\langle k_n : n < \omega\rangle$ by letting $k_0 = 1$ and taking $k_{n+1} > k_n!|h(e_n)|$; and let $X = \{\sum x_n e_n \in \mathbb{Z}^\omega : \forall n(x_n = 0 \vee x_n = k_n!)\}$. We have that $|X| = 2^{\aleph_0}$. Then, since the function h takes values in $\mathbb{Z}$ and $|\mathbb{Z}| = \aleph_0$, there exist $\sum x_n e_n$ and $\sum y_n e_n$ such that $\sum x_n e_n \neq \sum y_n e_n$ and $h(\sum x_n e_n) = h(\sum y_n e_n)$. Since $\sum x_n e_n \neq \sum y_n e_n$ there must be a least m such that $x_m \neq y_m$. Then, $h((x_m - y_m)e_m) = -h(\sum_{i>m}(x_i - y_i)e_i)$, as h is a homomorphism. Since for all $n < \omega$, either $x_n = 0$ or $x_n = k_n!$ and $x_m \neq y_m$, either $x_m = 0$ and $y_m = k_m!$ or the contrary, so $x_m - y_m = $. Therefore, $h((x_m - y_m)e_m) = k_m! h(e_m)$. Since $k_{m+1} > k_m!|h(e_m)|$, it cannot happen that $h(e_m) = k_{m+1}c$ for an arbitrary $c \in \mathbb{Z}$, so k_{m+1} does not divide $h((x_m - y_m)e_m)$. But k_{m+1} divides $h(\sum_{i>m}(x_i - y_i)e_i)$ because for every $i > m$, if $x_i - y_i \neq 0$, then $x_i - y_i = k_i!$, to which k_{m+1} divides. This is a contradiction, so we are done. ■

Definition 2.15. A group G is *slender* if every homomorphism h from the Baer-Specker group into G is such that $h(e_n) = 0$ for all but finitely many n.

Specker's Lemma then shows that $\mathbb{Z}$ is a slender group. Specker's Lemma can be easily extended to homomorphisms from $\mathbb{Z}^\kappa$ into $\mathbb{Z}$, with κ uncountable. This provides a first example of a situation in which one can naturally jump to uncountable groups.

Corollary 2.16. *Let κ be an uncountable cardinal. If $h : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ is a homomorphism, then $h(e_\alpha) = 0$ for all but finitely many $\alpha < \kappa$.*

Proof. Suppose the opposite. Let A be a set of the form $\{\alpha_n : h(e_{\alpha_n}) \neq 0, n < \omega\}$. The restriction $h \restriction \mathbb{Z}^A : \mathbb{Z}^A \longrightarrow \mathbb{Z}$ is still a homomorphism. Since A is isomorphic to $\mathbb{Z}$, so are $\mathbb{Z}^A$ and $\mathbb{Z}^\omega$. This contradicts Lemma 2.14. ■

$\prod_{n<\omega} \mathbb{Z}$ simply denotes the set of ω-sequences of integers. We will denote by $\prod_{n \geq m} \mathbb{Z}$ the set of ω-sequences of integers whose first m elements are 0.

Lemma 2.17. *For every homomorphism $h : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}$, there exists some $m < \omega$ such that $h[\prod_{n \geq m} \mathbb{Z}] = \{0\}$.*

Proof. Towards a contradiction, for each m take an element $a \in \Pi_{n \geq m}$ with $h(a) \neq 0$. For each $r \in \prod_{n<\omega} \mathbb{Z}$, let $z(r) = (\sum_{m \leq n} r_m a_m)_{n<\omega} \in \prod_{n<\omega} \mathbb{Z}$. Clearly, $z(r) \in \Pi_{n<\omega}\mathbb{Z}$. Now, let the function $f : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}$ given by $f(r) = h(z(r))$. Note that f is an homomorphism and that for all $n < \omega$, $f(e_n) = h(a_n) \neq 0$, which contradicts Lemma 2.14. ■

Recall that the direct sum $\bigoplus_{i \in I} A_i$ is the subgroup of $\prod_{i \in I} A_i$ consisting of all I-sequences $(a_i)_{i \in I}$ with $a_i \in A_i$ for all $i \in I$ and $a_i = 0$ for all but finitely many $i \in I$.

Corollary 2.18. *$Hom(\prod_{n<\omega} \mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{n<\omega} \mathbb{Z}$.*

Proof. The mapping $\theta : Hom(\prod_{n<\omega} \mathbb{Z}, \mathbb{Z}) \longrightarrow \bigoplus_{n<\omega} \mathbb{Z}$ given by $\theta(h) = \sum_{n \leq m} r_n e_n$ with $r_n = h(e_n)$ and m such that $h[\prod_{n \geq m} \mathbb{Z}] = \{0\}$ is clearly an isomorphism. ■

The group $\mathbb{Z}^{<\omega}$ of finite sequences of integers is isomorphic to the subgroup A of $\mathbb{Z}^{\omega}$ of sequences of integers with all but finitely-many coordinates being 0, whenever the context is clear we will identify A with $\mathbb{Z}^{<\omega}$ with no further comment. The equivalence classes of the quotient $\mathbb{Z}^{\omega}/\mathbb{Z}^{<\omega}$ consists of vectors differing in just finitely many coordinates. Later on we will see that we are just simply dealing with ultrapowers.

Corollary 2.19. *$Hom(\mathbb{Z}^{\omega}/\mathbb{Z}^{<\omega}, \mathbb{Z}) = \{0\}$.*

Proof. Let $h \in Hom(\mathbb{Z}^{\omega}/\mathbb{Z}^{<\omega}, \mathbb{Z})$ and let h' be the function given by $h'(\sum r_n e_n) = h'([\sum r_n e_n])$, where $[\sum r_n e_n]$ is the equivalence class of $\sum r_n e_n$. Clearly, h' is an element in $Hom(\mathbb{Z}^{\omega}, \mathbb{Z})$, so let $m \in \omega$ such that $h'[\prod_{n\geq m} \mathbb{Z}] = \{0\}$. Then, $h([\sum r_n e_n]) = h([\sum_{n<m} r_n e_n]) + h([\sum_{n\geq m} r_n e_n]) = 0$, because $h([\sum_{n<m} r_n e_n]) = h([0])$ and $h([\sum_{n\geq m} r_n e_n]) = h'(\sum_{n\geq m} r_n e_n) = 0$. ■

According to Corollary 2.19, the only possible homomorphism between $\mathbb{Z}^{\omega}/\mathbb{Z}^{<\omega}$ and $\mathbb{Z}$ is the trivial one. We now wonder if this would still remain true if instead of considering the quotient of a product of countable-many copies of $\mathbb{Z}$ we consider the product of uncountable-many copies. The following result is due to Katsuya Eda and lies on results of Jerzy Łoś.

Theorem 2.20 (Eda, 1982)**.** *$Hom(\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ if and only if there exists an ω_1-complete non-principal ultrafilter on κ.*

Proof. We first prove the right to left implication. Let $\mathcal{F}$ be an ω_1-complete non-principal ultrafilter on κ and let the mapping $h : \mathbb{Z}^{\kappa}/\mathbb{Z}^{<\omega} \longrightarrow \mathbb{Z}$ with $h([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) = n$ if and only if $\{\alpha : r_\alpha = n\} \in \mathcal{F}$. We see that f is well-defined. Since $\mathcal{F}$ is ω_1-complete ultrafilter and $\{\{\alpha : r_\alpha = n\} : n \in \omega\}$ is a partition of κ in ω-many pieces, there is some $n \in \omega$ such that $\{\alpha : r_\alpha = n\} \in \mathcal{F}$, so f is defined on every class $[\sum_{\alpha<\kappa} r_\alpha e_\alpha]$. Let now $\sum_{\alpha<\kappa} s_\alpha e_\alpha \in [\sum_{\alpha<\kappa} r_\alpha e_\alpha]$, so both differ in just finitely many coordinates, so since $\{\alpha : r_\alpha = n\} \in \mathcal{F}$ and $\mathcal{F}$ is non-principal, then $\{\alpha : s_\alpha = n\} \in \mathcal{F}$. Indeed, if $\mathcal{F}$ was principal, since it is an ultrafilter, there would be some $\beta \in \kappa$ such that $\beta \in X$ for every $X \in \mathcal{F}$. Then, $\{\alpha : r_\alpha = n\} - \{\beta\} \notin \mathcal{F}$. To see that f is an homomorphism just let two different classes $[\sum_{\alpha<\kappa} r_\alpha e_\alpha]$ and $[\sum_{\alpha<\kappa} s_\alpha e_\alpha]$. First note that $[\sum_{\alpha<\kappa} r_\alpha e_\alpha] + [\sum_{\alpha<\kappa} s_\alpha e_\alpha] = [\sum_{\alpha<\kappa}(r_\alpha + s_\alpha)e_\alpha]$. Let $n_r = \{\alpha : r_\alpha = n_r\}$ and $n_s = \{\alpha : s_\alpha = n_s\}$ and assume that $n_r, n_s \in \mathcal{F}$. Then, $\{\alpha : r_\alpha = n_\alpha \text{ and } s_\alpha = n_s\} = n_r \cap n_s \in \mathcal{F}$. But $n_0 \cap n_1 \subseteq \{\alpha : r_\alpha + s_\alpha = n_r + n_s\}$, so $\{\alpha : r_\alpha + s_\alpha = n_r + n_s\} \in \mathcal{F}$. Therefore, $f([\sum_{\alpha<\kappa}(r_\alpha + s_\alpha)e_\alpha]) = n_r + n_s = f([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) + f([\sum_{\alpha<\kappa} s_\alpha e_\alpha])$. Note also that since for every $n \in \mathbb{Z}$, $f(n \sum_{\alpha<\kappa} e_\alpha) = n$, f is surjective.

For the converse suppose $f : \mathbb{Z}^\kappa/\mathbb{Z}^{<\omega} \longrightarrow \mathbb{Z}$ is a non-zero homomorphism. For every $Y \subseteq \kappa$ we denote by $\mathbb{Z}^\kappa \restriction Y$ the set of κ-sequences of integers whose non-zero coordinates are indexed by elements in Y. Let $S' = \{Y \subseteq \kappa : f[\mathbb{Z}^\kappa \restriction Y/\mathbb{Z}^{<\omega}] \neq 0\}$. By assumption, f is a non-zero homomorphism. Then $\kappa \in S'$, as $\mathbb{Z}^\kappa \restriction \kappa = \mathbb{Z}^\kappa$ and for every $Z \subseteq Y$ with $Y \in S'$, either $Z \in S'$ or $Y \setminus Z \in S'$ or both.

Claim. Every set of pairwise disjoint elements of S' is finite.

Proof of the claim. Towards a contradiction, let $\{Y_n : n \in \omega\} \subseteq S'$ with $Y_n \cap Y_m = \emptyset$ whenever $n \neq m$. For each $n \in \omega$, let $a^{(n)} \in \mathbb{Z}^\kappa \restriction Y_n$ with $f([a^{(n)}]) \neq 0$. Such an $a^{(n)}$ always exists. Let the function $h : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}$ given by $h(r) = f([\sum_{n\in\omega} r_n a^{(n)}])$. Then, $h(e_n) = f([a^{(n)}]) \neq 0$ for all $n \in \omega$, contradicting Specker's Lemma. ■

Let $S = \{Y \in S' : \forall Z \subseteq Y(\text{either } Z \in S' \text{ or } Y \setminus Z \in S' \text{ but not both})\}$. Equivalently, S might be seen as the set $\{Y \in S' : \forall Z \subseteq Y(f[\mathbb{Z}^\kappa \restriction Z/\mathbb{Z}^{<\omega}] = \{0\} \vee f[\mathbb{Z}^\kappa \restriction (Y \setminus Z)/\mathbb{Z}^{<\omega}] = \{0\})\}$. Note that S is not empty. Assume otherwise that $S = \emptyset$. Then, for every $Y \in S'$ there is some $Z \subseteq Y$ such that both Z and $Y \setminus Z$ belong to S'. Then, since $\kappa \in S'$ we take $Y_0 \subseteq \kappa$ such that both Y_0 and $\kappa - Y_0$ belong to S'. Then, let $Y_1 \subseteq \kappa - Y_0$ with $Y_1, (\kappa - Y_0) - Y_1 \in S'$ and so on. This way we build a family of ω-many pairwise-disjoint elements of S', which contradicts the claim above. Now, let $D = \{X \subseteq \kappa : X \cap Y \in S'\}$ for an arbitrary $Y \in S$.

Claim. D is an ω_1-complete non-principal ultrafilter on κ.

Proof of the claim. We first check that D is a filter. It is clear that $\emptyset \notin D$ and that D is upwards closed. Let $X_0, X_1 \in D$ and assume that $X_0 \cap X_1 \notin D$. Since D is upwards closed we might assume with no loss of generality that $X_0, X_1 \subseteq Y$. Then, $X_0 \in S'$, so $f[\mathbb{Z}^\kappa \restriction X_0/\mathbb{Z}^{<\omega}] \neq \{0\}$. Since $f([\mathbb{Z}^\kappa \restriction X_0/\mathbb{Z}^{<\omega}]) = f([\mathbb{Z}^\kappa \restriction (X_0 \cap X_1)/\mathbb{Z}^{<\omega}]) + f([\mathbb{Z}^\kappa \restriction (X_0 \setminus X_1)/\mathbb{Z}^{<\omega}])$ and, by assumption, $f([\mathbb{Z}^\kappa \restriction (X_0 \cap X_1)/\mathbb{Z}^{<\omega}]) = \{0\}$, we have that $f([\mathbb{Z}^\kappa \restriction (X_0 \setminus X_1)/\mathbb{Z}^{<\omega}]) \neq \{0\}$. Analogously, we can prove that $f([\mathbb{Z}^\kappa \restriction (X_1 \setminus X_0)/\mathbb{Z}^{<\omega}]) \neq \{0\}$, which implies that $f([\mathbb{Z}^\kappa \restriction Y \setminus (X_0 \setminus X_1)/\mathbb{Z}^{<\omega}]) \neq \{0\}$ as $X_1 \setminus X_0 \subseteq Y \setminus (X_0 \setminus X_1)$. But this contradicts that $Y \in S$ because $X_0 \setminus X_1 \subseteq Y$ and both $X_0 \setminus X_1$ and $Y \setminus (X_0 \setminus X_1)$ belong to S'.

To see that D is an ultrafilter, let $X \subseteq \kappa$ and suppose that $X \notin D$. Since $f[\mathbb{Z}^\kappa \restriction Y/\mathbb{Z}^{<\omega}] \neq \{0\}$ and $f[\mathbb{Z}^\kappa \restriction Y/\mathbb{Z}^{<\omega}] = f[\mathbb{Z}^\kappa \restriction (X \cap Y)/\mathbb{Z}^{<\omega}] + f[\mathbb{Z}^\kappa \restriction (Y \setminus X)/\mathbb{Z}^{<\omega}] = f[\mathbb{Z}^\kappa \restriction (Y \setminus X)/\mathbb{Z}^{<\omega}]$, then $Y \setminus X \in D$.

Now, let $\{X_n : n \in \omega\} \subseteq D$ and assume, for the sake of a contradiction, that $\bigcap_{n\in\omega} X_n \notin D$. Without loss of generality we might assume that $X_{n+1} \subseteq X_n \subseteq Y$ for all $n \in \omega$ and $\bigcap_{n\in\omega} X_n = \emptyset$. Since for each $n \in \omega$, $X_n \in D$, $f[\mathbb{Z}^\kappa \restriction X_n/\mathbb{Z}^{<\omega}] \neq \{0\}$. Let $a^{(n)} \in \mathbb{Z}^\kappa \restriction X_n$ with $f([a^{(n)}]) \neq 0$ and let $h : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}$ be as in the previous claim, which is a well-defined homomorphism because $\bigcap_{n<\omega} X_n = \emptyset$. Then $h(e_n) = f([a^{(n)}]) \neq 0$ for each $n \in \omega$, which contradicts Specker's Lemma. This shows that D is ω_1-complete. To see that D is non-principal, note $\{\alpha\} \notin D$ for any $\alpha \in \kappa$ because for every $a \in \mathbb{Z}^\kappa \upharpoonright \{\alpha\}$, then $[a] = [0]$, so $f[\mathbb{Z}^\kappa \upharpoonright \{\alpha\}] = \{0\}$. ∎

Consequently, if there is a non-zero homomorphism $f : \mathbb{Z}^\kappa/\mathbb{Z}^{<\omega} \longrightarrow \mathbb{Z}$, there exists an ω_1-complete non-principal ultrafilter on κ. ∎

Consequently, if κ is the least cardinal with $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$, then κ is the least cardinal having an ω_1-complete non-principal ultrafilter. By Proposition 2.13, this ultrafilter is κ-complete, so κ is measurable. That is:

Corollary 2.21. *There exists a cardinal κ such that $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ if and only if there exists a measurable cardinal. The least measurable cardinal is the least cardinal κ for which $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ holds.*

In particular, the question of whether there exists a cardinal κ such that the group $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\omega}, \mathbb{Z}) \neq \{0\}$ is independent of ZFC. Eda's Theorem can be extended to homomorphisms from $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ into $\mathbb{Z}$.

Theorem 2.22. *$Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) \neq \{0\}$ if and only if there exists an ω_1-complete uniform ultrafilter on κ.*

Proof. The same proof of Eda's Theorem works, although small changes are required. For the right to left implication we define the function $f : \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \longrightarrow \mathbb{Z}$ given by $f([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) = n$ if and only if $\{\alpha : r_\alpha = n\} \in \mathcal{F}$. However, for f to be well-defined $\mathcal{F}$ we need $\mathcal{F}$ to be uniform instead of non-principal. For the claims we can just consider families of κ-many pairwise-disjoint sets and define the function $h : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ analogously. Contradictions will still arise as Specker's Lemma still holds for those functions, as showed in Corollary 2.16. ∎

By Lemma 2.12 every set of a κ-complete non-principal ultrafilter on κ is of cardinality κ. It is also clear that κ-complete implies ω_1-complete. Therefore, if κ is measurable, $Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) \neq \{0\}$ and the least cardinal for which that holds is measurable.

Corollary 2.23. *$Hom(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \mathbb{Z}) = \{0\}$ for all κ below the first measurable.*

We finish this section with a generalization of Lemma 2.17 and Corollary 2.18.

Lemma 2.24. *For every homomorphism $h : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ there is a finite subset I of κ such that $h[\mathbb{Z} \upharpoonright (\kappa \setminus I)] = \{0\}$ if and only if there is no ω_1-complete non-principal ultrafilter on κ.*

Proof. By contraposition, if $\mathcal{F}$ is a ω_1-complete non-principal ultrafilter on κ, the function $h : \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}$ given by $h(\sum_{\alpha<\kappa} r_\alpha e_\alpha) = n$ if and only if $\{\alpha : r_\alpha = n\} \in \mathcal{F}$ is a non-zero homomorphism as for every $I \subseteq \kappa$, $h[\mathbb{Z}^\kappa \restriction (\kappa \setminus I)] = \mathbb{Z}$. Conversely, let h be a mapping from $\mathbb{Z}^\kappa$ to $\mathbb{Z}$ such that there is no finite subset I of κ with $h[\mathbb{Z} \restriction (\kappa \setminus I)] = \{0\}$. Let $S' = \{Y \subseteq \kappa : f[\mathbb{Z}^\kappa \restriction Y] \neq 0\}$ and $S = \{Y \in S' : \forall Z \subseteq Y(\text{either } Z \in S' \text{ or } Y \setminus Z \in S' \text{ but not both})\}$. Now, fix $Y \in S$ and let $D = \{X \subseteq \kappa : X \neq \emptyset \wedge X \cap \in S'\}$. D is an ω_1-complete ultrafilter. Since all cofinite sets are in D, it is non-principal. ■

Corollary 2.25. *If there is no ω_1-complete non-principal ultrafilter on κ, then the group $Hom(\prod_{\alpha\in\kappa} \mathbb{Z}, \mathbb{Z})$ is isomorphic to $\bigoplus_{\alpha\in\kappa} \mathbb{Z}$. In particular, if κ is measurable, then $Hom(\prod_{\alpha\in\kappa} \mathbb{Z}, \mathbb{Z}) \not\cong \bigoplus_{\alpha\in\kappa} \mathbb{Z}$.*

Proof. Let $\theta : Hom(\prod_{\alpha\in\kappa} \mathbb{Z}, \mathbb{Z}) \longrightarrow \bigoplus_{\alpha\in\kappa} \mathbb{Z}$ given by $\theta(h) = \sum_{\alpha\in I} r_\alpha e_\alpha$ with $r_\alpha = h(e_\alpha)$ and $I \subseteq \kappa$ such that $h[\mathbb{Z} \restriction (\kappa \setminus I)] = \{0\}$. θ is an isomorphism. ■

2.3 Strongly compact and δ-strongly compact cardinals

Definition 2.26. An uncountable cardinal κ is said to be *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

The given characterization of strongly compact cardinals is due to Keisler and Tarski. As we shall see, strongly compact cardinals are measurable cardinals, hence its existence cannot be proven from ZFC. Let us first see some preliminary results.

Definition 2.27. Let κ be a cardinal. A *κ-additive measure* on a set S is a function $\theta : \mathcal{P}(S) \longrightarrow [0, 1]$ such that

(1) $\theta(S) = 1$,

(2) $\theta(\{x\}) = 0$ for every $x \in S$, and

(3) $\theta(\bigcup_{\alpha<\kappa} X_\alpha) = \sum_{\alpha<\kappa} \theta(X_\alpha)$ for every family $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(S)$ of pairwise-disjoint sets.

Note that if κ has a κ-complete non-principal ultrafilter $\mathcal{F}$, the function $\theta : \mathcal{P}(\kappa) \longrightarrow \{0, 1\}$ given by $\theta(X) = 1$ iff $X \in \mathcal{F}$ is a κ-additive measure. Clearly, $\theta(\kappa) = 1$ and $\theta(\{x\}) = 0$ because $\mathcal{F}$ is non-principal. To see (3) let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\kappa)$ be a family of pairwise-disjoint sets. We see that $\theta(\bigcup_{\alpha<\kappa} X_\alpha) = \sum_{\alpha<\kappa} \theta(X_\alpha)$. Clearly, if $X_\alpha \in \mathcal{F}$ for some $\alpha < \kappa$ (note that, since $\mathcal{F}$ is an ultrafilter, this α must be unique), then $\bigcup_{\alpha<\kappa} X_\alpha \in \mathcal{F}$. Therefore, if $\sum_{\alpha<\kappa} \theta(X_\alpha) = 1$, then $\theta(\bigcup_{\alpha<\kappa} X_\alpha) = 1$. By contraposition, if $\sum_{\alpha<\kappa} \theta(X_\alpha) = 0$, there is no $\alpha < \kappa$ such that $\theta(X_\alpha) = 1$. Then $\kappa - X_\alpha \in \mathcal{F}$ for every $\alpha < \kappa$. By κ-completeness, $\bigcap_{\alpha<\kappa} \kappa - X_\alpha \in \mathcal{F}$, so $\kappa - \bigcup_{\alpha<\kappa} X_\alpha \in \mathcal{F}$, that is, $\bigcup_{\alpha<\kappa} X_\alpha \notin \mathcal{F}$ and $\theta(\bigcup_{\alpha<\kappa} X_\alpha) = 0$.

Remark. If θ is a κ-additive measure, $\theta(X) = 0$ for every X of cardinality less than κ. Then, every κ admitting a κ-additive measure is regular. Indeed, let $\{X_\alpha : \alpha < \lambda\} \subseteq \kappa$ with $\lambda < \kappa$ and $|X_\alpha| < \kappa$ for every $\alpha < \lambda$. Then $|\bigcup_{\alpha<\lambda} X_\alpha| < \kappa$ because $\theta(\bigcup_{\alpha<\lambda} X_\alpha) = 0$ and $\theta(\kappa) = 1$.

Lemma 2.28 (Ulam, 1930). *For any λ there is a collection of sets $\{A^\xi_\alpha | \alpha < \lambda^+ \wedge \xi < \lambda\} \subseteq \mathcal{P}(\lambda^+)$ satisfying:*

(1) $A^\xi_\alpha \cap A^\xi_\beta = \emptyset$ *whenever $\alpha < \beta < \lambda^+$ and $\xi < \lambda$; and*

(2) $|\lambda^+ - \bigcup_{\xi<\lambda} A^\xi_\alpha| \leq \lambda$ *for each $\alpha < \lambda^+$*

Proof. For each $\mu < \lambda^+$ let $f_\mu : \lambda \longrightarrow \mu + 1$ be a surjective map and for each $\alpha < \lambda^+$ and $\xi < \lambda$ let $A^\xi_\alpha = \{\mu < \lambda^+ : f_\mu(\xi) = \alpha\}$. (1) follows immediately from (2). For (2) note that $(\lambda^+ - \bigcup_{\xi<\lambda} A^\xi_\alpha) \subseteq \alpha$. ■

The collection of sets $\{A^\xi_\alpha : \alpha < \lambda^+ \wedge \xi < \lambda\}$ is known as *Ulan matrix.*

Lemma 2.29 (Ulam, 1930). *If there is a κ-additive measure on κ, then κ is weakly inaccessible.*

Proof. From the remark, κ is already known to be regular. Let us see that it is a limit cardinal. Towards a contradiction, assume that $\kappa = \lambda^+$. Consider an Ulam matrix and let θ be a κ-additive measure on κ. Then, by (2) above, for each $\alpha < \lambda^+$ there is a $\xi_\alpha < \lambda$ such that $\theta(A^{\xi_\alpha}_\alpha) > 0$, so there must be some $\xi < \lambda$ such that $\xi = \xi_\alpha$ for many λ^+-many α's. Then, by (1) above, there is an uncountable set T whose elements are of θ-measure greater than 0 for which there is no $X, Y \in T$ with $\theta(X \cap Y) > 0$, which is impossible. ■

Theorem 2.30 (Keisler-Tarski, 1964). *Strongly compact cardinals are regular.*

Proof. Assume that κ is singular. Let $\mathcal{F}'$ be a κ-complete ultrafilter on κ^+ extending the κ-complete Fréchet filter $\mathcal{F} = \{X \subseteq \kappa^+ : |\kappa^+ - X| < \kappa^+\}$. Note that $\mathcal{F}$ is non-principal and so is $\mathcal{F}'$. By Lemma 2.8, $\mathcal{F}'$ is κ^+-complete, so κ^+ is measurable for it has a non-principal κ^+-complete ultrafilter, hence a κ^+-additive measure. By 2.29, κ^+ is weakly inaccessible which is impossible because κ^+ is not a limit. ■

Theorem 2.31 (Erdös-Tarski, 1943). *Strongly compact cardinals are measurable.*

Proof. By the previous, κ is regular so $\{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is κ-complete. Since it can be extended to a κ-complete ultrafilter, κ is measurable. ■

Consequently, the existence of strongly compact cardinals is independent of ZFC. We introduce now a weaker large cardinal notion.

Definition 2.32 (Bagaria-Magidor, 2013)**.** Let $\delta < \kappa$ be uncountable cardinals which might be singular. We say that κ is *δ-strongly compact* if for every set S, every κ-complete filter in S can be extended to a δ-complete ultrafilter on S. An uncountable limit cardinal κ is *almost strongly compact* if κ is δ-strongly compact for every uncountable cardinal $\delta < \kappa$.

Clearly, if κ is κ-strongly compact, then it is strongly compact. Note as well that if λ is a cardinal greater than κ and κ is δ-strongly compact, then, since every λ-complete filter is κ-complete, λ is δ-strongly compact as well. Note that if κ is a regular ω_1-strongly compact cardinal, since $\{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is a κ-complete non-principal ultrafilter, it extends to a ω_1-complete non-principal ultrafilter, so, by Proposition 2.13, there exists a measurable cardinal less than or equal to κ.

Proposition 2.33. *If κ is ω_1-strongly compact and λ is the first measurable, then κ is λ-strongly compact.*

Proof. Assume not. Then there is a λ-complete filter on a set I which is not extended to a ω_1-complete ultrafilter or, equivalently, no ω_1-complete ultrafilter extends a λ-complete filter. Let $\mathcal{F}$ be a ω_1-complete ultrafilter. Since it doesn't extend a λ-complete filter, by Lemma 2.11 there is a partition $\{X_\alpha : \alpha < \beta\}$ of I with $\beta < \lambda$ such that $X_\alpha \notin \mathcal{F}$ for any $\alpha < \beta$. Then, the set $\mathcal{U} = \{X \subseteq \beta : \bigcup\{X_\alpha : \alpha \in X\} \in \mathcal{F}\}$ is non-principal ω_1-complete ultrafilter on β. We first see that it is an ultrafilter. Let $Y \notin \mathcal{U}$. Then, $\bigcup_{\alpha\in Y} X_\alpha \notin \mathcal{F}$, so $S \setminus \bigcup_{\alpha\in Y} X_\alpha \in \mathcal{F}$, because $\mathcal{F}$ is an ultrafilter. Since $\{X_\alpha : \alpha < \beta\}$ is a partition of S, $S \setminus \bigcup_{\alpha\in Y} X_\alpha = \bigcup_{\beta\setminus Y} X_\alpha$, hence $\beta \setminus Y \in \mathcal{U}$. From this it easily follows that $\mathcal{F}$ is non-principal, because $X_\alpha \notin \mathcal{F}$ for any $\alpha \in \beta$. It remains to show that $\mathcal{U}$ is ω_1-complete. Let $\{Y_n : n \in \omega\} \subseteq \mathcal{U}$, so that $\bigcup_{\alpha\in Y_n} X_\alpha \in \mathcal{F}$ for every $n \in \omega$. Then, since $\mathcal{F}$ is ω_1-complete, $\bigcap_{n\in\omega}(\bigcup_{\alpha\in Y_n} X_\alpha) \in \mathcal{F}$. Note that if $a \in \bigcap_{n\in\omega}(\bigcup_{\alpha\in Y_n} X_\alpha)$, then $a \in \bigcup_{\alpha\in Y_n} X_\alpha$ for every $n \in \omega$. But then, $a \in \bigcup_{\alpha\in\bigcap Y_n} X_\alpha$, so $\bigcap_{n\in\omega}(\bigcup_{\alpha\in Y_n} X_\alpha) \subseteq \bigcup_{\alpha\in\bigcap Y_n} X_\alpha$, hence $\bigcup_{\alpha\in\bigcap Y_n} X_\alpha \in \mathcal{F}$ and $\bigcap_{n\in\omega} Y_n \in \mathcal{U}$. The existence of a non-principal ω_1-complete ultrafilter on β with $\beta < \lambda$ contradicts that λ is the first measurable. ■

Magidor proved in [Mag76] that it is consistent that the least measurable is ω_1-strongly compact. In that case, if κ was such a cardinal, κ would actually be strongly compact by Proposition 2.33.

Notation 2.34. *Let S be a non-empty set. $\mathcal{P}_\kappa(S)$ denotes the set of subsets of S with cardinality less than κ, that is, $\mathcal{P}_\kappa(S) = \{x \subseteq S : |x| < \kappa\}$. By X_a we denote the set $\{x \in \mathcal{P}_\kappa(S) : a \in x\}$.*

Let S be a non-empty set and let κ be a regular δ-strongly compact. Then, since κ is regular, the family of sets $\{X_a : a \in S\}$ generates a κ-complete filter $\mathcal{F}$ on $\mathcal{P}_\kappa(S)$ by closing upwards and under intersection of κ-many elements. Indeed, by closing under intersection of κ-many elements, we first get from the collection of sets $\{X_a : a \in S\}$ a family of sets of the form $X_A = \{x \in \mathcal{P}_\kappa(S) : \exists A \in \mathcal{P}_\kappa(S)(A \subseteq x)\}$. Then, by closing upwards we get $\mathcal{F} = \{Y \subseteq \mathcal{P}_\kappa(S) : \exists A \in \mathcal{P}_\kappa(S)(X_A \subseteq Y)\}$. Clearly, $\mathcal{F}$ is κ-complete. Then, since κ is δ-strongly compact, $\mathcal{F}$ can be extended to a δ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(S)$.

Definition 2.35. A δ-complete ultrafilter $\mathcal{F}$ on $\mathcal{P}_\kappa(S)$ containing the sets X_a for each $a \in S$ is called a *δ-complete fine measure* on $\mathcal{P}_\kappa(S)$. The *fineness* condition is that $X_a \in \mathcal{F}$ for all $a \in S$.

Then, by the previous we have the following.

Proposition 2.36. *If κ is a regular δ-strongly compact cardinal, then for every set there is a δ-complete fine measure on $\mathcal{P}_\kappa(S)$.*

The converse also holds. In fact, the result is still true if we drop the condition of κ being regular. This is a particular case of Theorem 2.42, which provides a nice characterization of δ-strongly compact. To go into detail we first need to introduce the useful notion of ultraproducts and ultrapowers and some important results.

Let $\mathcal{F}$ be an ultrafilter over a set I and let $\{X_i : i \in I\}$ be a collection of first-order structures. By $\prod_{i\in I} X_i$ we denote the set of functions f with domain I such that $f(i) \in X_i$ for every $i \in I$. Define the relation

$$f \sim_{\mathcal{F}} g \text{ if and only if } \{i \in I : f(i) = g(i)\} \in \mathcal{F},$$

which is easily seen to be an equivalence relation whose equivalence classes are denoted by $[f]_{\mathcal{F}}$ for each $f \in \prod_{i\in I} X_i$. We construct a first-order structure with universe $\prod_{i\in I} X_i/\mathcal{F}$ in the language of the structures $\{X_i : i \in I\}$ by interpreting

$R^{\prod_{i\in I} X_i/\mathcal{F}}([f_1]_{\mathcal{F}}, \ldots, [f_n]_{\mathcal{F}})$ if and only if $\{i \in I : R^{X_i}(f_1(i), \ldots, f_n(i))\} \in \mathcal{F}$ for every relation symbol R,

$F^{\prod_{i\in I} X_i/\mathcal{F}}([f_1]_{\mathcal{F}}, \ldots, [f_n]_{\mathcal{F}}) = f_{\mathcal{F}}$ with $f_{\mathcal{F}}(i) = F^{X_i}(f_1(i), \ldots, f_n(i))$ for all $i \in I$ for every function symbol F, and

$c^{\prod_{i\in I} X_i/\mathcal{F}} = [f]_{\mathcal{F}}$ where $f(i) = c^{X_i}$ for all $i \in I$, for every constant symbol c.

Definition 2.37. Let $\mathcal{F}$ be a filter on I and let $\{X_i : i \in I\}$ be a collection of classes. The *reduced product* of $\{X_i : i \in I\}$ by $\mathcal{F}$ is the quotient $\prod_{i\in I} X_i/\sim_{\mathcal{F}}$, which we denote by $\prod_{i\in I} X_i/\mathcal{F}$. If $\mathcal{F}$ is an ultratilfer, the reduced product of $\{X_i : i \in i\}$ is instead called the *ultraproduct* of $\{X_i : i \in i\}$ by $\mathcal{F}$. If $X_i = X$ for every $i \in I$, we denote $\prod_{i\in I} X$ by X^I and the ultraproduct, which i this case we call *ultrapower* of X by $\mathcal{F}$, by $Ult_{\mathcal{F}}(X)$.

Next theorem shows how important ultraproducts are.

Theorem 2.38 (Łoś, 1955)**.** *Let $\mathcal{F}$ be an ultrafilter on a set I and let $\{X_i : i \in I\}$ be a collection of first-order structures in the language $\mathcal{L}$. Let $\varphi(x_1, \dots, x_n)$ be a $\mathcal{L}$-formula and let $f_1, \dots, f_n \in \prod_{i\in I} X_i$. Then,*

$$(*) \quad \prod_{i\in I} X_i/\mathcal{F} \vDash \varphi([f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}}) \text{ if and only if } \{i \in I : X_i \vDash \varphi(f_1(i), \dots, f_n(i))\} \in \mathcal{F}.$$

Proof. We go by induction on the complexity of φ. We first start with the atomic formulas:

$x_1 = x_2$:
$\prod_{i\in I} X_i/\mathcal{F} \vDash [f]_{\mathcal{F}} = [g]_{\mathcal{F}}$ if and only if $f \sim g$ if and only if $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$ if and only if $\{i \in I : X_i \vDash f(i) = g(i)\} \in \mathcal{F}$.

For a predicate $R(x_1, \dots, x_n)$:
$\prod_{i\in I} X_i/\mathcal{F} \vDash R([f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$ if and only if $R^{\prod_{i\in I} X_i/\mathcal{F}}([f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$ if and only if $\{i \in I : R^{X_i}(f_1(i), \dots, f_n(i))\} \in \mathcal{F}$ if and only if $\{i \in I : X_i \vDash R(f_1(i), \dots, f_n(i))\} \in \mathcal{F}$

Now assume that $(*)$ holds for φ and ψ. Recall that since $\mathcal{F}$ is an ultrafilter on I, $X \in \mathcal{F}$ if and only if $I \setminus X \notin \mathcal{F}$. Since the logical connectives $\vee, \to$ and $\leftrightarrow$ can be written in terms of $\neg$ and $\wedge$, it is enough proving only those cases.

$\neg\varphi$: $\prod_{i\in I} X_i/\mathcal{F} \vDash \neg\varphi([f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$ if and only if $\prod_{i\in I} X_i/\mathcal{F} \nvDash \varphi([f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$ if and only if, by Induction Hypothesis, $\{i \in I : X_i \nvDash \varphi(f_1(i), \dots, f_n(i))\} \in \mathcal{F}$.

$\varphi \wedge \psi$: $\prod_{i\in I} X_i/\mathcal{F} \vDash \varphi \wedge \psi$ if and only if $\prod_{i\in I} X_i/\mathcal{F} \vDash \varphi$ and $\prod_{i\in I} X_i/\mathcal{F} \vDash \psi$ if and only if, by Induction Hypothesis, $\{i \in I : X_i \vDash \varphi\} \in \mathcal{F}$ and $\{i \in I : X_i \vDash \psi\} \in \mathcal{F}$ if and only if $\{i \in I : X_i \vDash \varphi \wedge \psi\} \in \mathcal{F}$.

For the existencial quantifier it is enough proving that if $(*)$ holds for $\varphi(y, x_1, \dots, x_n)$, it also holds for $\exists y\varphi$. Assume that $\prod_{i\in I} X_i/\mathcal{F} \vDash \exists y\varphi(y, [f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$. Then, there exists some $g \in \prod_{i\in I} X_i$ such that $\prod_{i\in I} X_i/\mathcal{F} \vDash \varphi(g_{\mathcal{F}}, [f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$, so $\{i \in I : X_i \vDash \varphi(g(i), f_1(i), \dots, f_n(i))\} \in \mathcal{F}$, hence $\{i \in I : \exists y\varphi(y, f_1(i), \dots f_n(i))\} \in \mathcal{F}$. Conversely, if $\{i \in I : \exists y\varphi(y, f_1(i), \dots f_n(i))\} \in \mathcal{F}$, for each $i \in I$ let $x_i \in X_i$ be such that $X_i \vDash \varphi(x_i, f_1(i), \dots, f_n(i))$ if such x_i exists and arbitrary otherwise. Let the function $g \in \prod_{i\in I} X_i$ be given by $g(i) = x_i$. Then, $\{i \in I : X_i \vDash \varphi(g(i), f_1(i), \dots, f_n(i))\} \in \mathcal{F}$, hence $\prod_{i\in X} X_i/\mathcal{F} \vDash \exists y\varphi(y, [f_1]_{\mathcal{F}}, \dots, [f_n]_{\mathcal{F}})$. ■

As a consequence, the ultrapower of a model M is elementarily equivalent to M, meaning this that they satisfy the same sentences. Indeed, by Łoś' Theorem, if σ is a sentence in the language of M, and $\mathcal{F}$ is an ultrafilter on I, since $M_i = M$ for every $i \in I$, then $\{i \in I : M \vDash \sigma\}$ is either empty or I, so $Ult_{\mathcal{F}}(M) \vDash \sigma$ if and only if $M \vDash \sigma$.

Definition 2.39. Let M be an $\mathcal{L}$-structure and let $\mathcal{F}$ be an ultrafilter on I. The *canonical embedding* $j_{\mathcal{F}} : M \longrightarrow Ult_{\mathcal{F}}(M)$ is the mapping given by $j(x) = [f]_{\mathcal{F}}$ where $f : I \longrightarrow M$ is the constant function $f(i) = x$ for all $i \in I$.

Given two $\mathcal{L}$-structures M and N, an *elementary embedding* is just a mapping $f : M \longrightarrow N$ such that for every formula $\varphi(\bar{x})$, where $\bar{x}$ denotes an arbitrary tuple of coordinates, and every $\bar{a} \in M^{|\bar{a}|}$, $M \vDash \varphi(\bar{a})$ if and only if $N \vDash \varphi(f(\bar{a}))$, where $f(\bar{a}) = (f(a_i))_{i \in |\bar{a}|}$. that is, an elementary embedding is an embedding preserving all formulas. It follows from Łoś' Theorem that the canonical embedding $j : M \longrightarrow Ult_{\mathcal{F}}(M)$ is an elementary embedding. Indeed, if $x \in M$, $Ult_{\mathcal{F}}(M) \vDash \varphi(j(x))$ if and only if $Ult_{\mathcal{F}}(M) \vDash \varphi([f]_{\mathcal{F}})$ (with f the constant function described above) if and only if $M \vDash \varphi(x)$.

Defining the ultrapower of the universe V of all sets by an ultrafilter $\mathcal{F}$ on a set I might be a bit more cumbersome. Note that if $f \in V^I$, the equivalence class $[f]_{\mathcal{F}} = \{g \in V^I : \{i \in I : g(i) = f(i)\} \in \mathcal{F}\}\}$ is a proper class. To overcome this problem, we let $[f]^0_{\mathcal{F}} = \{g \in [f]_{\mathcal{F}} : \forall h(h \in [f]_{\mathcal{F}} \rightarrow rk(g) \leq rk(h)))\}$, that is, the members of minimal rank of $[f]_{\mathcal{F}}$. Since $[f]^0_{\mathcal{F}}$ is a set for every $f \in V^I$, we can take the domain of the ultrapower to be the set $V^I/\mathcal{F} = \{[f]^0_{\mathcal{F}} : f : S \longrightarrow V\}$ and define the membership relation $\in_{\mathcal{F}}$ by $[f]^0_{\mathcal{F}} \in_{\mathcal{F}} [g]^0_{\mathcal{F}}$ if and only if $\{i \in I : f(i) = g(i)\} \in \mathcal{F}$.

Definition 2.40. If $\mathcal{F}$ is an ultrafilter on a set I, the *ultrapower of* V *by* $\mathcal{F}$ is the model $\langle V^I/\mathcal{F}, \in_{\mathcal{F}}\rangle$.

Łoś' Theorem still applies to this context, so $\langle V^I/\mathcal{F}, \in_{\mathcal{F}}\rangle$ is elementarily equivalent to V. Moreover, under some extra assumptions on the corresponding ultrafilter, the ultrapower of V by an ultrafilter $\mathcal{F}$ is isomorphic to an inner model of ZFC. Indeed, next result shows that the assumption of $\mathcal{F}$ being ω_1-complete is equivalent to the relation $\in_{\mathcal{F}}$ being well-founded.

Proposition 2.41. *$\mathcal{F}$ is ω_1-complete if and only if $\in_{\mathcal{F}}$ is well-founded.*

Proof. For the left to right direction, assume there is a countable (hence infinite) decreasing $\in_{\mathcal{F}}$-sequence $\{[f_n]^0_{\mathcal{F}} : n \in \omega\}$. Then, $[f_{n+1}]^0_{\mathcal{F}} \in_{\mathcal{F}} [f_n]^0_{\mathcal{F}}$ for every $n \in \omega$, hence $\{i \in I : f_{n+1}(i) \in f_n\} \in \mathcal{F}$ for every $n \in \omega$. Since $\mathcal{F}$ is an ω_1-complete ultrafilter on I, $\bigcap_{n \in \omega}\{i \in I : f_{n+1}(i) \in f_n(i)\} \in \mathcal{F}$, so it is non-empty. This leads to a infinite descending $\in$-sequence, which is not possible. Conversely, assume that $\mathcal{F}$ is not ω_1-complete and let $\{X_n : n \in \omega\} \subseteq \mathcal{F}$ such that $\bigcup_{n\in\omega} X_n \notin \mathcal{F}$. For each $k \in \omega$ we define the function $g_\kappa : S \longrightarrow V$ given by $g_\kappa(i) = n - k$ if $i \in (\bigcap_{m<n} X_m) \setminus X_n$ and $n \geq k$ or $g_k(i) = 0$ otherwise. It is easy to see that $\{i \in I : g_{k+1}(i) \in g_k(i)\} \supseteq \bigcap_{m\leq k} X_m - \bigcap_{n\in\omega} Xn \in \mathcal{F}$ for $k \in \omega$, hence $\{[g_n]^0_{\mathcal{F}} : n \in \omega\}$ is a infinite descending $\in_{\mathcal{F}}$-sequence, contradicting our assumption. ■

By the previous and the Mostowski Collapse, Theorem 1.23, if $\mathcal{F}$ is ω_1-complete, the ultrapower $\langle V^I/\mathcal{F}, \in_{\mathcal{F}}\rangle$ of V by $\mathcal{F}$ is isomorphic to a unique transitive model $\langle M, \in\rangle$. Therefore, for every $[f]^0_{\mathcal{F}}, [g]^0_{\mathcal{F}} \in \langle V^I/\mathcal{F}, \in_{\mathcal{F}}\rangle$, $[f]^0 \in_{\mathcal{F}} [g]^0$ if and only if $\pi([f]) \in \pi([g])$, where π stands for the collapsing map. We will denote each $[f]^0$ simply by $[f]$. Similarly, to simplify notation we will denote $\pi([f])$ by $[f]$ and $Ult_{\mathcal{F}}(V)$ will denote the ultrapower of V by $\mathcal{F}$. By an abuse of notation we will sometimes denote $Ult_{\mathcal{F}}(V)$ to the transitive class M to which it is isomorphic whenever $\mathcal{F}$ is ω_1-complete. Since the canonical embedding is an elementary embedding, so is the mapping $j = j_{\mathcal{F}} \circ \pi : V \longrightarrow Ult_{\mathcal{F}}(V) \cong M$, with M a transitive class. Therefore, if α is an ordinal, so is $j(\alpha)$ and if $\alpha < \beta$, then $j(\alpha) < j(\beta)$. Consequently, $\alpha < j(\alpha)$ for every ordinal α. Moreover, again by elementarity, $j(\alpha + 1) = j(\alpha) + 1$. It is also clear that $j(n) = n$ for every natural number. It is an easy consequence of $\mathcal{F}$ being ω_1-complete that $j(\omega) = \omega$, for if $[f] < \omega$, that is, if $f(x) < \omega$ for almost all $x \in S$, then there would be some $n \in \omega$ such that $f(x) = n$ for almost all $x \in S$. Using a similar argument, one can prove that if $\mathcal{F}$ is κ-complete, then $j(\delta) = \delta$ for every $\delta < \kappa$. The following theorem, the last of this section, characterizes δ-strongly compact cardinals in terms of δ-complete fine measures and elementary embeddings. Recall that the critical point of an elementary embedding is the first ordinal which does not map to itself.

Theorem 2.42 (Bagaria-Magidor, 2013)**.** *The following are equivalent for any uncountable cardinals $\delta < \kappa$:*

(1) *κ is δ-strongly compact.*

(2) *For every α greater than or equal to κ there exists a definable elementary embedding $j : V \longrightarrow M$, with M transitive, and a critical point greater than or equal to δ, such that j is definable in V, and there exists $D \in M$ such that $j''\alpha \subseteq D$ and $M \models$ "$|D| < j(\kappa)$", where $j''\alpha = \{j(\beta) : \beta \in \alpha\}$.*

(3) *For every set I there exists a δ-complete fine measure on $\mathcal{P}_\kappa(I)$.*

Proof. (1)⇒(2): Let κ be δ-strongly compact and let $\alpha \geq \kappa$. Suppose that there is a δ-complete fine measure $\mathcal{F}$ on $\mathcal{P}_\kappa(\alpha)$. Let $j_{\mathcal{F}} : V \longrightarrow M$ be the corresponding ultrapower embedding. By Proposition 2.41, $Ult_{\mathcal{F}}(V)$ is well-founded, hence isomorphic to a transitive class M. Let $\pi : Ult_{\mathcal{F}}(V) \longrightarrow M$ be the collapsing map and let $j = \pi \circ j_{\mathcal{F}}$. We show that j satisfies the conditions in (2). Define $D := \pi([h]_{\mathcal{F}})$, where $h : \mathcal{P}_\kappa(\alpha) \longrightarrow V$ is given by $h(a) = a$ for all $a \in \mathcal{P}_\kappa(\alpha)$. Since $X_a = \{x \in \mathcal{P}_\kappa(\alpha) : a \in x\} \in \mathcal{F}$, $j''\alpha = \{j(\beta) : \beta \in \alpha\} \subseteq D$. Indeed, if $x \in j''\alpha$, then $x = j(\beta)$ for some $\beta \in \alpha$. Since $j(\beta) = \pi(j_{\mathcal{F}}(\beta))$, $j(\beta) \in \pi([h]_{\mathcal{F}})$ if and only if $j_{\mathcal{F}}(\beta) \in_{\mathcal{F}} h$. This in turn happens if and only if $\{x \in \mathcal{P}_\kappa(\alpha) : j_{\mathcal{F}}(x) \in h(x)\} = \{x \in \mathcal{P}_\kappa(\alpha) : \beta \in x\} \in \mathcal{F}$. But $\{x \in \mathcal{P}_\kappa(\alpha) : \beta \in x\} = X_\beta$, so $j(\beta) \in \pi([h]_{\mathcal{F}})$, hence $j''\alpha \subseteq D$. Also, note that for every $x \in \mathcal{P}_\kappa(\alpha)$, $|h(x)| = |x| < \kappa$, so, by Łoś' Theorem, $Ult_{\mathcal{F}}(V) \vDash |[h]_{\mathcal{F}}| < j_{\mathcal{F}}(\kappa)$, hence $M \vDash |D| < j(\kappa)$.

The existence of δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$ for every $\alpha \geq \kappa$ under the assumption that κ is regular has been proved already, so we show that it always exists such a fine measure for κ singular. Note that if $\kappa \leq \beta < \alpha$ and $\mathcal{F}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$, then $\{X \subseteq \mathcal{P}_\kappa(\beta) : \{Y \in \mathcal{P}_\kappa : Y \cap \beta \in X\} \in \mathcal{F}\}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\beta)$, so fix $\alpha \geq \kappa$ and assume, with no loss of generality, that α is regular. The successor κ^+ is regular and it is clearly δ-strongly compact as well, so there is a δ-complete fine measure $\mathcal{F}^*$ on $\mathcal{P}_{\kappa^+}(\alpha)$. Let $j_* = \pi^* \circ j_{\mathcal{F}^*}$, where $j_{\mathcal{F}^*} : V \longrightarrow Ult_{\mathcal{F}^*}(V)$ is the corresponding ultrapower embedding and $\pi^* : Ult_{\mathcal{F}^*}(V) \longrightarrow N$ the corresponding collapsing map. The critical point of j_* is greater than or equal to δ. Define $D^* := \pi^*([h^*]_{\mathcal{F}^*})$ with $h^* : \mathcal{P}_{\kappa^+}(\alpha) \longrightarrow V$ is given by $h^*(a) = a$ for all $a \in \mathcal{P}_{\kappa^+}(\alpha)$. Again, one can show that $D^* \in N$, $j_*''\alpha \subseteq D^*$, and $N \vDash |D| < j(\kappa^+)$. Since $j(\kappa^+) = j(\kappa)^+$ by elementarity, $N \vDash |D| < j(\kappa)^+$. Now, if $\beta = \sup(j_*''\alpha)$, then $\beta \cap D^*$ is cofinal in β[1]. Since $N \vDash |D| < j(\kappa)^+$. By elementarity, since κ is singular, so is $j(\kappa)$. Then, $cf(\beta) < j(\kappa)$. Let C be a club of β such that $ot(C) = cf(\beta)$. Note that $j_*''\alpha$ contains all limit points of β of cofinality ω. Since $cf(\beta)$ is uncountable, $C \cap j_*''\alpha$ is unbounded in β. Therefore, the set $I = \{\gamma \in \alpha : j(\gamma) \in C\}$ is unbounded in α, so it has cardinality α. Now let $\mathcal{U} = \{X \subseteq \mathcal{P}_\kappa(I) : j_*(I) \cap C \in j_*(X)\}$, which is a δ-complete fine measure on $\mathcal{P}_\kappa(I)$. Since $|I| = \alpha$, $\mathcal{U}$ induces a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$, and we are done.

(2)⇒(3): Let us assume, with no loss of generality, that I is an ordinal α greater than or equal to κ. Let $j : V \longrightarrow M$ and D be as defined before and let $\mathcal{F} = \{X \subseteq \mathcal{P}_\kappa(\alpha) : D \in j(X)\}$, which is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$. Since $M \vDash |D| < j(\kappa)$, $\mathcal{F}$ is well-defined.

(3)⇒(1): Let $\mathcal{F}$ be a κ-complete filter over a set I. Again, we assume with no loss of generality that $\mathcal{F}$ is a κ-complete filter over $\alpha = |I|$. By assumption, there exists a δ-complete fine measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\mathcal{F})$. Let $j : V \longrightarrow Ult_{\mathcal{U}}(V)$ be the corresponding ultrapower embedding and let $\pi : Ult_{\mathcal{U}}(V) \longrightarrow M$ be the collapsing map, with M a

[1] A subset A of an ordinal α is said to be *cofinal in* α if $\sup(A) = \alpha$.

transitive model. Define D as before. Again, $j''\mathcal{F} \subseteq D$ and $M \models |D| < j(\kappa)$. By elementarity, $\mathcal{F}$ is $j(\kappa)$-complete in M, so $\bigcap(j(\mathcal{F}) \cap D) \neq \emptyset$. Let $\mathcal{V} = \{X \subseteq \alpha : a \in j(X)\}$, where a is a fixed element in $\bigcap(j(\mathcal{F}) \cap D)$. The set $\mathcal{V}$ is a δ-complete non-principal ultrafilter on α and it contains $\mathcal{F}$, because if $X \in \mathcal{F}$, then $j(X) \in D \cap j(\mathcal{F})$, so $a \in j(\mathcal{F})$. ■

2.4 The Dugas-Göbel cardinal

Let $\mathcal{A}$ be the category of abelian groups. A functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a *radical* if for every $G \in \mathcal{A}$, then $T(G/T(G)) = 0$, where 0 is the trivial group.

Definition 2.43. Let $X \in \mathcal{A}$. The *radical singly generated by* X, denoted by $\mathcal{R}_X$, is the functor $R_X : \mathcal{A} \longrightarrow \mathcal{A}$ given by $R_X(G) = \bigcap\{Ker(f) : f \in Hom(G, X)\}$.

Since the intersection of subgroups is a subgroup, $R_X(G)$ is a subgroup of G for every X, G. Note also that R_X is indeed a radical as $R_X(G/R_X(G)) = 0$ for all X, G. To see this it is enough to show that if $a \in R_X(G/R_X(G))$, then $a = 0$. So let $a \in R_X(G/R_X(G))$. Then, there is some $g \in G$ such that $a = g + R_X(G)$ and $f(a) = 0$ for all $f \in Hom(G/R_X(G), X)$. Let $h \in Hom(G, X)$. Note that if $\pi : G \longrightarrow G/R_X(G)$ is the canonical projection, then there exists a unique $f \in Hom(G/R_X(G), X)$ such that $h(g) = f(\pi(g))$. By assumption $f(\pi(g)) = f(a) = 0$. Since h was arbitrary, $g \in R_X(G)$, hence $a = 0$.

Definition 2.44. A group X is said to be *torsionless* if and only if the canonical homomorphism $X \longrightarrow X^{**}$ given by $x \mapsto (y \mapsto y(x))$, with $y \in Hom(X^*, \mathbb{Z})$ is injective.

We will denote the canonical homomorphism above by σ_X. The following result shows that the radical singly generated by $\mathbb{Z}$ is a useful tool to determine whether a given group G is torsionless.

Proposition 2.45. *G is torsionless if and only if $R_{\mathbb{Z}}(G) = 0$. Then, for any group G, $G/R_{\mathbb{Z}}(G)$ is torsionless.*

Proof. If G is torsionless, the mapping σ_G is injective. Let $x \in G$, then $\sigma_G(x) : Hom(G, \mathbb{Z}) \longrightarrow \mathbb{Z}$ such that $f \longmapsto f(x)$ for every $f \in Hom(G, \mathbb{Z})$. Since σ_G is injective, $\sigma_G(x) = 0$ if and only if $x = 0$. But then, $\bigcap\{Ker(f) : f \in Hom(G, \mathbb{Z})\} = \{0\}$, so $R_{\mathbb{Z}} = 0$. Conversely, if $R_{\mathbb{Z}}(G) = 0$ and $x, y \in G$ are such that $\sigma_G(x) = \sigma_G(y)$, then for every $f \in Hom(G, \mathbb{Z})$, $f(x) = f(y)$ so $f(x - y) = 0$ and $x - y = 0$ by assumption, so $x = y$, hence σ_G is injective. The rest follows immediately by the fact that R_X is a radical functor for every X. ■

Proposition 2.46. *$Hom(A, X) = \{0\}$ if and only if $R_X(A) = A$. In particular, $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) = \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for all κ smaller than the first measurable.*

Proof. Assume that $Hom(A, X) = \{0\}$. Since $R_X(A) \subseteq A$ for every A, let $a \in A$. Note that $R_X(A) = \bigcap\{Ker(f) : f \in Hom(A, X)\}$. Therefore, by assumption, since the only homomorpshim from A into X is the zero-function, $a \in R_X(A)$. On the other way around, if $R_X(A) = A$ then $f(a) = 0$ for every $f \in Hom(A, X)$ and every $a \in A$, hence $f = 0$. The rest follows by 2.23. ■

A group X is said to be *strongly cotorsion-free* if and only if $R_X(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) = \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for all uncountable regular cardinal κ up to the first measurable. Proposition 2.46 tells us that $\mathbb{Z}$ is strongly cotorsion-free. One then naturally wonders whether $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) = \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for κ greater than or equal to the first measurable. The following answers this in the negative.

Proposition 2.47. *If κ is greater than or equal to the first measurable cardinal λ, then $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda}) = 0$.*

Proof. Let $[a] \in \mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda}$ different to $[0]$. Then, there is some $X \subseteq \kappa$ of cardinality λ such that $a_\alpha \neq 0$ for all $\alpha \in X$. Since X is of cardinality λ, the first measurable, there is an ω_1-complete non-principal ultrafilter on X. Let $\mathcal{F}$ be this filter. Define the homomorphism $f : \mathbb{Z}^\kappa/\mathbb{Z}^{<\lambda}$ as in the proof of Eda's Theorem, that is, $f([\sum_{\alpha<\kappa} r_\alpha e_\alpha]) = n$ if and only if $\{\alpha : r_\alpha = n\} \in \mathcal{F}$. Clearly, $[a] \notin Ker(f)$. ■

Definition 2.48. Let κ be a cardinal and $X \in \mathcal{A}$, then

$$R_X^\kappa(A) = \sum\{R_X(B) : B \subseteq A, |B| < \kappa\}.$$

That is, $R_X^\kappa(A)$ is the group generated by the set of generators consisting of all elements in $R_X(B)$ with $B \subseteq A$ and $|B| < \kappa$. It is easy to see that $R_X(B) \subseteq R_X(A)$ whenever $B \subseteq A$. Consequently, for every A we have $R_X^\kappa(A) \subseteq R_X(A)$. Also, if A is of cardinality less than κ, it is clear that $R_X^\kappa(A) = R_X(A)$. This motivates the following definition.

Definition 2.49. Let X be a group. The *Dugas-Göbel cardinal for X* is the least cardinal κ such that $R_X = R_X^\kappa$. We say that R_X satisfies the *cardinal condition* whenever there exists such a cardinal.

It may happen that for some group X the Dugas-Göbel cardinal doesn't exist. Indeed, in ZFC the singly generated radical of non-zero strongly cotorsion-free groups does not satisfy the cardinal condition. More precisely:

Theorem 2.50. *If there are no measurable cardinals then R_X does not satisfy the cardinal condition for every non-zero strongly cotorsion-free X.*

Before proving Theorem 2.50, let us see the following important lemma. Recall that in the previous section we have defined the reduced product of a family of structures of the same language. Of course, this applies to the situation in which the family of structures consists of (abelian) groups, so let I be an infinite set and $\{X_i : i \in I\}$ be a family of I-many (abelian) groups. Recall that if $x \in \prod_{i\in I} X_i$, then $supp(x) = \{i \in I : x_i \neq 0\}$. Let $\mathcal{F}$ be a filter on I. Then, $X_{\mathcal{F}} = \{x \in \prod_{i\in I} X_i : I \setminus supp(x) \in \mathcal{F}\}$ is a subgroup of $\prod_{i\in I} X_i$. To see this, note that $x \in X_{\mathcal{F}}$ if and only if $\{i \in I : x_i = 0\} \in \mathcal{F}$. Then, if $x, y \in X_{\mathcal{F}}$, since $I \setminus supp(x + y) = \{i \in I : x_i + y_i = 0\} \supseteq \{i \in I : x_i = 0 \wedge y_i = 0\} = \{i \in I : x_i = 0\} \cap \{i \in I : y_i = 0\} = (I \setminus supp(x)) \cap (I \setminus supp(y))$, it follows that $I \setminus supp(x+y) \in \mathcal{F}$, so $x+y \in X_{\mathcal{F}}$. Similarly, $x \in X_{\mathcal{F}}$ if and only if $-x \in \mathcal{F}$. Then, the reduced product of $\{X_i : i \in I\}$ with respect to $\mathcal{F}$ is the quotient group $\prod_{i\in I} X_i/X_{\mathcal{F}}$. We denote it by $\prod_{i\in I} X_i/\mathcal{F}$.

Lemma 2.51 (Wald-Łoś Lemma)**.** *Let I be an infinite set, $\{X_i : i \in I\}$ a family of I-many non-trivial groups and $\mathcal{F}$ a λ-complete filter over I for some infinite cardinal λ. Then, every subgroup Y of $\prod_{i\in I} X_i/\mathcal{F}$ with $|Y| < \lambda$ is embeddable into $\prod_{i\in I} X_i$.*

Proof. Let Y be a subgroup of $\prod_{i\in I} X_i/\mathcal{F}$ of size less than λ and let $\bar{y}$ be a representative of each $y \in Y$. For every two $a, b \in Y$, we let the sets $Y_{a,b} = \{i \in I : \bar{a}_i + \bar{b}_i = \overline{(a+b)}_i\}$. Note that $Y_{a,b} \in \mathcal{F}$ for all $a, b \in Y$. Now, since $\mathcal{F}$ is λ-complete and $|Y| < \lambda$, then $Z = \bigcap_{a,b\in Y} Y_{a,b} \in \mathcal{F}$. Then, the function $f : Y \longrightarrow \prod_{i\in I} X_i$ given by $f(a)_i = \bar{a}_i$ if $i \in Z$, $f(a)_i = 0$ otherwise, is well defined and it is easy to see that f is an embedding. ■

The Wald-Łoś Lemma holds in a more general context. Namely, it also holds for families $\{M_i : i \in I\}$ of R-modules, generalizing the particular case we have taken. Recall that abelian groups are $\mathbb{Z}$-modules. The proof goes the same way, although Z must be defined as $\bigcap_{a,b\in Y} Y_{a,b} \cap \bigcap_{a\in Y, r\in R}\{i \in I : r\bar{a}_i = \bar{a}_{ri}\}$.

Recall that if $X_i = X$ for all $i \in I$ we denote $\prod_{i\in I} X_i$ by X^I. Note that $\mathbb{Z}^{<\kappa} = \{x \in \mathbb{Z}^\kappa : |supp(x)| < \kappa\}$ can be seen as $\mathbb{Z}^\kappa_{\mathcal{F}} = \{x \in \mathbb{Z}^\kappa : \kappa \setminus supp(x) \in \mathcal{F}\}$ where $\mathcal{F}$ is the filter consisting of the subsets $A \subseteq \mathbb{Z}^\kappa$ whose complement has cardinality less than κ. In this case, we will keep the notation $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ for the reduced product, as we have done in section 2.2. Note as well that if κ is regular, since $\mathcal{F}$ is non-principal, $\mathcal{F}$ is also κ-complete.

Proof of Theorem 2.50. Suppose the contrary and let λ such that $R^\lambda_X = R_X$ for a non-zero strongly cotorsion-free group X. Let $\kappa \geq \lambda$ be a regular cardinal. By assumption on X, $Hom(\mathbb{Z}^\kappa/\mathbb{Z}, X) = \{0\}$, that is, $R_X(\mathbb{Z}^\kappa/\mathbb{Z}) = \mathbb{Z}^\kappa/\mathbb{Z}$. Note that $\mathbb{Z}^\kappa$ is torsionless. To see this, assume that $\sigma_{\mathbb{Z}^\kappa}(x) = \sigma_{\mathbb{Z}^\kappa}(y)$ and consider the projections from each factor to $\mathbb{Z}$. By Proposition 2.45, $R_{\mathbb{Z}}(\mathbb{Z}^\kappa) = 0$. Every $B \subseteq \mathbb{Z}^\kappa/\mathbb{Z}$ with $|B| < \lambda$ is embeddable into $\mathbb{Z}^\kappa$ by the Wald-Łoś Lemma. Therefore, $R_{\mathbb{Z}}(B) = 0$ and $R^\lambda_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) = 0$. But $R_{\mathbb{Z}}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) \neq 0$ as seen in Proposition 2.46; contradiction. ∎

Corollary 2.52. *If it exist, the Dugas-Göbel cardinal of a strongly cotorsion-free group X is greater than or equal to the first measurable cardinal.*

In particular, since $\mathbb{Z}$ is strongly cotorsion-free, $R_{\mathbb{Z}}$ doesn't satisfy the cardinal condition in absence of some large cardinal assumptions. Therefore one can wonder whether being κ measurable, it holds that $R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$. Note that for every group X, if $\kappa < \lambda$ and $R^\kappa_X = R_X$, then $R^\lambda_X = R_X$ as well, so the question is equivalent to asking whether $R_{\mathbb{Z}} = R^\kappa_{\mathbb{Z}}$ holds when κ is the first measurable. The following result will clarify the situation.

Theorem 2.53 (Dugas, 1985)**.** *If κ is a strongly compact cardinal and X is an abelian group of cardinality smaller than κ, then $R_X = R^\kappa_X$.*

We see however a different version of this theorem which uses weaker large cardinal assumptions. It is due to Eda and Abe (see [EA87]) although the proof we provide is due to Bagaria and Magidor (see [BM13]).

Theorem 2.54 (Dugas-Eda-Abe, 1987)**.** *If κ is δ-strongly compact an X is an abelian group of cardinality smaller than δ, then $R_X = R^\kappa_X$. Hence, if κ is almost strongly compact, then $R_X = R^\kappa_X$ for every X of cardinality less than κ.*

Proof. (Bagaria-Magidor, 2013) Let κ be a δ-strongly compact cardinal and let X be an abelian group of cardinality smaller than δ. By taking an isomorphic copy of X if necessary, we may assume that $X \in H_\delta$. Let A be an arbitrary group. Since $R^\kappa_X(A) \subseteq R_X(A)$, we just have to prove that if $a \notin R^\kappa_X(A)$, then $a \notin R_X(A)$, so let $a \in A$ and assume that $a \notin R^\kappa_X(A)$. For each $B \in \mathcal{P}_\kappa(A)$ with $a \in \langle B\rangle$, the smallest group containing B, fix a homomorphism $f_B : \langle B\rangle \longrightarrow X$ such that $f_B(a) \neq 0$. If $a \notin B$, set $f_B(a) = 0$. Since κ is a δ-strongly compact cardinal, there is a δ-complete fine measure on $\mathcal{P}_\kappa(A)$. Let $\mathcal{F}$ be such δ-complete fine measure and let $j : V \longrightarrow Ult_{\mathcal{F}}(V)$ be the corresponding ultrapower embedding (see Definition 2.36). Since $\mathcal{F}$ is δ-complete, $Ult_{\mathcal{F}}(V)$ is well-founded and if π is its Mostowski collapse, $\pi \circ j = id^{H_\delta}_V$, that is, the identity of V on H_δ, so $j(X) \cong X$. In $Ult_{\mathcal{F}}(V)$, the domain of the function $[f] := [\langle f_B : B \in \mathcal{P}_\kappa(A)\rangle]$ is the subgroup $[\prod_{B\in\mathcal{P}_\kappa(A)}\langle B\rangle]$ of $j(A)$ and takes values in $j(X)$. Now, since $\mathcal{F}$ is fine, then $\{B \in \mathcal{P}_\kappa(A) : b \in B\} \in \mathcal{F}$ for every $b \in A$. Equivalently, $\{B \in \mathcal{P}_\kappa(A) : b \in \langle B\rangle\} \in \mathcal{F}$ for every $b \in A$. Since $j(b) = [g]$ where $g : \mathcal{P}_\kappa(V) \longrightarrow V$ is the constant function with value b, that is, a vector of length $\mathcal{P}_\kappa(V)$

with all its coordinates equal to b, then for each $b \in A$, $j(b) \in [\prod_{B\in\mathcal{P}_\kappa(A)}\langle B\rangle]$. In particular, $j(a) \in [\prod_{B\in\mathcal{P}_\kappa(A)}\langle B\rangle]$. Moreover, $[f](j(a)) \neq 0$ because $f_B(a) \neq 0$ for any $B \in \mathcal{P}_\kappa(A)$ with $a \in \langle B\rangle$. Let $\pi \restriction j(X) : j(X) \longrightarrow X$ be the isomorphism between $j(X)$ and X, it follows that $\pi \restriction j(X) \circ [f] \circ j \restriction A : A \to X$ doesn't take a to 0, so $a \notin R_X(A)$. The rest follows immediately. ■

Corollary 2.55. *If κ is ω_1-strongly compact and λ is the first measurable, then $R_X = R_X^\kappa$ for every X with size less than λ.*

Proof. By Proposition 2.33, if κ is ω_1-strongly compact and λ is least measurable λ then κ is λ-strongly compact. Then, the previous applies. ■

There is another important consequence of the previous theorem.

Proposition 2.56. *κ is ω_1-strongly compact if and only if $R_\mathbb{Z} = R_\mathbb{Z}^\kappa$.*

Proof. The left to right direction is immediate, for $\mathbb{Z}$ is of countable cardinality. For the right to left direction let S be an arbitrary set and let $\mathcal{F}$ be a κ-complete filter on S. By the Wald-Łoś Lemma, every subgroup B of $\mathbb{Z}^S/\mathcal{F}$ of cardinality less than κ is embeddable into $\mathbb{Z}^S$. By Proposition 2.45 $R_\mathbb{Z}(\mathbb{Z}^S) = 0$, so $R_\mathbb{Z}(B) = 0$ for every subgroup B with $|B| < \kappa$. Therefore, since $R_\mathbb{Z} = R_\mathbb{Z}^\kappa$, it follows that $R_\mathbb{Z}(\mathbb{Z}^S) = 0$. Arguing as in the left to right direction of the proof of Eda's Theorem, but taking a non-zero homomorphism f from $\mathbb{Z}^S/\mathcal{F}$ into $\mathbb{Z}$, we get a ω_1-complete ultrafilter extending $\mathcal{F}$, showing that κ is ω_1-strongly compact. ■

Then, the question asked before, whether $R_\mathbb{Z} = R_\mathbb{Z}^\kappa$ holds with κ the least measurable, is equivalent to the question whether the first ω_1-strongly cardinal is the first measurable. By building a model of ZFC in which the first ω_1-strongly compact cardinal is singular, they showed that it is consistent that the first ω_1-strongly compact cardinal is in between the the first measurable cardinal and the first strongly compact cardinal. However, the methods used are highly technical and go far beyond the scope of this work. May the reader be interested, see [BM13], section 6.

Chapter 3

The Whitehead's problem

In 1938, Kurt Gödel proved that in the constructible universe L the Continuum Hypothesis and the Axiom of Choice hold. Since L is a model of ZF, both CH and AC are therefore consistent with ZF. Two decades later, Paul J. Cohen developed the method of forcing with which he constructed a model of ZF where $\neg CH$ and $\neg AC$ hold, showing this way that AC and CH are undecidable in ZF. In this chapter we shall see another example of undecidability in the context of infinite abelian groups. Namely, we shall study Shelah's proof on the undecidability of the Whitehead's problem. To do this we will proceed in a similar manner as Gödel and Cohen did. On the one hand, we will show that the Whitehead's problem has a positive solution in $ZFC + V = L$. On the other, we will show that its solution is negative in $ZFC + MA$, where MA stands for Martin's Axiom.

3.1 W-groups and Stein's Theorem

In this section we introduce Whitehead's problem and see its solution for the countable case. We also study some features of the Whitehead groups.

Definition 3.1. A group A is said to be a *Whitehead-group*, W-group for short, if every homomorphism π onto A with $Ker(\pi) \cong \mathbb{Z}$ splits.

By Theorem 1.33, every homomorphism onto a free group splits. Consequently, every free group is a W-group. The Whitehead's Problem asks whether the converse is true. The countable case was answered in the positive by Stein in 1951. We shall study his proof, for it will provide a useful guide to find a positive answer for the uncountable case under the assumption of the Axiom of Constructibility. Some preliminary results will be needed. Recall that in Definition 1.41 we introduced the Ext of a free resolution $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$, defined by

$$Ext(A, C) = Hom(F_0, C)/Im(f_0')$$

where $f_0' : Hom(F_1, C) \longrightarrow Hom(F_0, C)$ is the homomorphism induced by f_0. Next theorem shows that W-groups can be characterized in terms of the vanishing of $Ext(\cdot, \mathbb{Z})$.

Theorem 3.2. *A group A is a W-group if and only if $Ext(A, \mathbb{Z}) = 0$.*

Proof. Let A be a W-group and let $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$ be a free resolution. Let φ be an arbitrary function in $Hom(F_0, \mathbb{Z})$. Let $B = (\mathbb{Z} \oplus F_1)/I$ where $I = \{(\varphi(y), -f_0(y)) : y \in F_0\}$. This way we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_0 & \xrightarrow{f_0} & F_1 & \xrightarrow{f_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle id_A} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\pi_1} & B & \xrightarrow{g_1} & A & \longrightarrow & 0 \end{array}$$

where π_1 and π_2 are the natural projections on the corresponding coordinates and $g_1((x, y) + I) = f_1(y)$ for every $(x, y) \in B$. Note that the bottom line is an exact sequence. Indeed, to see that π_1 is injective let $x \in Ker(\pi_1)$. Then $(x, 0) \in I$ so there exists some $y \in F_0$ such that $\varphi(y) = x$ and $f_0(y) = 0$. Since f_0 is injective, $y = 0$, hence $x = 0$. Also, to see that g_1 is surjective, let $x \in A$. Since f_1 is surjective, there is some $y \in F_1$ such that $f_1(y) = x$. But then $g_1(\pi_2(y)) = x$. To finish, let us see that $Ker(g_1) = Im(\pi_1)$. Let $(x, y) + I \in Im(\pi_1)$. Then, there is some $a \in \mathbb{Z}$ such that $\pi_1(a) = (a, 0) + I = (x, y) + I$. Then, $g_1((x, y) + I) = g_1((a, 0) + I) = f_1(0) = 0$, so $(x, y) + I \subseteq Ker(g_1)$, which shows that $Im(\pi_1) \subseteq Ker(g_1)$. Now let $(x, y) + I \in Ker(g_1)$. Then, $f_1(y) = g_1((x, y) + I) = 0$, hence $y \in Ker(f_1) = Im(f_0)$, so there is an $a \in F_0$ such that $f_0(a) = y$ (note that a is unique because f_0 is injective). Then $(x, y) + I = (x, f_0(a)) + I = (x + \varphi(a)) + I$. Then $\pi_2(x + \varphi(a)) = (x, y) + I$, so $(x, y) + I \in Im(\pi_1)$. It is easy to see that $Ker(g_1) \cong \mathbb{Z}$. Then, since the bottom line is exact and A is a W-group, there exists a homomorphism $\rho : A \longrightarrow B$ such that $\rho \circ g_1 = id_A$. By Proposition 1.38, there is a homomorphism $\tau : B \longrightarrow \mathbb{Z}$ such that $\pi_1 \circ \tau = id_{\mathbb{Z}}$. If we let $\psi = \tau \circ \pi_2$, then $f_0'(\psi) = \varphi$, so $Im(f_0') = Hom(F_0, \mathbb{Z})$, hence $Ext(A, \mathbb{Z}) = 0$.

Conversely, suppose $Ext(A, \mathbb{Z}) = 0$ and let the exact sequence $0 \to \mathbb{Z} \xrightarrow{\pi_1} B \xrightarrow{g} A \to 0$. Let $0 \to F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} A \to 0$ be a free resolution and let $\pi_2 : F_1 \longrightarrow B$ be a surjective homomorphism such that $\pi_1 \circ \pi_2 = f_1$ (the homomorphism π_2 exists because F_1 is a projective group, see Proposition 1.39). Then there is a homomorphism $\varphi : F_0 \longrightarrow \mathbb{Z}$ such that we have a commutative diagram like the one above. By assumption, there exists some $\psi : F_1 \longrightarrow \mathbb{Z}$ such that $f_0'(\psi) = \psi \circ f_0 = \varphi$. Note that if $Ker(\pi_2) \subseteq Ker(\psi)$. Indeed, if $x \in Ker(\pi_2)$, $f_1(x) = g_1(\pi_2(x)) = 0$, so $x \in Ker(f_1) = Im(f_0)$ and there exists some $y \in F_0$ such that $f_0(y) = x$. Then, $\psi(x) = \psi(f_0(y)) = \varphi(y) = 0$. Note that ψ induces a mapping $\tau : B \longrightarrow \mathbb{Z}$ such that $\tau \circ \pi_1 = id_{\mathbb{Z}}$. So the bottom line splits, hence A is a W-group. ∎

Corollary 3.3.

(1) *A subgroup of a W-group is a W-group.*

(2) *Every W-group is torsion-free.*

(3) *If B_0 is a subgroup of B_1 such that B_1 is a W-group but B_1/B_0 is not, then there exists a homomorphism $\psi : B_0 \longrightarrow \mathbb{Z}$ which does not extend to a homomorphism $\psi' : B_1 \longrightarrow \mathbb{Z}$.*

Proof. For (1) assume A is a W-group and let B be an arbitrary subgroup of A. Then, the sequence $0 \to B \xrightarrow{i} A \xrightarrow{\pi} A/B \to 0$, with i the inclusion and π the projection, is exact. Then, by Theorem 1.42, there is an exact sequence $Ext(A, \mathbb{Z}) \to Ext(B, \mathbb{Z}) \to 0$. Since A is a W-group, then $Ext(A, \mathbb{Z}) = 0$. Therefore, $Ext(B, \mathbb{Z}) = 0$, so B is a W-group. We prove (2) by contraposition. First note that if $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is the canonical projection, although $Ker(\pi) \cong \mathbb{Z}$, π does not split, so $\mathbb{Z}/n\mathbb{Z}$ is not a W-group for any $n > 0$. Now let A be a non-torsion-free group. Then, there is $a \in A$ such that $\langle a \rangle$ is a non-zero finite cyclic group, hence isomorphic to some $\mathbb{Z}/n\mathbb{Z}$, so $\langle a \rangle$ isn't a W-group. Since subgroups of W-groups are W-groups and $\langle a \rangle$ is a subgroup of A, A is not a W-group. To see (3) let the exact sequence $0 \to B_0 \xrightarrow{i} B_1 \xrightarrow{\pi} B_1/B_0 \to 0$, with i the inclusion and π the projection. By Theorem 1.42 there is an exact sequence $Hom(B_1, \mathbb{Z}) \xrightarrow{i'} Hom(B_0, \mathbb{Z}) \to Ext(B_1/B_0, \mathbb{Z}) \to Ext(B_1, \mathbb{Z})$. By assumption and Theorem 3.2, $Ext(B_1, \mathbb{Z}) = 0$ and $Ext(B_1/B_0, \mathbb{Z}) \neq 0$, which implies that i' is not surjective, which means that there is a homomorphism from B_0 into $\mathbb{Z}$ which does not extend to a homomorphism from B_1 into $\mathbb{Z}$. ■

The main result of this section is Stein's proof for the countable case of Whitehead's Problem. Let us first introduce some terminology and a few useful results.

Definition 3.4. A subgroup B of a torsion-free group A is called *pure subgroup* if the quotient group A/B is torsion-free. If B is a subgroup of A, the *pure closure* of B in A is the subgroup $B' = \{a \in A : na \in B \text{ for some } n \neq 0\}$.

Remark. The actual definition of pure subgroup of A does not need A to be torsion-free but it is enough for our purposes. For more details see [Kap69], p. 14.

It is clear that the pure closure of a subgroup B of A is a pure subgroup. Note that if A is free, so is B' by Theorem 1.31. Also, if B is finitely-generated, since B and B' have the same dimension, so is B'. As a consequence, if A is free, then every finitely-generated subgroup B of A is contained in a finitely-generated pure subgroup of A. The following result shows that the converse also holds for countable torsion-free groups.

> **Pontryagin's condition:** A is a countable torsion-free group such that every finitely-generated subgroup of A is contained in a finitely-generated pure subgroup of A.

Theorem 3.5 (Pontryagin's Criterion)**.** *If A satisfies the Pontryagin's condition, then A is free.*

Proof. Let $\{a_n : n < \omega\}$ be an enumeration of A. By induction, we define a smooth chain $\{B_n : n < \omega\}$ of finitely-generated pure subgroups of A. We start with $B_0 = 0$. If B_n has been already defined, we let B_{n+1} be a finitely-generated subgroup of A containing $B_n \cup \{a_n\}$. It is clear that $\bigcup_{n<\omega} B_n = A$. Moreover, since B_n is pure in A, B_{n+1}/B_n is torsion-free, and it is finitely-generated because B_{n+1} is finitely-generated. Therefore, by Theorem 1.36, B_{n+1}/B_n is free. By Theorem 1.35, A is free. ■

Later on in this chapter we will construct a torsion-free group of cardinality $\aleph_1$ which satisfies a stronger condition without being free, showing that the Pontryagin's Criterion is not true if A is not countable.

Definition 3.6. Let C be a set (or a group) of the form $B \times \mathbb{Z}$. By π we will denote the projection of C onto the first factor. If B is a group, we define a $(B, \mathbb{Z})$*-group* to be a group C whose underlying set is $B \times \mathbb{Z}$ such that the projection onto the first factor $\pi : C \longrightarrow B$ is a homomorphism and $(0, n) + (0, m) = (0, n + m)$ for all integers n, m.

An easy example of a $(B, \mathbb{Z})$-group is $B \oplus \mathbb{Z}$. It is easy to see that for any $(B, \mathbb{Z})$-group, $Ker(\pi) \cong \mathbb{Z}$. Therefore, if π does not split, then B is not a W-group.

Lemma 3.7 (See [Ekl76], Lemma 4.3)**.** *Let B_0 be a subgroup of B_1 such that B_1 is a W-group but B_1/B_0 is not. Let C_0 be a $(B_0, \mathbb{Z})$-group and ρ a splitting homomorphism for the projection onto the first factor $\pi : C_0 \longrightarrow B_0$. Then there is a $(B_1, \mathbb{Z})$-group C_1 which is an extension of C_0 such that ρ does not extend to a splitting homomorphism for $\pi : C_1 \longrightarrow B_1$.*

Proof. By the previous remark, $\pi : C_0 \longrightarrow B_0$ splits, there is a homomorphism $\rho : B_0 \longrightarrow C_0$ such that $\pi \circ \rho = id_{B_0}$. Then, $\rho(b) = (b, m)$ for any $b \in B_0$ and a fixed $m \in \mathbb{Z}$, so the mapping $\tau : B_0 \oplus \mathbb{Z} \longrightarrow C_0$ given by $\tau(b, z) = \rho(b) + (0, n)$ is an isomorphism. We may assume that $C_0 = B_0 \oplus \mathbb{Z}$ and $\rho(b) = (b, 0)$ for every $b \in B_0$. Let $C_1' = B_1 \otimes \mathbb{Z}$ and let $\psi : B_0 \longrightarrow \mathbb{Z}$ a homomorphism which does not extend to a homomorphism $\psi' : B_1 \longrightarrow \mathbb{Z}$, which exists by Corollary 3.3 (3). Let the homomorphism $\gamma : C_0 \longrightarrow C_1'$ given by $\gamma(b, n) = (b, n + \psi(b))$. Since ψ cannot be extended to a homomorphism from B_1 into $\mathbb{Z}$, there is no splitting homomorphism $\rho_1' : B_1 \longrightarrow C_1'$ for $\pi : C_1' \longrightarrow B_1$ whose resitriction to B_0 is $\gamma \circ \rho$ for otherwise, by letting φ be $\pi \circ \rho_1$, for any $b \in B_0 \subseteq B_1$,

$\varphi(b) = \psi(b)$, contradiction. Now, define the mapping $f : C_1' \longrightarrow B_1 \times \mathbb{Z}$ taking each pair (b, n) to itself if $b \notin B_0$ and to $(b, n - \psi(b))$ otherwise. Note that f is a bijection such that $f \circ \gamma$ is the inclussion of $B_0 \oplus \mathbb{Z}$ into $B_0 \times \mathbb{Z}$. Now we let C_1 be the group whose underlying set is $B_1 \times \mathbb{Z}$ with the addition $u + v = f(f^{-1}(u) + f^{-1}(v))$. Then C_1 is an extension of C_0 and there is no splitting homomorphism $\rho_1' : B_1 \longrightarrow C_1$ for $\tau : C_1 \longrightarrow B_1$ extending $\rho : B_0 \longrightarrow C_0$. ■

The following is the main theorem of this section and it answers positively the Whitehead's Problem in its countable case.

Theorem 3.8 (Stein, 1951)**.** *Every countable W-group is free.*

Proof. Let A be a countable W-group. By Corollary 3.3, A is torsion-free. Therefore, it is just enough that A satisfies Pontryagin's Criterion. We go by contradiction. Assume that there is a finitely-generated subgroup B_0 of A which is not contained in a finitely-generated pure subgroup of A. Let B be the pure closure of B_0 in A, whih is not finitely-generated by the choice of B_0. Then, B is the union of a strictly-increasing chain of finitely-generated groups $B_0 \subsetneq B_1 \subsetneq \ldots \subsetneq B_n \subsetneq \ldots$. Also, note that since B is the pure closure of B_0, B/B_0 is a torsion group. Now, by induction on n we built a chain $C_0 \subsetneq C_1 \subsetneq \ldots \subsetneq C_n \subsetneq \ldots$ of groups such that C_n is a torsion-free $(B_n, \mathbb{Z})$-group for every $n < \omega$. Before we start with the construction, note that if S is the set of generators of B_0 and C is torsion-free, any homomorphism $\rho : B \longrightarrow C$ is completely determined by its values on S. Indeed, let b be an arbitrary element in B, then there is some $n \neq 0$ such that $nb \in B_0$. Since ρ is determined by its values on the elements of S and C is torsion-free, the equation $nx = \rho(nb)$ has a unique solution in C, which is $x = \rho(b)$. We list all set-mappings $\{g_n : n < \omega\}$ with $g_n : S \longrightarrow S \times \mathbb{Z}$ such that $\pi \circ g = id_S$. There are countable-many of them as S is finite and $\mathbb{Z}$ is countable. Let $C_0 = B_0 \oplus \mathbb{Z}$ and suppose that C_n has been already defined. If g_n extends to a splitting homomorphism ρ for $\pi_n : C_n \longrightarrow B_n$, we let C_{n+1} be an extension of C_n such that ρ dos not extend to a splitting homomorphism for $\pi_{n+1} : C_{n+1} \longrightarrow B_{n+1}$. Since B_{n+1}/B_n is torsion, it is not a W-group by Corollary 3.3 (2), hence Lemma 3.7 applies, and such C_{n+1} exists. If the first case does not hold, we just let ρ be any splitting homomorphism for $\pi_n : C_n \longrightarrow B_n$ and define C_{n+1} in the same way as before. Since B_n is finitely generated and torsion-free, it is free by Theorem 1.36, so such ρ exists. This finishes the construction of the chain, so let $C = \bigcup_{n<\omega} C_n$. Clearly, C is a torsion-free $(B, \mathbb{Z})$-group. Now assume for the sake of contradiction that $\pi : C \longrightarrow B$ splits and let $\rho : B \longrightarrow C$ be a splitting homomorphism for π. Then, $\rho \restriction S = g_n$ for some $n < \omega$. Then, by construction, $\rho \restriction B_n$ is a splitting homomorphism for $\pi_n : C_n \longrightarrow B_n$ which extends g_n and extends, in turn, to a splitting homomorphism for $\pi_{n+1} : C_{n+1} \longrightarrow B_{n+1}$, but this contradicts the construction of C_{n+1}. Therefore, $\pi : C \longrightarrow B$ does not split. As noticed before Definition 3.6, $Ker(\pi) = \mathbb{Z}$. Since B is a subgroup of a W-group, it is a W-group, by Corollary 3.3 (1). But $\pi : C \longrightarrow B$ does not split, which is a contradiction. We conclude that A is free. ■

3.2 $\aleph_1$-free groups and the Chase's condition

Stein answers Whitehead's problem in the positive in its countable case. However, the same reasoning cannot work for the uncountable case for Stein's proof lies on Pontryagin's Criterion, which just applies to countable (torsion-free) groups. Moreover, the recursive constructions of B and C run over ω. In this section we generalize some of the previous notions and results.

Definition 3.9. A group A is said to be $\aleph_1$-*free* if all of its countable subgroups are free.

There are examples of $\aleph_1$-groups which are not free. For instance, Baer and Specker proved that the direct product of an infinite set of infinite cyclic groups is $\aleph_1$-group but not free (see [Fuc70], Theorem 19.2). The following is an easy consequence of Stein's theorem.

Proposition 3.10. *Every W-group is $\aleph_1$-free.*

Proof. Let A be a W-group. By Corollary 3.3 (1), every subgroup of A is a W-group. In particular, every countable subgroup of A is a W-group. Then, by Stein's Theorem, every countable subgroup of A is free. ■

Proposition 3.11. *If A is an $\aleph_1$-free group, then it is torsion-free and every finite subset of A is contained in a finitely-generated pure subgroup.*

Proof. It is clear that A is torsion-free, for otherwise there would exist a finite torsion group, hence a countable non-free group. Now let S be a finite subset of A and let $\langle S\rangle$ be the group generated by S. Let $\langle S\rangle_*$ be the pure closure of $\langle S\rangle$ and assume for the sake of contradiction that it is not finitely-generated. Then, there exists a countably generated subgroup N of $\langle S\rangle_*$ which is not finitely-generated. Recall that the rank of an abelian group is the cardinality of a maximal linearly independent subset. We prove that the rank of N is finite. This, together with the fact that it is countably generated, would imply that N is not free, contradicting that A is $\aleph_1$-free. So let T be a linearly independent set in N, since N is a subgroup of $\langle S\rangle_*$, for every $t \in T$ there exists some $n \in \mathbb{N}$ such that $nt \in \langle S\rangle$, so every $t \in T$ can be written as the linear combination of elements in S. Therefore, every linearly independent subset of N has cardinality less than or equal to $|S|$, thus N has finite rank. We conclude that $\langle S \rangle_*$ is finitely-generated. ■

The converse of Proposition 1.34 also holds (see [EM02], 2.3 Theorem, p. 98). We notice that $\aleph_1$-free groups may be seen as a generalization of torsion-free groups. Indeed, by Theorem 1.36, every finitely-generated subgroup of a torsion-free group is free. In this case, being an $\aleph_1$-free group implies that not only finitely-generated but countable subgroups are free. The following is a generalization of pure subgroups.

Definition 3.12. A subgroup of an $\aleph_1$-free group A is called $\aleph_1$-*pure subgroup* if A/B is $\aleph_1$-free.

As said above, Pontryagin's condition just applies to countable torsion-free groups. We generalize it to this more general context.

> **Chase's condition:** A is an $\aleph_1$-free group such that every countable subgroup of A is contained in a countable $\aleph_1$-pure subgroup of A.

Chase's condition can be expressed in terms of ascending chains, as next lemma shows.

Lemma 3.13. *If A is a group of cardinality $\aleph_1$, A satisfies the Chase's condition if and only if A is the union of a smooth chain of countable free groups $\{A_\alpha : \alpha < \omega_1\}$ such that $A_0 = 0$ and for each $\alpha < \omega_1$, $A_{\alpha+1}$ is $\aleph_1$-pure in A.*

Proof. The right to left direction is easy, for if A is the union of a smooth chain of countable free subgroups as in the statement, then for any countable subgroup B of A there is some $\alpha < \omega_1$ such that $B \subset A_{\alpha+1}$. For the other direction, assume that A is a group of cardinality $\aleph_1$ which satisfies the Chase's condition. We list all the elements of A in a sequence of length ω_1 so that $A = \{a_\alpha : \alpha < \omega_1\}$. We define each A_α by induction. Let $A_0 = 0$. Suppose that A_β has been already defined for all $\beta < \alpha$ we consider two cases. If α is a limit, we just let $A_\alpha = \bigcup_{\beta\alpha} A_\beta$, which is still countable for it is the union of countable-many countable sets. If $\alpha = \beta + 1$ let A_α be a countable $\aleph_1$-pure subgroup of A containing $A_\beta \cup \{a_\beta\}$, which exists by the assumption on A. ■

Note that for every group it is always possible to build a smooth chain of groups without further requirements about the properties they should exhibit. If A is a group of cardinality $\aleph_1$, we will denote by E_A the set of all ordinals $\alpha < \omega_1$ such that A_α is not $\aleph_1$-pure in A, where $\{A_\alpha : \alpha < \omega_1\}$ is the smooth chain of countable free groups of the previous lemma. Next theorem shows that free groups can be characterized by the stationarity of this set. To prove it we first need the following new concepts.

Definition 3.14. A subset C of an infinite limit ordinal α is said to be *unbounded* if for every $\beta < \alpha$ there is some $\gamma \in C$ such that $\beta < \gamma$. We say that it is *closed* if and only if for every limit ordinal $\beta < \alpha$, if $C \cap \beta$ is unbounded in β, then $\beta \in C$. If C is a closed and unbounded subset of an ordinal α, we say that C is a *club*. A subset S of an ordinal α is said to be *stationary* if it intersects all club subsets of α.

A function $f : \omega_1 \longrightarrow \omega_1$ is said to be *normal* if it is an strictly increasing function such that for any limit ordinal α in the domain, $f(\alpha) = \sup\{f(\beta) : \beta < \alpha\}$. Note that, since the image of every normal function is a club, every stationary subset S of ω_1 intersects it.

Theorem 3.15 (Chase's Criterion)**.** *A group A of cardinality $\aleph_1$ is free if and only if E_A is not a stationary subset of ω_1.*

Proof. Assume first that E_A is not a stationary subset of ω_1. Then there is a club subset C of ω_1 such that $E_A \cap C = \emptyset$. For every club subset C of ω_1 there is a normal function $f : \omega_1 \longrightarrow \omega_1$ such that $Im(f) = C$. Let f be a normal function on ω_1 such that $E_A \cap Im(f) = \emptyset$ and let $A^*_\alpha = A_{f(\alpha)}$. Since $Im(f)$ is unbounded and for any limit ordinal α in the domain, $f(\alpha) = \sup\{f(\beta) : \beta < \alpha\}$s, $\{A^*_\alpha : \alpha < \omega_1\}$ is a smooth chain such that $\bigcup_{\alpha \in OR} A^*_\alpha$. By the choice of f, $Im(f) \cap E = \emptyset$, so A^*_α is $\aleph_1$-pure for every $\alpha < \omega_1$, thus $A^*_{\alpha+1}/A^*_\alpha$ is free for every $\alpha < \omega_1$. By Theorem 1.35, A is free. Conversely, suppose that A is a free group and let S be a basis of A. We define a smooth chain $\{S_\alpha : \alpha < \omega_1\}$ of subsets of S and a normal function $f : \omega_1 \longrightarrow \omega_1$ such that for every $\alpha < \omega_1$, S_α is a basis of $A_{f(\alpha)}$. We let $S_0 = 0$ and $f(0) = 0$. Suppose S_β and $f(\beta)$ have been already defined for every $\beta < \alpha$. If α is a limit, we let $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and $f(\alpha) = \sup\{f(\beta) : \beta < \alpha\}$. Since $A_{f(\alpha)} = \bigcup_{\beta<\alpha} A_{f(\beta)}$, it is clear that S_α is a basis of $A_{f(\alpha)}$. If $\alpha = \beta + 1$, let S^*_0 be a countable subset of S such that $S_\beta \subsetneq S^*_0$ (this is possible because S is uncountable) and let σ^*_0 be an ordinal such that $S^*_0 \subseteq A_{\sigma^*_0}$. Note that $f(\beta) < \sigma^*_0$. Now, let S^*_1 be a countable subset of S such that $A_{\sigma^*_0}$ is contained in the group generated by S^*_1. We repeat this argument inductively and we get a chain of countable subsets $X_0 \subsetneq S^*_0 \subseteq S^*_1 \subseteq \ldots \subseteq S^*_n \subseteq \ldots$ and a sequence of ordinals $f(\beta) < \sigma^*_0 \leq \ldots \leq \sigma^*_n \leq \ldots$ such that for every $n < \omega$, $S^*_n \subseteq A_{\sigma^*_n} \subseteq \langle S^*_{n+1} \rangle$. Then, let $S_\alpha = \bigcup_{n<\omega} S^*_n$ and $f(\alpha) = \sup\{\sigma^*_n : n < \omega\}$, so that S_α is a basis of $A_{f(\alpha)}$. Then, the sequence built is the smooth chain $\{S_\alpha : \alpha < \omega_1\}$ of subsets of S and the normal function $f : \omega_1 \longrightarrow \omega_1$ such that for every $\alpha < \omega_1$, S_α is a basis of $A_{f(\alpha)}$. Note that E_A is not stationary in ω_1 because for every $\alpha < \omega_1$, $f(\alpha) \notin E_A$ because $A/A_{f(\alpha)}$ is isomorphic to the free group generated by $X \setminus X_\alpha$, so $A_{f(\alpha)}$ is $\aleph_1$-pure in A. This finishes the proof. ∎

Chase's Criterion will be used in next section.

3.3 The undecidability of the Whitehead's problem

Our aim is to prove that the Whitehead's problem is undecidable in ZFC. To do this we will show that every W-group of size $\aleph_1$ is free in the constructible universe L, which we also study here. Then, we will show that a negative answer for W-groups of size $\aleph_1$ is also consistent with ZFC. Indeed, under the assumption of Martin's Axiom there is always a W-group of size $\aleph_1$ which is not free.

3.3.1 The Axiom of Constructibility

In Definition 1.24 we have seen how V is built by transfinite recursion on the ordinals. Recall that we start with $V_0 = \emptyset$ and for limit ordinals we just let V_α be the union of all V_β with $\beta < \alpha$. In the successor case, $V_{\alpha+1}$ is taken to be the power set of V_α. This move is rather problematic, in a sense. Indeed, note that $|V_\omega| = \aleph_0$. However, by the well-known Cantor's Theorem, $|V_{\omega+1}| = |\mathcal{P}(V_\omega)| > |V_\omega|$, that is, $V_{\omega+1}$ has uncountable-many sets. Since there are countable-many formulas of the language of set theory, it follows that there are uncountable-many sets in $V_{\omega+1}$ that cannot be defined, not even with parameters. To avoid this situation, instead of taking the power set of V_α to get $V_{\alpha+1}$, one can just let $V_{\alpha+1}$ be the set of the subsets of V_α that are definable, with or without parameters, in V_α. This is the idea of the Constructible Universe, which we introduce here. We show that $\langle L, \in\rangle$ is a model of ZFC with interesting combinatorial properties such as the $\diamondsuit$-principle. We refer the intereted reader to [Jech03], 13, pp. 175-200; and [Kun13] II.6 pp. 134-144.

Definition 3.16. A set X is *definable* over a model $\langle M, \in\rangle$ if there exists a first-order formula φ in the language of set theory and some *parameters* $a_1, \ldots, a_n \in M$ such that $X = \{x \in M : \langle M, \in\rangle \vDash \varphi(x, a_1, \ldots, a_n)\}$. We let $\mathcal{D}(M) = \{X \subseteq M : X \text{ is definable over } \langle M, \in\rangle\}$

It is easy to see that if X is a transitive set, then $X \subseteq D(X)$. Indeed, if $x \in X$ then $x \subseteq X$. The set x can be defined in X as $x = \{y \in X : y \in x\}$, so $x \in D(X)$. Moreover, if X is transitive, then $D(X)$ is also transitive, for if $x \in D(X)$ and $y \in x \subseteq X \subseteq D(X)$, then $y \in D(X)$.

Definition 3.17. The Constructible Universe is the union $L = \bigcup_{\alpha \in OR} L_\alpha$, where:

$$L_0 = \emptyset,$$

$$L_{\alpha+1} = \mathcal{D}(L_\alpha),$$

$$L_\alpha = \bigcup_{\beta<\alpha} L_\beta, \text{ if } \alpha \text{ is a limit.}$$

As next result show $\langle L_\alpha : \alpha \in OR\rangle$ is a cumulative hierarchy.

Proposition 3.18. *For every $\alpha, \beta \in OR$,*

(1) *L_α is transitive and if $\alpha \leq \beta$, then $L_\alpha \subseteq L_\beta$.*

(2) *$\alpha = L_\alpha \cap OR$.*

Proof. To prove (1) we see by transfinite induction that L_α is transitive for every $\alpha \in OR$. The basic and limit cases being trivial, we just have to focus on the successor case. But this follows from the fact that if X is transitive, so is $D(X)$. Indeed, if L_α is transitive, so is $D(L_\alpha) = L_{\alpha+1}$, and we are done. By induction on β we prove that for every $\alpha \leq \beta$, then $L_\alpha \subseteq L_\beta$. Being again the basic and limit cases trivial, assume that it holds for β. Since L_β is transitive, $L_\beta \subseteq D(L_\beta) = L_{\beta+1}$. We prove (2) by transfinite induction as well, being again the basic and limit cases clear. Assume that $L_\alpha \cap OR = \alpha$. Then, $\alpha = \{x \in L_\alpha : L_\alpha \vDash \varphi(x)\} \in D(L_\alpha) = L_{\alpha+1}$, where $\varphi(x)$ says that x is an ordinal, which is first-order expressible. Then, $\alpha + 1 = L_{\alpha+1} \cap OR$. ■

Theorem 3.19 (Gödel, 1938)**.** *$L \vDash ZFC$.*

Proof. By Proposition 3.18 (2) it easily follows that L satisfies Infinity because, since $\omega + 1 = L_{\omega+1} \cap OR$, $\omega \in L_{\omega+1} \subset L$. By Lemma 1.25, since L is a transitive proper class, it satisfies Extensionality and Foundation. We first prove that Separation holds in L. To see this, let φ be a formula without y free. The formula φ may have x, z free, along with a vector v of n-many free variables, so let φ be written as $\varphi(x, z, v)$. Fix $z, v \in L$ and let $y = \{x \in z : L \vDash \varphi(x, z, v)\}$. Fix $\alpha \in OR$ such that $z, v \in L_\alpha$. Then, by the Reflection Theorem 1.27 we can take an ordinal $\beta > \alpha$ such that for all $a \in L_\beta$, $L_\beta \vDash \varphi(a, z, v)$ if and only if $L \vDash \varphi(a, z, v)$. Then, $y = \{x \in L_\beta : L_\beta \vDash \psi(x, z, v)\} \in L_{\beta+1} \subset L$ where $\psi(x, z, v) = \varphi(x, z, v) \wedge x \in z$. Now that we have that Reflection holds in L, it is easy to see that Union, Pairing, Power Set and Separation hold in L as well. Indeed, if $x, y \in L_\alpha$, then $\bigcup x = \{y \in L_\alpha : \exists z \in x(y \in z)\} \in L_{\alpha+1} \subset L$ so Union holds; $\{x, y\} = \{z \in L_\alpha : (z = x \vee z = y)\} \in L_{\alpha+1} \subset L$ so Pairing holds; $\mathcal{P}(x) = \{y \in L_\alpha : \exists z(z \subseteq x \wedge z = y)\} \in L_{\alpha+1} \subset L$ so Power Set holds. To see that L satisfies Replacement, let f be a definable function with domain $a \in L$. Then $rg(f) = \{b \in L : L \vDash \exists x \in a\varphi(x, b)\}$, where $\varphi(x, y)$ is the defining formula for f. Then $rg(f) \in L$. It remains to check that L satisfies the Axiom of Choice. We actually prove something stronger. Namely, we show that there is a definable well-ordering in L, which implies that every set in L has a choice function, that is, that L satisfies the Axiom of Choice.

Lemma 3.20. *There is a definable well-ordering of L.*

Proof of the lemma. We go by induction. Assume we have already defined a well-ordering $<_\beta$ of L_β for all $\beta < \alpha$ in such a way that $<_\beta \subseteq <_{\beta'}$ whenever $\beta < \beta'$. Suppose that $\alpha = \beta + 1$. Note that

$$\mathcal{D}(L_\beta) = \bigcup_{n<\omega} (\bigcup_{k<\omega} \mathcal{D}(k, L_\beta, n) \cup \{z : \exists m > 0 \exists (b_0, \dots, b_{m-1}) \in L_\beta^m \ \exists R \in \mathcal{D}(L_\beta, n+m)$$
$$(z = \{(a_0, \dots, a_{n-1}) \in L_\beta^n : (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}) \in R\})\}),$$

where

(i) $\mathcal{D}(0, L_\beta, n) = \{Diag(L_\beta, n, i, j) : i, j < n\} \cup \{Triang(L_\beta, n, i, j) : i, j < n\}$, where, in turn, $Diag(L_\beta, n, i, j) = \{x \in L_\beta^n : x_i = x_j\}$, $Triang(L_\beta, n, i, j = \{x \in L_\beta^n : x_i \in x_j\}$;

(ii) $\mathcal{D}(k+1, L_\beta, n) = \mathcal{D}(k, L_\beta, n) \cup \{L_\beta^n \setminus R : R \in \mathcal{D}(k, L_\beta, n)\} \cup \{R \cap S : R, S \in \mathcal{D}(k, L_\beta, n)\} \cup \{Proj(R, i) : R \in \mathcal{D}(k, L_\beta, n), i < n+1\}$, where $Proj(R, i)$ stands for the projection of the set $S \in L_\beta^n$ on its i-th coordinate; and

(iii) $\{z : \exists m > 0 \exists (b_0, \dots, b_{m-1}) \in L_\beta^m \ \exists R \in \mathcal{D}(L_\beta, n+m)(z = \{(a_0, \dots, a_{n-1}) \in L_\beta^n : (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}) \in R\})\}$ is the set of subsets of L_β^n defined with parameters in L_β.

Suppose $\alpha = \beta + 1$. We define a well-ordering in L_α. For each $n < \omega$ we can well-order $\bigcup_{k<\omega} \mathcal{D}(k, L_\beta, n)$ in a natural way. We order it first with respect to k. For a given $k+1$ we order $\mathcal{D}(k+1, L_\beta, n)$ with respect to the operations of complementation, intersection and projection using the already defined well-orderings of $\bigcup_{k<\omega} \mathcal{D}(k, L_\beta, n)$ for all $n < \omega$. We denote by $<_{\beta,n}$ this well-ordering of $\mathcal{D}(L_\beta, n)$. We also order lexicographically, in acordante with $<_\beta$, all finite sequences of elements of L_β. We denote this by $<_\beta^{lex}$. So now we define a well-ordering of L_α given by $x <_\alpha y$ if and only if

(a) $x, y \in L_\beta$ and $x <_\beta y$, or

(b) $x \in L_\beta$ and $y \notin L_\beta$, or

(c) $x, y \notin L_\beta$ with x and y being $x = \{x_0 : (x_0, a_0, \dots, a_{m_x-1}) \in R_x\}$ and $y = \{x_0' : (x_0', a_0', \dots, a_{m_y-1}') \in R_y\}$, respectively, where $(a_0, \dots, a_{m_x-1})$ and $(a_0', \dots, a_{m_x-1}')$ are the $<_\beta^{lex}$-least possible, and R_x and R_y are the $<_{\beta,m_x+1}$-least possible and the $<_{\beta,m_y+1}$-least possible, respectively; and

  (a') $(a_0, \dots, a_{m_x-1}) <_\beta^{lex} (a_0', \dots, a_{m_y-1}')$, or

  (b') $(a_0, \dots, a_{m_x-1}) = (a_0', \dots, a_{m_y-1}')$ and $R_x <_{\beta,m_x+1} R_y$.

Now let $<_L$ be the union of the well-orderings defined in each L_α, that is, $<_L = \bigcup_{\alpha \in OR} <_\alpha$. It remains to show that $<_L$ is a well-ordering for L. We first show that it is linear. Since $<_{\alpha+1}$ is a linear ordering of $L_{\alpha+1}$ such that $<_\alpha \subseteq <_{\alpha+1}$ for all ordinals

α, linearity is clear. So let us prove that $<_L$ is well-ordered. To see this, let a be a non-empty set in L. Note that if $x <_L y$, then $\rho^L(x)$, that is, the least ordinal α such that $x \in L_{\alpha+1}$, called the rank of x in L, is greater than or equal to $\rho^L(y)$. So let A_0 be the subset of A consisting of all elements of A with minimal rank. If A_0 contains only one element, this is the minimal element of A, so we are done. Otherwise, let x be the $<_{\alpha+1}$-least element of A_0 in $L_{\alpha+1}$, where $\rho^L(x) = \alpha$. Then x is the $<_L$-least element of A. ■

If $x \in L$, then $<_L \restriction x$ is a well-ordering of x, so in L every set has a well-ordering. Therefore, since L is a model of ZF, AC holds in L, hence L is a model of ZFC. ■

The assumption that the universe of all sets is actually L is known as the *Axiom of Constructibility*. Our aim in this section is to prove that Whitehead groups of cardinality $\aleph_1$ are free under the assumption that $V = L$. The fact that L satisfies the *Jensen's diamond principle* will play a key role in the proof.

$\diamondsuit$: There exists a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that $S_\alpha \subseteq \alpha$ for all α and such that for every $S \subseteq \omega_1$, the set $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary.

We say that a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ satisfying the conditions of $\diamondsuit$ is a *$\diamondsuit$-sequence.*

Theorem 3.21 (Jensen, 1972)**.** $L \vDash \diamondsuit$.

Proof. We build a $\diamondsuit$-sequence in L. By induction on $\alpha < \omega_1$ we define sequence of pairs (S_α, C_α) such that $S_\alpha \subseteq \alpha$ and C_α is a club subset of α. Let $S_0 = C_0 = \emptyset$ and let $S_{\alpha+1} = C_{\alpha+1}$ for all α. If α is a limit ordinal, we let (S_α, C_α) be the $<_L$-least pair such that $S_\alpha \subseteq \alpha$, C_α is a club subset of α and $S_\alpha \cap \xi \neq S_\xi$ for all $\xi \in C_\alpha$, if such pair exists. If it does not exist such a pair, we just let $S_\alpha = C_\alpha = \alpha$. Let us prove that the sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. Assume the opposite. Then, there exists $X \subset \omega_1$ such that $\{\alpha : X \cap \alpha = S_\alpha\}$ is not stationary, that is, such that there exists a club $C \subseteq \omega_1$ such that $X \cap \alpha \neq S_\alpha$ for all $\alpha \in C$. Let (X, C) be the $<_L$-least pair satisfying that. Note that since $\langle S_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-sequence of pairs of subsets of ω_1, it belongs to L_{ω_2}, which is a model of $ZFC^- + V = L$, that is, a model of ZFC without Power Set and $V = L$. The well-order $<_L$ is absolute for all transitive models of $ZFC^- + V = L$ for it requires only ZF^-, so the sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is, in L_{ω_2}, the $<_L$-least pair such that $S_\alpha \subseteq \alpha$, C_α is a club subset of α and $S_\alpha \cap \xi \neq S_\xi$ for all $\xi \in C_\alpha$, if such pair exists and such that $S_\alpha = C_\alpha = \alpha$, if it does not. Analogously, $(X, C) \in L_{\omega_2}$ is the $<_L$-least pair with $X \subseteq \omega_1$ and C a club of ω_1 such that $X \cap \alpha \neq S_\alpha$ for all $\alpha \in C$. By Theorem 1.22, let N be a countable elementary submodel of L_{ω_2}. Then N satisfies the same sentences than L_{ω_2}, so $\langle S_\alpha : \alpha < \omega_1 \rangle$ and (X, C) are in N, for they are definable in L_{ω_2} with no parameters. It is easy to see that $\omega_1 \cap N$ is an ordinal. To see this, as every subset of ordinals is an ordinal if it is transitive, we just have to show that $\omega_1 \cap N$ is transitive. But since the intersection of transitive sets is transitive, it is enough to prove that N is transitive. So let $a \in N$ and let f be the $<_L$-least mapping from ω_1 onto a. Since f is definable in L_{ω_2} from N, $f \in N$. Then $f(\alpha) \in N$ for every $\alpha \in \omega_1$, so $a \subseteq N$. Let us then let $\delta = \omega_1 \cap N$. The transitive collapse of N is L_γ for some $\gamma < \omega_1$ by the Gödel's Condensation Lemma (see [Jech03], Lemma 13.17). Let $\pi : N \longrightarrow L_\gamma$ be the collapsing map. We can show by induction on $OR \cap N$ that $\pi(\delta) = \omega_1$. Moreover, for every $b \subseteq \omega_1$ which is in N, then $\pi(b) = b \cap \delta$. Therefore, $\pi(X) = X \cap \delta$, $\pi(C) = C \cap \delta$ and $\pi(\langle S_\alpha : \alpha < \omega_1 \rangle) = \langle S_\alpha : \alpha < \delta \rangle$. We then have that L_δ satisfies that $(X \cap \delta, C \cap \delta)$ is the $<_L$-least pair (Z, D) with (Z, D) such that $Z \subseteq \delta$, D is a club of δ and $Z \cap \xi \neq S_\xi$ for all $\xi \in D$. By absoluteness this holds in L, so $X \cap \delta = S_\delta$. Since $C \cap \delta$ is unbounded in δ and C is closed, $\delta \in C$, which would imply that $X \cap \delta \neq S_\delta$, which is a contradiction. ■

It is easy to see that $\diamondsuit$ implies the Continuum Hypothesis. Indeed:

Theorem 3.22. $\diamondsuit \rightarrow CH$

Proof. Since $\aleph_1 \leq 2^{\aleph_0}$, we show that there is a one-to-one function from $P(\omega)$ to ω_1. Let $S \in P(\omega)$, that is, $S \subseteq \omega \in \omega_1$. By $\diamondsuit$ there exists a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ for every α such that for every $S \subseteq \omega_1$ the set $\{\alpha : \alpha \cap S = S_\alpha\}$ is stationary. Therefore, $\{\alpha : \alpha \cap S = S_\alpha\}$ intersects the tail set $C_\omega = \{\beta < \omega_1 : \beta > \omega\}$, which is a club in ω_1. Let $a \in C_\omega$ such that $a \cap S = S_a$. Since $a \in \omega_1$ such that $a > \omega$ and $S \subseteq \omega$ then $S \subseteq a$ so $S = a \cap S = S_a$. Let $S' \subseteq \omega$ such that $a \cap S' = S_a$. Again, $S' = S_a$ hence $S' = S$. Therefore the map f which sends every $S \in P(\omega)$ to the least element a in C_ω such that $a \cap S = S_a$ is a one-to-one function from $P(\omega)$ into ω_1, so $2^{\aleph_0} = |P(\omega)| \leq |\omega_1| = \aleph_1$. Thus, $2^{\aleph_0} = \aleph_1$. ■

More generally, as Gödel proved in 1938, the Generalized Continuum Hypothesis holds in L.

Theorem 3.23 (Gödel, 1938)**.** $V = L \longrightarrow GCH$.

Proof. It is enough to see that $2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$. So let $a \in L$ be an element in $\mathcal{P}(\omega_\alpha)$. By Reflection, let λ be an ordinal big enough to satisfy the conjunction θ of the finitely-many axioms of ZFC needed to construct L and such that $a \in L_\lambda$. By Löwenheim-Skolem-Tarski, let $N \in L$ be an elementary substructure of L_λ of countable cardinality such that $a \in N$. Let M be its Mostowski collapse. Since it is a transitive set satisfying θ, then $M = L_\alpha$ with $\alpha = M \cap OR < \omega_{\alpha+1}$. Since if $x \in y$, then $x <_L y$, $a \in M = L_\alpha$

with $\alpha < \omega_{\alpha+1}$, hence $a \in L_{\omega+1}$. We conclude that $2^{\aleph_\alpha} = |\mathcal{P}(\omega_\alpha)| \leq |L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$, hence $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\alpha \in OR$, and $L \vDash GCH$. ■

The Diamond Principle admits a generalization by changing ω_1 for any other regular cardinal κ. Also, if E is a stationary subset of a regular cardinal κ, then we have the following more general principle.

$\diamondsuit(E)$: There exists a sequence $\langle S_\alpha : \alpha \in E\rangle$ such that $S_\alpha \subseteq \alpha$ for all α and such that for every $S \subseteq \kappa$, the set $\{\alpha \in E : S \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

Theorem 3.21's proof can be generalized to show that the principle $\diamondsuit(E)$ holds in L for any cardinal κ and any stationary set $E \subset \kappa$. We notice that, as the Continuum Hypothesis holds in L, ω_1 is a regular cardinal. The $\diamondsuit(E)$ principle is stronger than what we actually need for our purposes. The following, a consequence of the fact that $\diamondsuit(E)$ holds in L, shall be enough.

Lemma 3.24 (V=L)**.** *Let B be the union of a strictly-increasing smooth chain of countable sets $\langle B_\alpha : \alpha < \omega_1\rangle$ and let $Y \subseteq \omega_1$. Let E be a stationary subset of ω_1. Then there is a sequence of functions $\langle g_\alpha : B_\alpha \longrightarrow B_\alpha \times Y : \alpha \in E\rangle$ such that for any function $h : B \longrightarrow B \times X$ satisfying that $h(B_\alpha) \subseteq B_\alpha \times Y$ for all α, there is an ordinal $\alpha \in E$ such that $h \restriction B_\alpha = g_\alpha$.*

Proof. Let $C_\alpha = B_\alpha \times (B_\alpha \times Y)$ and $C = B \times (B \times Y)$ and let $\{S_\alpha : \alpha \in E\}$ be a sequence such that $S_\alpha \subseteq C_\alpha$ for all $\alpha \in E$ and such that for any subset X of C the set of $\alpha \in E$ with $X \cap C_\alpha = S_\alpha$ is stationary in ω_1. Since $\diamondsuit(E)$ holds in L and $\{C_\alpha : \alpha < \omega_1\}$ is a strictly increasing smooth chain of countable sets, this sequence exists. As a subset of $B_\alpha \times (B_\alpha \times Y)$, S_α could be a function from B_α to $B_\alpha \times Y$. In that case, call it g_α. Otherwise, let g_α be an arbitrar function from B_α to $B_\alpha \times Y$. Let h be a function from B into $B \times Y$, hence a subset of $B \times (B \times Y)$. By $\diamondsuit(E)$, $h \cap C_\alpha = B_\alpha$. But since $h(B_\alpha) \subseteq B_\alpha \times Y$ for all α, then $h \cap C_\alpha = h \restriction B_\alpha = g_\alpha$. ■

This is enough to prove that, in L, every W-group of size $\aleph_1$ is free.

Theorem 3.25 (Shelah, 1974)**.** *$ZFC+V = L$ implies that every W-group of cardinality $\aleph_1$ is free.*

Proof. Let A be a W-group of cardinality $\aleph_1$. By Corollary 3.10, A is $\aleph_1$-free. We first see that A satisfies the Chase's condition. We go by contradiction. Suppose that A does not satisfy the Chase's condition, there is a countable subgroup B of A such that for every countable subgroup C of A containing B, the quotient group A/C is not $\aleph_1$-free, or, equivalently, there is a countable subgroup B of A such that for every countable subgroup C of A containing B there exists a countable subgroup C' of A containing C such that C'/C is not free. Note that there is at least one of such subgroups C for otherwise there would not be any counterexample. We can construct by transfinite induction a strictly increasing smooth chain $\{B_\alpha : \alpha < \omega_1\}$ of countable groups such that $B_{\alpha+1}/B_\alpha$ is not free for every $\alpha < \omega$. We let $B_0 = B$. Suppose B_β have been defined for every $\beta < \alpha$. If α is a limit we just let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. If $\alpha = \beta + 1$, since, by induction, B_β is a countable subgroup of A containing B, there is a countable subgroup C'' containing B_β such that C''/B_β, so let $B_\alpha = C''$. It is clear that the set $E(B) = \{\alpha < \omega_1 : B_{\alpha+1}/B_\alpha$ is not free$\}$ is a stationary subset of ω_1.

Lemma 3.26 (V=L)**.** *Let B be the union of a strictly increasing smooth chain $\{B_\alpha : \alpha < \omega_1\}$ of countable free groups such that $E(B)$ is stationary in ω_1. Then B is not a W-group.*

Proof of the lemma. By induction, we built a smooth chain of groups $\{C_\alpha : \alpha < \omega_1\}$ where each C_α is a $(B_\alpha, \mathbb{Z})$-group and the union C is a $(B, \mathbb{Z})$-group such that $\pi : C \longrightarrow B$ does not split. To do this we mimic the construction in Stein's theorem. However, in this case we need Lemma 3.24, for it assures that there is a set of functions $\{g_\alpha : B_\alpha \longrightarrow B_\alpha \times \mathbb{Z} : \alpha \in E(B)\}$ such that for any function $h : B \longrightarrow B \times \mathbb{Z}$ satisfying that $h(B_\alpha) \subseteq B_\alpha \times \mathbb{Z}$ for all α, there is an ordinal $\alpha \in E(B)$ such that $h \restriction B_\alpha = g_\alpha$. The construction then goes almost the same way. We start with C_0, which we let to be any $(B_0, \mathbb{Z})$-group. Suppose that C_β has been defined for all $\beta < \alpha$. For limit ordinals we take unions. If $\alpha = \beta + 1$, we consider two cases. If $\beta \in E(B)$ and $g_\beta : B_\beta \longrightarrow B_\beta \times \mathbb{Z}$ is a splitting homomorphism for $\pi_\beta : C_\beta \longrightarrow B_\beta$, we let C_α to be an extension of C_β such that g_β does not extend to a splitting field for $\pi_\alpha : C_\alpha \longrightarrow B_\alpha$, which exists by Lemma 3.7 because it is not free (because $B_{\beta+1}/B_\beta$ is not), hence not a W-group. Otherwise, if $\beta \notin E(B)$ or the mapping g_β is not a splitting field for π_β, we let C_α be any $(B_\alpha, \mathbb{Z})$-group extending C_α. Let $C = \bigcup_{\alpha<\omega_1} C_\alpha$. As in Stein's theorem we note that if $\rho : B \longrightarrow C$ was a splitting homomorphism for $\pi : C \to B$, there would be some $\beta \in E(B)$ such that $\rho \restriction B_\beta = g_\beta$ which is impossible by the construction of $C_{\beta+1}$, so π does not split and we are done. ■

Let $B^* = \bigcup_{\alpha<\omega_1} B_\alpha$. By the lemma, since $\{B_\alpha : \alpha < \omega_1\}$ is a strictly increasing smooth chain of countable free groups and $E(B)$ is stationary in ω_1, B^* is not a W-group. However, subgroups of W-groups are W-groups, which implies that A is not a W-group. This is a contradiction, so A satisfies the Chase's condition.

By Lemma 3.13, A is the union of a smooth chain $\{A_\alpha : \alpha < \omega_1\}$ of countable free groups such that $A_{\alpha+1}$ is $\aleph_1$-pure in A. We prove that the set $E = \{\mu < \omega_1 :$

a_μ is not $\aleph_1$-pure in $A\}$, is equal to $E' = \{\mu < \omega_1 : A_{\mu+1}/A_\mu$ is not free$\}$. Since $E' \subseteq E$ is clear we just have to show that $E \subseteq E'$. Suppose that $\mu \notin E'$. Then, by Corollary 1.34 and the Second Homomorphism Theorem, for every $\xi > \mu$, A_ξ/A_μ is free because $(A_\xi/A_\mu)/(A_{\mu+1}/A_{\xi+1}) \cong A_\xi/A_{\mu+1}$. Then A/A_μ is $\aleph_1$-free for all its countable subgroups are in A_ξ/A_μ for some $\xi > \mu$. Therefore, $\mu \notin E$ and $E = E'$. By the lemma, E' is not stationary in ω_1, as A is, by assumption, a W-group. But then, since $E = E'$, the Chase's Criterion implies that A is free. ■

We conclude that a positive solution of the Whitehead's Problem is consistent with ZFC.

3.3.2 Martin's Axiom

In this section we prove that assuming Martin's Axiom there is a W-group of size $\aleph_1$ which is not free. The consistency of Martin's Axiom was proved by Donald A. Martin, based on the work of Robert Solovay and Stanley Tennenbaum on the consistency of the existence of Suslin trees. The proof, for which the powerful technique of forcing is used, can be seen in the Appendix. However, the reader who is willing to accept on faith the consistency of Martin's Axiom will have no problem to follow this section.

Definition 3.27. A *partial ordering* is a pair $\langle P, \leq \rangle$ such that $P \neq \emptyset$ and $\leq$ is a reflexive, antisymmetric and transitive relation on P.

We will normally denoted $\langle P, \leq \rangle$ simply by P and call it *poset*. The elements of P are sometimes called *conditions*. If $p, q \in P$ and $p \leq q$ we say that p *extends* q.

Definition 3.28. Let P be a poset. A subset $D \subseteq P$ is *dense* if for every $p \in P$ there exists $q \in D$ such that $q \leq p$. It is *dense below* p if for every $q \leq p$ there is some $r \in D$ such that $r \leq q$. D is said to be *open* if $p \in D$ and $q \in D$ for every $q \in P$ such that $q \leq p$.

Note that if D is dense below p and $q \leq p$, then D is dense below q and that D is dense if and only if D is dense below p for every p.

Definition 3.29. Two elements $p, q \in P$ are *compatible* if there exists an element $r \in P$ such that $r \leq p, q$. If p and q are not compatible, they are *incompatible*.

Definition 3.30. A subset $C \subseteq P$ is a *chain* if for every $p, q \in C$, either $p \leq q$ or $p \leq q$. It is an *antichain* if every two $p, q \in C$ are incompatible. A partial ordering P is *ccc* (has the *countable chain condition*) if all of its antichains are countable.

It is easy to see that if A is a maximal antichain of P, then $D = \{p : p \leq q \text{ with } q \in A\}$ is dense open, and that every dense open subset contains a maximal antichain.

Definition 3.31. Let P be a poset and let $G \subseteq P$. G is said to be a *filter* if

(1) $G \neq \emptyset$,

(2) every two elements of G are compatible, and

(3) if $p \in G$ and $p \leq q$, then $q \in G$.

A *generic filter with respect to a family of dense subsets* of P is a filter whose intersection with every dense subset of that familiy is non-empty.

The Martin's Axiom is a generalization of the Baire Category Theorem, which states that in every compact Hausdorff space the intersection of $\aleph_1$-many dense open sets is non-empty.

> **Martin's Axiom (MA):** For every ccc poset P and every family of $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of P, there is a filter $G \subseteq P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

The Martin's Axiom statement is always true in ZFC for families of ω-many dense substes. Further, if P is a poset and $\{D_n : n < \omega\}$ is a family of dense subsets of P. Then, for every $p \in P$ there exists a filter $G \subseteq P$ such that $p \in G$ and G is generic for $\{D_n : n < \omega\}$. Note as well that the restriction to ccc posets is necessary for otherwise the axiom would be false. For instance, let P to be the set of functions with domain a finite set of natural numbers and range a subset of ω_1 ordered by reverse inclusion. For every $n \in \omega$, the set D_n of all f such that $n \in dom(f)$ is dense, and for every $\alpha \in \omega_1$, the set E_α of all f with $\alpha \in rg(f)$. Then, if G were a generic filter for the family $\{D_n : n \in \omega\} \cap \{E_\alpha : \alpha \in \omega_1\}$, then $\bigcup G$ would be a function from ω onto ω_1, which is impossible. By a similar argument one gets the following.

Theorem 3.32 (MA)**.** *Let A and B be sets of cardinality less than $2^{\aleph_0}$ and let P be a family of functions from a subset of A into B such that*

(1) *for every $a \in A$ and every $f \in P$, there exists $g \in P$ with $dom(g) = a$ extending f, and*

(2) *for every uncountable subset P' of P, there exist $f_1, f_2 \in P'$ and $f_3 \in P$ such that $f_1 \neq f_2$ and f_3 extends both f_1 and f_2.*

Then there exists a function $g : A \longrightarrow B$ such that for every finite subset F of A there exists f in P with F contained in the domain of f and $g \restriction F = f \restriction F$.

Note that (2) above means that P is ccc. We have seen that CH holds in L. The situation is just the opposite under the assumption of Martin's Axiom. Actually, in their proof of th consistency of MA, Solovay and Tennenbaum proved that $MA + 2^{\aleph_0} = \kappa$ is consistent with ZFC for every regular cardinal κ greater than $\aleph_1$.

Theorem 3.33 (Martin-Solovay, 1970)**.** *$MA \to \neg CH$.*

Proof. It is enough proving that for any family $\{f_\alpha : \alpha < \omega_1\}$ of functions from ω into $2 = \{0, 1\}$, there is a function $g : \omega \longrightarrow 2$ which is not in the family. To see this, let P be the partial ordering consisting of the functions $p : S \longrightarrow 2$ with S a finite subset of ω such that $p \leq q$ if and only if p extends q. First note that P is ccc for it is countable. Also, p and q are compatible if and only if they agree in $dom(p) \cap dom(q)$ for then $p \cup q$ extends p and q. For every $n < \omega$ let the set $D_n = \{p \in P : n \in dom(p)\}$. It is easy to see that D_n is dense. Let also $E_\alpha = \{p \in P : \exists n \in dom(p)$ such that $p(n) \neq f_\alpha\}$, which is also dense. By MA, there exists a filter $G \subseteq P$ such that $G \cap D_n \neq \emptyset$ for all $n < \omega$ and $G \cap E_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. But then $\bigcup G$ is a function with domain ω, as G meets every D_n and different to f_α for all $\alpha < \omega_1$ for G meets every E_α. So $\bigcup G = g$ and we are done. ■

The following theorem, proved in ZFC, will be key to answer the Whitehead's Problem in the negative.

Theorem 3.34. *There is a group A of size $\aleph_1$ which satisfies the Chase's condition but is not free.*

Proof. By induction, we build a smooth chain $\{A_\alpha : \alpha < \omega_1\}$ of countable groups A_α such that

(1) A_α is free for every $\alpha < \omega_1$,

(2) $A_\alpha / A_{\beta+1}$ is free for every $\beta < \alpha < \omega_1$, and

(3) $A_{\alpha+1}/A_\alpha$ is not free for every limit ordinal α.

Let A_0 be the trivial group and assume that $\{A_\alpha : \alpha < \delta\}$ with $\delta < \omega_1$ is a smooth chain satisfying (1), (2) and (3). To define A_δ we consider three different cases. If δ is a successor ordinal of a successor ordinal, we let $A_\delta = A_\alpha \oplus \mathbb{Z}$, which clearly satisfies (1). It satisfies (2), too. Indeed, since $A_\alpha/A_{\beta+1}$ is a free subgroup of $A_\delta/A_{\beta+1}$, and $(A_\delta/A_{\beta+1})/(A_\alpha/A_{\beta+1}) \equiv A_\delta/A_\alpha$ is free as well, then (2) holds by Corollary 1.34. If δ is a limit ordinal, we let $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. Let $\{\sigma_n : n < \omega\}$ be an increasing sequence of ordinals whose limit is δ such that σ_n is a successor ordinal for every $n < \omega$. Then, $A_\delta = \bigcup_{n<\omega} A_{\omega_n}$, and $A_{\sigma_{n+1}}/A_{\sigma_n}$ is free for every $n < \omega$. Then, by Theorem 1.35 A_δ is free and so is A_δ/A_{σ_n} for every $n < \omega$, so (1) holds. By Corollary 1.34, since $A_{\mu+1}$ is contained in A_{σ_n} for some $n < \omega$ for every $\mu < \delta$, (2) holds. And, in the third case, if δ is the successor ordinal of a limit α, we let $\{\sigma_n : n < \omega\}$ as above although we require σ_0 to be 0. By the proof of Theorem 1.35 we know that there is a smooth chain of sets $\{X_n : n < \omega\}$ such that X_n is a basis of A_{σ_n}. For each $n > 1$ let $x_n \in X_n \setminus X_{n-1}$.

Define $Y_n = X_n \setminus \{x_n\}$ and let B be the subgroup of A_α generated by $\bigcup_{n<\omega} Y_n$. Let $P = \prod_{n<\omega} \langle x_n \rangle$ and define $A_{\alpha+1}$ to be the subgroup of $B \oplus P$ generated by A_α and $\{z_m : 1 \leq m < \omega\}$ where z_m is the element in P of the form $z_m = \sum_{n \geq m} \frac{n!}{m!} x_n$. The union $\bigcup_{n<\omega} Y_n \cup \{z_m : 1 \leq m < \omega\}$ is a basis of $A_{\alpha+1}$, hence (1) holds. Now, note that for each $k < \omega$ the quotient $A_{\alpha+1}/A_{\sigma_k}$ is isomorphic to the subgroup of $A_{\alpha+1}$ generated by $\bigcup_{n>k}(Y_n \setminus Y_k) \cup \{z_m : k+1 \leq m < \omega\}$. Therefore, again by Corollary 1.34, $A_{\alpha+1}/A_{\sigma_k}$ is free. To see that (3) holds, note that $m!z_m - z_1 \in A_\alpha$ for every $m \geq 1$, hence $z_1 + A_\alpha$ is a non-zero element of $A_{\alpha+1}/A_\alpha$, divisible by n for every $n < \omega$. Free groups have no such elements, so (3) holds.

Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Since A is the union of ω_1-many countable groups, it is of size $\aleph_1$. Note that for every $\alpha < \omega_1$, the group $A_{\alpha+1}$ is $\aleph_1$-pure in A for every countable group $A/A_{\alpha+1}$ is in $A_\beta/A_{\alpha+1}$ for a sufficiently large β. By the construction, $A_\beta/A_{\alpha+1}$ is free and every subgroup of a free group is free. Therefore, $A_{\alpha+1}$ is $\aleph_1$-pure for every $\alpha < \omega_1$. By Lemma 3.13, A satisfies the Chase's condition. By (3), the set $E_A = \{\alpha < \omega_1 : A_\alpha \text{ is not } \aleph_1\text{-pure}\}$ is stationary in ω_1, for it is the set of limit ordinals of ω_1. By the Chase's Criterion A is not free, so we are done. ■

Our next step is proving that, assuming $MA + 2^{\aleph_0} > \aleph_1$, any group of cardinality $\aleph_1$ which satisfies Chase's condition is a W-group. This would imply, according to the previous theorem, that there exists a W-group which is not free, answering the Whitehead's Problem in the negative.

Theorem 3.35 (Shelah, 1974). *$ZFC + MA + 2^{\aleph_0} > \aleph_1$ implies that any group of cardinality $\aleph_1$ which satisfies Chase's condition is a W-group. In particular, $ZFC + MA + 2^{\aleph_0} > \aleph_1$ implies that there is a W-group of cardinality $\aleph_1$ which is not free.*

Proof. Let A be a group of size $\aleph_1$ satisfying Chase's condition and let $\pi : B \longrightarrow A$ be a surjective homomorphism such that $Ker(\pi) = \mathbb{Z}$. We have to prove that π splits. Let P be the set of all homomorphisms $\varphi : S \longrightarrow B$ such that $\pi \circ \varphi = id_S$ with S a finitely-generated pure subgroup of A. Note first that P is not empty. Indeed, let S' be a basis of the pure subgroup S, which is free for it is finitely-generated and A is torsion-free, as it satisfies Chase's condition hence it is $\aleph_1$-free. Let $f : S' \longrightarrow B$ be the set-function sending x to some y where $\pi(y) = x$, which exists because π is surjective. Then, f extends in the natural way to a homomorphism $\varphi_f : S \longrightarrow B$ which satisfies the required conditions.

Claim. Let $\varphi \in P$ and let F be a finite subset of A. There is a function $\varphi' \in P$ such that φ' extends φ and $F \subseteq dom(\varphi')$.

Proof of the claim. Let S be a finitely-generated pure subgroup of A and let φ be a function in P whose domain is S. By Proposition 3.11, since A is $\aleph_1$-free there is a finitely-generated pure subgroup S' containing $S \cup F$. Clearly, S'/S is a finitely-generated torsion free group, so it is free. By Corollary 1.34, there is a basis of S which extends to a basis of S'. That is, there is a basis of S' of the form $X \cup Y$ where X is a basis of S. For each $x \in X$, let $\varphi'(x) = \varphi(x)$ and for each $y \in y$ let $\varphi'(y) = b_y$ where b_y is some element in B such that $\pi(b_y) = y$. This way, $\varphi' \in P$. ■

Claim. If P' is an uncountable subset of P, then there is a free subgroup A' of A which is pure in A and an uncountable subset P'' of P' such that $dom(\varphi) \subseteq A'$ for every $\varphi \in P''$.

Proof of the claim. Let $P' = \{\varphi_\alpha : S_\alpha \longrightarrow B : \alpha < \omega_1\}$ be an uncountable subset of P. We may assume by taking an uncountable subset of P' if necessary that there is an m such that the basis of each S_α is of cardinality m. It is easy to see that there is a pure subgroup T of A which is maximal with respect to the property that T is contained in uncountable many S_α (which might be the trivial group), so we may assume that T is contained in S_α for all α. As in the claim above, let $X \cup Y_\alpha$ be a finite basis of S_α for each α where X is the basis of T. Now we let the desired free subgroup A' be the union of a smooth chain $\{A_\alpha : \alpha < \omega_1\}$ with $A_0 = T$ such that for each $\alpha < \omega_1$, A_α is a pure subgroup of A and $A_{\alpha+1}/A_\alpha$ is free. By Theorem 1.35, A' is free and, since the union of pure subgroups is pure, A' is also pure. We construct this chain by induction on $\alpha < \omega_1$ and, as already said, we let $A_0 = T$. Assume that $\{A_\mu : \mu < \alpha\}$ has been already defined. If α is a limit ordinal we let $A_\alpha = \bigcup_{\mu<\alpha} A_\alpha$. So let us assume that $\alpha = \beta + 1$ is a successor ordinal and let $\{\sigma_{\mu+1} : \mu < \alpha\}$ be a strictly increasing sequence of ordinals such that $Y_{\sigma_{\alpha+1}}$ is a subset of $A_{\mu+1}$. Let C_β be a countable $\aleph_1$-pure subgroup of A containing A_β, which exists because A satisfies Chase's condition. Note that there exists $\sigma_\alpha > \sigma_{\mu+1}$ for all $\mu < \alpha$ such that $\langle Y_{\sigma_\alpha} \rangle \cap C_\beta = 0$ for otherwise there would be an element $c \in C_\beta$ and uncountable many $\tau < \omega_1$ such that $c \in \langle Y_\tau \rangle$, so the pure closure of $T + \langle c \rangle$ would contradict the maximality of T. Let now A_α be the pure closure of $A_\beta + \langle Y_{\sigma_\alpha} \rangle$. Since $(Y_{\sigma_\alpha}) \cap C_\beta = 0$, then $A_\alpha \cap C_\beta = A_\beta$, so A_α/A_β is isomorphic to a countable subgroup of A/C_β hence it is free, as C_β is $\aleph_1$-pure in A. Let $P'' = \{\varphi_{\sigma_{\mu+1}} : \mu < \omega_1\}$. This is an uncountable subset of P' such that $dom(\varphi) \subseteq A'$ for every $\varphi \in P''$, so we are done. ■

Now, by Theorem 3.32 (here is where MA is needed) there is a function $g : A \longrightarrow B$ such that for every finite subset F os A there exists $f \in P$ with $F \subseteq dom(f)$ such that $g \restriction F = f \restriction F$. Clearly, g is an homomorphism such that $\pi \circ g = id_A$, hence π splits, hence A is a W-group. ■

The main result of this chapter follows.

Theorem 3.36 (Shelah, 1974). *The Whitehead's Problem is undecidable in ZFC. More precisely, the Whitehead's Problem has a positive solution under the assumption of the Axiom of Constructibility, which is consistent with ZFC; and it has a negative solution under the assumption of Martin's Axiom, also consistent with ZFC. Therefore, both a positive and a negative solution to the Whitehead's Problem are consistent with ZFC.*

We notice that the models used in the proof are such that CH holds in one of them while $\neg CH$ holds in the other. Shelah's Theorem left open the question of whether CH is sufficient to imply that W-groups are free. After trying to interest the set-theoretic community in the problem by stating a combinatorial analog, he took the problem by himself. This would result not only in an independence result in 1976 but in the beginning of the so-called proper forcing, being this an example of how other fields of mathematics motivate deep advances in set theory as well.

Appendix: Forcing and the consistency of Martin's Axiom

We briefly introduce the technique of forcing and prove the consistency of Martin's Axiom. Our exposition may lack of the rigour and detail that any standard textbook on the topic would have. Classical introductions to forcing are [Jech03], (see Chapter 14 (pp. 201-218)) and [Kun13] (see Chapter IV). For an approach based on admissible sets, Bagaria's course notes [Bag19] (see Chapter 4) are also a great introduction. This appendix is based on them. An introduction to iterated forcing and the proof of the consistency of MA can also be found in [Jech03] (see Chapter 16, pp. 267-273; the proof of the consistency of Martin's Axiom can be seen in [Jech03], Theorem 16.13) and [Bag19] (see Chapter 6). There is also a chapter devoted to Iterated Forcing in [Kun13] where the consistency of Martin's Axiom is discussed (see Chapter V). The following exposition is intended to provide an intuitive yet maybe rudimentary idea of what forcing is. Nevertheless, it should be enough to convince the reader about the consistency of MA with ZFC. May the reader be interested in further reading, we refer him or her to the mentioned references.

1.1 Forcing

The forcing technique was discovered by Paul Cohen, and it was first used to build a model where CH did not hold, and a model where AC did not hold neither. This together with Gödel consistency results on the CH and AC would prove the independence of both the Continuum Hypothesis and the Axiom of Choice from ZFC.

Given a countable transitive model M of set theory, forcing's main idea is to extend M to a larger transitive model N by adding a *generic* set G which is not in M. The new model is expected to exhibit some new features which the oringal model doesn't have. We refer to this larger transitive model N as the *generic extension* or *forcing extension* of M and denote it by $M[G]$.

1.1.1 Admissible sets

The method of forcing can be carried out in a weaker theory than ZFC. The Kripke-Platek set theory, denoted KP, consists of the universal closure of the axioms of *Extensionality, Pairing, Union, Foundation,* Δ_0-*Separation* and Δ_0-*Collection.* By Δ_0-Separation we mean the axiom of Separation restricted to Δ_0-formulas, that is, first-order formulas all whose quantifiers are bounded. Δ_0-formulas are absolute for transitive classes. The axiom of Δ_0-Collection is the schema $\forall x \in a \exists y \varphi(x, y) \to \exists z \forall x \in a \exists y \in z \varphi(x, y)$ for each Δ_0-formula φ where z does not occur free. KP proves Δ_0-Replacement.

Definition 1.1. M is an *admissible set* if it is a transitive model of KP.

Countable admissible models can be extended by means of forcing. We will see later on why the countable condition is necessary. Note that ZFC cannot prove the existence of any countable transitive model of ZFC, for it would be proving its own consistency. By taking a sufficiently big fragment of ZFC we can easily overcome this inconvenient. Indeed, $KP + Infinity + Power\ Set + AC$ proves that there exist uncountable many countable admissible sets. By ZFC^* we will denote a fragment of ZFC big enough to include KP.

Although introduced already in Chapter 3, we recall the following notions. A *partial ordering* is a pair $\langle P, \leq \rangle$ where P is a non-empty set and $\leq$ is a partial order on P. The elements of P are called *conditions.* We read $p \leq q$ as p *extends* q. If a partial ordering P is a set, we call it *poset.* Two conditions $p, q \in P$ are *compatible* if there exists another condition $r \in P$ such that $r \leq p, q$. If p, q are not compatible, they are *incompatible.* A subset D of a partial ordering P is said to be *dense* if for every condition $p \in P$ there exists $q \in D$ such that $q \leq p$. D is *dense below* a condition $p \in P$ if for every $q \leq p$ there exists $r \in D$ such that $r \leq q$. D is said to be *open* if it is downward closed. We say that a subset A of a partial ordering P is a *maximal antichain* if all its elements are *incompatible* and cannot be extended to a larger antichain subset. Assuming AC, every dense open subset of P contains a maximal antichain. We say that a partial ordering P satisfies the *countable chain condition*, denoted *ccc*, if all its antichains are countable. A *generic filter with respect to a family of dense subsets* of P is a filter whose intersection with every dense open subset of that family is non-empty. Given M a model of ZFC^*, we say that G is *generic over* M if it is a generic filter over a poset $P \in M$. Note that if $G \subseteq P$ is generic over M, then it contains just one element from each antichain in P. Clearly, if $a \in A \cap G$ there cannot be any $b \in A$ with $b \neq a$ such that $b \in G$ because then a and b would be compatible, which is not possible. This proves that $|A \cap G| \leq 1$.

To see that $A \cap G \neq \emptyset$, consider the set $D = \{p : p \leq q,\ \text{some } q \in A\}$. D is dense open, hence $G \cap D \neq \emptyset$. Now, if $a \in D \cap G$, $a \leq p$ for some $p \in A$ so $p \in G$, thus $G \cap A \neq \emptyset$. Therefore, the following.

Proposition 1.2. *Let M be a model of ZFC^*. If G is generic over M, then for every maximal antichain A of P that belongs to M, $|G \cap A| = 1$.*

In order to get a forcing extension $M[G]$ strictly larger than M we need the generic $G \subseteq P$ not to be an element of M. Easy examples show that it is not always the case that $G \in M$. Indeed, if $P = \{p\}$ for some $p \in M$ or P is a linear order, G is clearly an element in M. For a generic filter not to be an element of M we need the poset P to be perfect.

Definition 1.3. A partial ordering P is said to be *perfect* if for every $p \in P$ there exist $q, r \leq p$ such that q and r are incompatible.

Let P be a perfect poset. Then, every filter $G \subseteq P$ generic over M is not in M as an element. Suppose the contrary. Then, $D = \{p : p \notin G\} \in M$ as well and $G \cap M \neq \emptyset$. Conversely, if P is not perfect, let $G = \{q : \neg q \bot r\}$ where p is an element in P which is not extended by any compatible conditions. It is easy to see that G is a generic filter. This proves the following.

Proposition 1.4. *A partial ordering P is perfect if and only if $G \notin M$ for every generic filter.*

We require M to be a countable transitive model of ZFC^* because in that case the family of dense open subsets of any partial ordering $P \in M$ is countable as well. Indeed, since M is countable, so is the collection $\{D_n : n < \omega\}$ of all dense open subsets of P and we can let D_0 to be an open dense subset such that $p \in D_0$. Let $p_0 = p$. For each $n < \omega$, given p_n, let $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Then let G be the upward closure of $\{p_n : n < \omega\}$. Clearly, G is a generic filter over M containing p. We then have the following.

Proposition 1.5. *Let M be a countable admissible model and let P be a partial ordering in M. For every condition $p \in P$ there exists $G \subseteq P$ generic over M such that $p \in G$.*

The following is an easy example of non-trivial forcing.

Example. Let P be the poset of all partial functions $p : n \longrightarrow 2$ with $n \in \omega$ ordered by reversed inclusion. Since P is a perfect poset which belongs to any model of $KP + Infinity$, if M is a countable model of $KP + Infinity$, then there exists a generic filter over M. Let $f = \bigcup G$. It is easy to see that f is a function, being G a filter. Now, for ever $n < \omega$ let the set $D_n = \{p \in P : n \in dom(p)\}$, which is dense. Then, since $G \cap D \neq \emptyset$, $dom(f) = \omega$ so f is the characteristic function of a subset of ω which is not in M, which we call *Cohen generic real*. We have that $M[G]$ is a non-trivial forcing extension of M.

1.1.2 The generic model extension

Given a set A, we can define the class $L(A)$ in the same fashion we defined the constructible universe L as the union $\bigcup_{\alpha \in OR} L_\alpha(A)$ where $L_0(A) = tc(A)$, that is, the transitive closure of A; $L_{\alpha+1}(A) = D(L_\alpha(A))$ for every successor ordinal $\alpha + 1$ and, for α a limit, $L_\alpha(A) = \bigcup_{\beta<\alpha} L_\beta(A)$. Then, $L(A)$ is, informally, the class of all definable sets allowing parameters from A. Indeed, $L(A)$ is the least transitive model of ZF containing all the ordinals and all elements of A. As in the case of L, for every $\alpha, \beta \in OR$, $L_\alpha(A)$ is transitive and, if $\alpha < \beta$, then $L_\alpha(A) \subseteq L_\beta(A)$. Since admissible sets correctly compute L, if M is an admissible set, it correctly computes $L(A)$ whenever $A \subseteq M$. Actually, if M is an admissible set and $A \subseteq M$, $M[A]$ is the least admissible set such that $M \subseteq M[A]$ and $A \in M$ and it is of the form $L_\lambda(M \cup \{A\})$ for some $\lambda \geq OR \cap M$. Moreover, for every limit ordinal λ, $L_\lambda(M \cap \{A\}) = \bigcup_{\alpha<\lambda} L_\lambda((M \cup \{A\}) \cap V_\alpha)$ satisfies all axioms of KP with the possible exception of Δ_0-collection.

If $P \in M$ is a poset and $G \subseteq P$ is a filter generic over M, we let $M[G] = \bigcup_{\alpha<\lambda} L_\lambda((M \cap V_\alpha) \cup \{G\})$ with $\lambda = OR \cap M$. We will see that $M[G]$ is the least admissible set including M such that $G \in M$. We will see as well that $M[G]$ satisfies all axioms of ZFC which hold in the ground model M. This, however, is not trivial at all. Indeed, since the truth value in $M[G]$ of any given formula depends on the sets that belong to $M[G]$, a method is required in order to determine the formulas in the language of set theory which are true in $M[G]$. In particular, a method is required to prove that all axioms which hold in M also hold in $M[G]$. This is done by defining a relation $\Vdash$ between conditions p and formulas φ so that $p \Vdash \varphi$, read "p forces φ", if and only if for any generic G of M containing p, the formula φ holds in $M[G]$, that is, $M[G] \vDash \varphi$. Of course, the truth values of formulas in $M[G]$, particularly those with parameters, do not only depend on G and the sets in M but also on other sets that are obtained from elements in G and M. Since a set x belongs to $M[G]$ if and only if x is constructed from G and M in less than $\lambda = OR \cap M$ steps, for each $\alpha < \lambda$ we define in M the class of names of rank $\leq \alpha$ so that if $\tau \in M$ is a name of rank $\leq \alpha$, then the set in $M[G]$ named by τ is constructed from τ and G in $\leq \alpha + 1$ many steps.

Definition 1.6. Let M be an admissible set and P a partial ordering in M. We denote by M^P the class of P-*names*, defined by transfinite induction as follows:

(1) τ is a P-name of rank 0 if $\tau = \emptyset$.

(2) τ is a name of rank $\leq \alpha$, in symbols $\tau \in M^P_\alpha$ if $\tau = \{(\sigma, p) : p \in P \wedge \tau \in M^P_\beta$ with $\beta < \alpha\}$.

(3) τ is a P-name if $\tau \in M^P_\alpha$ for some $\lambda \in \sup(OR \cap M)$.

For instance $\sigma = \{(\emptyset, p)\}$ is a name of rank 1, while $\{(\sigma, p), (\emptyset, q)\}$ is a name of rank 2.

Definition 1.7. For every element $x \in M$, $\check{x} = \{(\check{y}, p) : p \in P \wedge y \in x\}$ is the *standard name* of x. The *standard name for the generic filter* G is $\dot{G} = \{(\check{p}, p) : p \in P\}$.

If G is a generic filter over M, the *interpretation* $i_G(\tau)$ of a name τ by G is defined by induction on the rank τ as follows:

(1) If $\tau \in M^P_0$, then $i_G(\tau) = \emptyset$,

(2) $i_G(\tau) = \{i_G(\sigma) : (\sigma, p) \in \tau \wedge p \in G\}$.

Note that it might happen that two different names have the same interpretation. If $p \in G$, then $\sigma = \{(\check{\emptyset}, p)\}$ and $\tau = \{(\check{\emptyset}, p), (\check{\emptyset}, q)\}$ are interpreted as $\{\emptyset\}$. However, if $p \notin G$ and $q \in G$, $i_G(\sigma) = \emptyset$ and $i_G(\tau) = \{\emptyset\}$.

We let $N = \{i_G(\tau) : \tau \in M^P\}$. Since $i_G(\check{x}) = x$ for every standard name $\check{x}$, it can be easily seen by induction on the rank that $M \subseteq N$. Moreover, since $\dot{G} = \{(\check{p}, p) : p \in \mathbb{P}\}$ and $i_G(\check{x}) = x$ for every standard name $\check{x}$, $i_G(\dot{G}) = \{i_G(\check{p}) : p \in G\} = \{p : p \in G\} = G$. It is easy to see that N satisfies Extensionality, Foundation, Pairing and Union.

1.1.3 The forcing relation

Our aim is to show that the class $N = \{i_G(\tau) : \tau \in M^P\}$ is $M[G]$, so we have to prove that N is an admissible set satisfying all axioms of ZFC that M satisfies. To do this, as already said, we define the *forcing relation* $\Vdash_P$, which we will simply write as $\Vdash$ if the context is clear. We shall go recursively starting with Δ_0-formulas (we can do this because admissible sets satisfy Σ_1-recursion, but let us ignore this details).

(1) If $\tau, \sigma \in M^P_0$, $p \Vdash \sigma = \tau$ for every $p \in P$, and $p \Vdash \sigma \in \tau$ for no $p \in P$.

(2) Let us assume that the forcing relation involving P-names of rank $\leq \alpha$ and atomic formulas is been already defined. Let $\tau, \sigma \in M^P_{\leq \alpha+1}$. Then,

   $p \Vdash \sigma \subseteq \tau$ if and only if $\{q \leq : q \leq p' \to \exists(\tau', p'') \in \tau(q \leq p'' \wedge q \Vdash \sigma' = \tau')\}$ is dense below p for very $(\sigma', p') \in \sigma$. We define $p \Vdash \tau \subseteq \sigma$ analogously.

$p \Vdash \sigma = \tau$ if and only if $p \Vdash \sigma \subseteq \tau$ and $p \Vdash \tau \subseteq \sigma$.

(3) Let $\sigma, \tau \in M^P$ and $p \in P$, $p \vDash \sigma \in \tau$ if and only if $\{q \leq p : \exists(\tau', p') \in \tau(q \leq p' \wedge q \Vdash \sigma = \tau')\}$ is dense below p.

(4) For any Δ_0-formulas φ, ψ for which $\Vdash$ has already been defined,

$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$,

$p \Vdash \neg\varphi$ if and only if for no $q \leq p$, $q \Vdash \varphi$,

$p \Vdash \exists x \in \tau\varphi(x)$ if and only if the set $\{q \leq p : \exists(\tau', p') \in \tau(q \leq p' \wedge q \Vdash \varphi(\tau'))\}$ is dense below p.

(5) If φ, ψ are arbitrary formulas for which $\Vdash$ has been already defined:

$p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$,

$p \Vdash \neg\varphi$ if and only if for no $q \leq p$, $q \Vdash \varphi$,

$p \Vdash \exists x\varphi(x)$ if and only if for every $q \leq p$ there exists some $r \leq q$ and a P-name τ such that $r \Vdash \varphi(\tau)$.

By induction both on the complexity on the formula φ and on the rank of the names $\sigma_1, \ldots, \sigma_m$ one can prove the following.

Theorem 1.8 (Forcing Theorem)**.** *If G is P-generic over M and $N = \{i_G(\tau) : \tau \in M^P\}$, then $N \vDash \varphi(i_G(\sigma_1), \ldots, i_G(\sigma_n))$ if and only if $\exists p \in G(M \vDash$ "$p \Vdash_P \varphi(\sigma_1, \ldots, \sigma_n)$") for all Δ_0-formula φ and every name $\sigma_1, \ldots, \sigma_n$.*

Remark. If M also satisfies σ_n-Separation and σ_n-Collection, then Forcing Theorem applies also to all σ_n-formulas.

Theorem 1.9 (Generic Model Theorem)**.** *If M is admissible, then $N := \{i_G(\tau) : \tau \in M^P\}$ is the least admissible set including M and containing G. Moreover: if $M \vDash$ Infinity, then $N \vDash$ Infinity; if $M \vDash \Sigma_n -$ Separation $+ \Sigma_n -$ Recursion, then $N \vDash \Sigma_n -$ Separation; if $M \vDash \Sigma_n -$ Collection$+\Sigma_n -$Recursion, then $N \vDash \Sigma_n -$ Collection; if $M \vDash$ Power Set, then $N \vDash$ Power Set; if $M \vDash AC$, then $N \vDash AC$.*

As a consequence, $N = M[G]$. Indeed, since N is admissible, for every $X \in N$, $L_\lambda(X) \subseteq N$. Since N includes $M \cup \{G\}$, $M[G] = \bigcup_{\alpha<\lambda} L_\lambda((M \cap V_\alpha) \cup \{G\})) \subseteq N$.

1.2 Iterated forcing

Let P be a forcing notion over a countable transitive model M of ZFC^* and let $G \subseteq P$ be a generic filter over M. Let Q be a forcing notion in $M[G]$. Then, we can forcing again with Q over $M[G]$, resulting in a new forcing extension $M[G][H]$, with $H \subseteq Q$

a generic filter over $M[G]$. This iterated process can be done directly from M in just one step. The forcing notion used is called the *iteration of* P *and* Q and is denoted by $P * \dot{Q}$. We note that since Q is a forcing notion in $M[G]$, it might happen that it does not belong to M. To have some control over Q from M, we fix a name $\dot{Q} = \langle \dot{Q}, \dot{\leq}_Q \rangle$ for Q so that some condition $p \in G$ forces $\dot{Q}$ to be a partial ordering in $M[G]$ and $i_G(\dot{Q}) = Q$. The conditions of $P * \dot{Q}$ are pairs $(p, \dot{q})$ such that $p \in P$, $\dot{q} \in dom(\dot{Q})$ and $p \Vdash_P$ "$\dot{q} \in \dot{Q}$. For any two $(p_1, \dot{q}_1), (p_2, \dot{q}_2) \in P * \dot{Q}$, $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq_P p_2$ and $p \Vdash_P$ "$\dot{q}_1 \leq_{\dot{Q}} \dot{q}_2$. The set $P * \dot{Q}$ with the defined order is a partial ordering in M and if $G \subseteq P$ is generic over M and $H \subseteq Q$ is generic over $M[G]$, then $G * \dot{H} = \{(p, \dot{q}) \in P * \dot{Q} : p \in G \wedge i_G(\dot{q}) \in H\}$ is generic over M. Moreover, since $G * \dot{H}$ can be easily constructed from G and H, $M[G * \dot{H}] \subseteq M[G][H]$ and, since G is the first projection of $G * \dot{H}$, it can be easily seen that $G \in M[G * \dot{H}]$. But then, since we have G and $G * \dot{H}$, we easily get H, so $M[G][H] \subseteq M[G * \dot{H}]$ and the following holds.

Theorem 1.10. $M[G * \dot{H}] = M[G][H]$.

Clearly, if S was a forcing notion in $M[G][H] = M[G * \dot{H}]$, one could repeat this whole process, obtaining a forcing extension $M[G][H][I]$ with $I \subseteq S$ generic over $M[G][H]$. This motivates the following definition.

Definition 1.11. P_{n+1} is an *iteration of length* $n + 1$ if $P_{n+1} = P_n * \dot{Q}_n$ where P_n is an iteration of length n such that $\Vdash_{P_n}$ "$\dot{Q}_n$ is a poset".

Therefore, if $P_{n+1} = (\ldots((Q_0 * \dot{Q}_1) * \dot{Q}_2) * \ldots * \dot{Q}_n)$ and $p \in P$, then p is of the form $(\ldots((q_0, \dot{q}_1), q_2), \ldots), q_n)$ where $(\ldots((q_0, \dot{q}_1) \ldots, q_{n-1}) \in (\ldots((Q_0 * \dot{Q}_1) * \dot{Q}_2) * \ldots * \dot{Q}_{n-1})$ and $(\ldots((q_0, \dot{q}_1), q_2), \ldots q_{n-1})$ forces (with respect to $(\ldots((Q_0 * \dot{Q}_1) * \dot{Q}_2) * \ldots * \dot{Q}_{n-1})$, that $\dot{q}_{n-1} \in \dot{Q}_n$. To simplify this cumbersome notation, we just write $(q_0, \dot{q}_1, \ldots, \dot{q}_n)$ for any element in P_{n+1}, where $q_0 \in Q_0 = P_0$ and $p \upharpoonright m \Vdash_{P_m} p_m \in \dot{Q}_m$ for every $m < n$. This discussion motivates the following.

Definition 1.12. We say that P_λ is a *forcing iteration with finite support of length* $\lambda > 0$ if:

(1) either $\lambda = 1$ and $\mathbb{P}_\lambda$ is a poset, or

(2) $\lambda = \alpha + 1$ with $\alpha > 0$ and P_α a forcing iteration with finite support of length α such that $P_\lambda = P_\alpha * \dot{Q}_\alpha$ where $\dot{Q}_\alpha$ is the P_α-name of a poset, or

(3) λ is a limit ordinal and

  (3.1) the elements of P_λ are λ-sequences $\langle p_\alpha : \alpha < \lambda \rangle$ such that for all but finitely-many $\alpha < \alpha$, $p_\alpha = 1$, where 1 is the maximal element in P_α, whose existence can be assumed without loss of generality, (3.2) for every $0 < \alpha < \lambda$, the poset P_α consisting of the elements $p \upharpoonright \alpha$ with $p \in P_\lambda$ is an iteration with finite support of length α, and

(3.3) the ordering $\leq_{P_\lambda}$ in P_λ is given by $p \leq_{P_\lambda} q$ if and only if for all $0 < \alpha\lambda$, $p \upharpoonright \alpha \leq_{P_\alpha} q \upharpoonright \alpha$.

It can be checked by induction on $\lambda > 0$ that if $p \in P_\lambda$, then $p \upharpoonright \alpha \Vdash_{P_\alpha}$ "$p_\alpha \in \dot{Q}_\alpha$" for every $0 < \alpha < \lambda$. Note that if P_λ is a forcing iteration with finite support of length λ, then for every $\alpha < \lambda$ we can see each P_α as the sub-poset of P_λ where each $\langle p_\beta : \beta < \lambda \rangle$ is identified with $\langle p'_\beta : \beta < \lambda \rangle$ where $p'_\beta = p_\beta$ for very $\beta < \alpha$ and $p'_\beta = 1$ for every $\beta \geq \alpha$.

Recall that a poset P is ccc if all its antichains are of countable size. More generally, a poset P is κ-cc with κ a regular cardinal κ if every antichain of P is of size less than κ.

Theorem 1.13. *If κ is an uncountable regular cardinal and P_λ is a forcing iteration with finite support of length λ such that $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is κ-cc" for every $\alpha < \lambda$, then P_λ is κ-cc.*

The following theorem shows that κ-cc posets are useful forcing notions whenever we do not want to collapse cardinals.

Theorem 1.14. *If P is a κ-cc poset in M and $G \subseteq P$ is generic over M, then all cardinals greater than or equal to κ remain cardinals in $M[G]$.*

1.3 Consistency of Martin's Axiom

Recall that Martin's Axiom states the following:

> For every ccc poset P and every family of $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of P, there is a filter $G \subseteq P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

To show its consistency, we build a model in which MA holds. Then, pick M to be a countable transitive model of ZFC^*. For each *ccc* poset $P \in M$, we force over M to add a generic filter for families of uncountable size of maximal antichains in P. Of course, we want to do this for every *ccc* poset, so we should force with all of them. However, the collection of *ccc* posets in M is a proper class. Note that if P is a *ccc* poset and $\langle A_\alpha : \alpha < \omega_1 \rangle \subseteq P$ is a family of maximal antichains, by Löwenheim-Skolem-Tarski, there is an elementary sub-poset Q of size $\leq \aleph_1$ such that $\langle A_\alpha : \alpha < \omega_1 \rangle \subseteq Q$. Since P is *ccc*, each A_α is countable, so Q is *ccc* as well. Also, since Q is isomorphic to a set whose set of conditions is a subset of ω_1, we may assume Q to be of this form.

Definition 1.15. If Q is a *ccc* poset whose set of conditions is a subset of ω_1, we say that Q is a *suitable* set.

In Chapter 3 we have proved that GCH is consistent, so we can assume M is a model of GCH. In this case, there are, up to isomorphism, only $\aleph_2$ many suitable sets. The idea, is to define the iteration using only suitable sets to get the desired extension in $\aleph_2$-many steps. But first note that if P is a partial ordering and τ is a P-name for a subset of some ordinal λ, then the set σ consisting of all pairs $(\check{\alpha}_q, q)$ where α_q is an ordinal and q is an element in a maximal antichain A below p such that $q \Vdash_P \tau' = \check{\alpha}_q$ for some $(\tau', p) \in \tau$. Therefore σ consists of the pairs of the form $(\check{\alpha}, p)$ with $\alpha < \lambda$. Clearly, P forces that $\sigma = \tau$. To this names we call them *nice names*. It is easy to see that if P is *ccc*, then every P-name for a subset of $\aleph_1$ has an equivalent nice name of size $\aleph_1$.

To summarize, if each iteration is *ccc*, no cardinals are collapsed, which mean that at any stage of the iteration, $\aleph_1$ and $\aleph_2$ remain as in the ground model. Also, if $P_\beta = \langle \dot{Q}_\alpha : \alpha < \beta \rangle$ is an iteration with finite support with $\beta \leq \omega_2$ such that for every $\alpha < \beta$, $\Vdash_{P\alpha}$ "$\dot{Q}_\alpha$ is a suitable partial ordering" we can assume that $\dot{Q}_\alpha$ is a P_α-name of size $\aleph_1$ for all $\alpha < \beta$, so by induction on β we get that if $\beta < \omega_2$, then P_β has size less than $\aleph_2$ and P_{ω_2}, the last stage in our iterated forcing, is of size $\aleph_2$. Therefore, there are at most $\aleph_2$ nice P_β-names of cardinality $\aleph_1$ for subsets of $\aleph_1$, from which the following holds.

Lemma 1.16. *Let M be a transitive model of $ZFC^* + GCH$ and let $P_\beta = \langle \dot{Q}_\alpha : \alpha < \beta \rangle$ be an iteration with finite support with $\beta \leq \omega_2$ such tht for every $\alpha < \beta$, $\Vdash_{P\alpha}$ "$\dot{Q}_\alpha$ is a suitable partial ordering". Then, for every P_β-generic filter G_β over M, $M[G_\beta] \vDash 2^{\aleph_1} = \aleph_2$.*

So let M be a countable transitive model of $ZFC^* + GCH$. Let $\pi : \omega_2 \longrightarrow (\omega_2 \times \omega_2)$ be a surjective function such that if $\pi(\alpha) = (\beta, \gamma)$, then $\beta \leq \alpha$ for every $\alpha < \omega_2$. Let P_0 be a suitable partial ordering and let $\langle \dot{Q}_\gamma : \gamma < \omega_2 \rangle$ be an enumeration of all nice P_0-names for suitable partial orderings. Now, suppose we have defined P_α and that for each $\beta \leq \alpha$ we have an enumeration $\langle \dot{Q}^\beta_\gamma : \gamma < \omega_2 \rangle$ of all nice P_β-names for suitable partial orderings. Then, $P_{\alpha+1} = P_\alpha * \dot{Q}^\beta_\gamma$ where $\pi(\alpha) = (\beta, \gamma)$. For the limit ordinals λ just let P_λ consists of all λ-sequences $\langle p_\alpha : \alpha < \lambda \rangle$ with $p_\alpha = 1$ for all but finitely-many $\alpha < \lambda$ and $p \restriction \alpha \in P_\alpha$ for all $\alpha < \lambda$.

Theorem 1.17 (Martin-Solovay, 1970)**.** *Let M and P_{ω_2} be as defined above and let $G \subseteq P_{\omega_2}$ be generic over M. Then MA holds in $M[G]$.*

Proof. Let Q be a *ccc* partial ordering in $M[G]$ and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a family of maximal antichains of Q. Since we may assume that Q is suitable, there is, as said before, a nice P_{ω_2}-name $\dot{Q}$ in M of cardinality less than or equal to $\aleph_1$. Also, since each A_α is a subset of ω_1 for each $\alpha < \omega_1$, they have a nice P_{ω_2}-name $\dot{A}_\alpha$ of size $\aleph_1$, too. Let β be an ordinal in ω_2 such that every p in the range of $\dot{Q}$ which is also in the range of $\dot{A}_\alpha$ for every $\alpha < \omega_1$, $supp(p) = \{\alpha < \omega_2 : p_\alpha \neq 1\} \subseteq \beta$. Let $\dot{Q} \restriction \beta$ be the set $\{(\check{\gamma} \restriction \beta, p \restriction \beta) : (\check{\gamma}, p) \in \dot{Q}\}$ where each $\check{\gamma} \restriction \beta$ is a P_β-name for γ. Let also $\dot{A}_\alpha \restriction \beta$ be $\{(\check{\gamma} \restriction \beta, p \restriction \beta) : (\check{\gamma}, p) \in \dot{A}_\alpha\}$ for each $\alpha < \omega_1$. This way $\dot{Q} \restriction \beta$ is a P_β-name for a suitable partial ordering and $\dot{A}_\alpha$ is a P_β-name for a maximal antichain of $\dot{Q} \restriction \beta$, so there is some $\gamma < \omega_2$ such that $\dot{Q} \restriction \beta = \dot{Q}^\beta_\gamma$. If $\pi(\alpha) = (\beta, \gamma)$ we may see $\dot{Q}^\beta_\gamma$ and $\dot{A}_\alpha \restriction \beta$ for all $\alpha < \omega_1$ as P_α-names. Since G_α is a P_α-generic filter over M and $i_{G_\alpha}(\dot{A}_\alpha \restriction \beta) = A_\alpha$ for all $\alpha < \omega_1$ and $i_{G_\alpha}(\dot{Q}^\beta_\gamma) = Q$, then $M[G_{\alpha+1}]$ is of the form $M[G_\alpha][H]$ with $H \subseteq Q$ generic over $M[G_\alpha]$. Therefore, H is generic for the family $\{A_\alpha : \alpha < \omega_1\}$, which also holds in $M[G]$, so we are done. ■

Note that defining an analogous iteration with length any regular cardinal κ greater than κ, $2^{\aleph_0} = \kappa$ would hold in the generic extension. As a consequence, the following holds.

Corollary 1.18. *If ZFC is consistent, so is* $ZFC + MA + 2^{\aleph_0} > \aleph_1$.

Bibliography

[Bag19] Joan Bagaria. Models of Set Theory. `https://www.icrea.cat/security/files/researchers/researcher-sections/mst2019-20.pdf`, 2019-20.

[BM13] Joan Bagaria, Menachem Magidor. Group radicals and strongly compact cardinals. *Transactions of the American Mathematical Society*, 366(4):1857-1877, 2013.

[Coh63] Paul J. Cohen. The Independence of the Continuum Hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, 50(6):1143-1148, 1963.

[Coh64] Paul J. Cohen The Independence of the Continuum Hypothesis II. *Proceedings of the National Academy of Sciences of the United States of America*, 51(1):105-110.

[Dug85] Mandfred Dugas. On reduced products of abelian groups. *Rendiconti del Seminario Matematico della Università di Padova*, 73:41-47, 1985.

[DG85] Mandfred Dugas, Rüdiger Göbel. On radicals and products. *Pacific Journal of Mathematics*, 118(1):79-104, 1985.

[Eda82] Katsuya Eda. A Boolean power and a direct product of abelian groups, *Tsukuba Journal of Mathematics*, 6(2):187-193, 1982.

[Eda83] Katsuya Eda. On a Boolean power of a torsion free abelian group, *Journal of Algebra*, 82(1):84-93, 1983.

[EA87] Katsuya Eda, Yoshihiro Abe. Compact cardinals and abelian groups. *Tsukuba Journal of Mathematics*, 11(2):353-360, 1987.

[Ekl76] Paul C. Eklof. Whitehead's problem is undecidable. *American Mathematical Monthly*, 83:775-788, 1976.

[Ekl77] Paul C. Eklof. Homological algebra and set theory. *Transactions of the American Mathematical Society*, 227: 207-225, 1977.

[Ekl97] Paul C. Eklof. Set theoretic generated by abelian group theory. *The Bulletin of Symbolic Logic*, 3(1):1-16, 1997.

[EM02] Paul C. Eklof, Alan H. Mekler. Almost Free Modules. Set theoretic-methods. Revised Edition. *North-Holland Mathematical Library. Elsevier Science B. V.*, 2002.

[FFMS] Solomon Feferman, Harvey M. Friedman, Penelope Maddy, John R. Steel. Does mathematics needs new axioms? *The Bulletin of Symbolic Logic*, 6(1):401-446, 2000.

[Fuc70] László Fuchs. Infinite Abelian Groups. Vol I. *Accademic Press, New York*, 1970.

[Göd40] Kurt Gödel. Consistency of the Continuum Hypothesis. *Princeton University Press*, 1940.

[Jech03] Thomas Jech. Set Theory. The Third Millennium Edition, revised and expanded. *Springer-Verlag Berlin Heidelberg*, 2003.

[Kan94] Akihiro Kanamori. The Higher Infinite. *Perspectives in Mathematical Logic, Springer-Verlag, Berlin*, 1994.

[Kap69] Irving Kaplansky. Infinite Abelian Groups (Revised Edition). *University of Michigan Press, Ann Arbor*, 1969.

[Kun13] Kenneth Kunen. Set Theory. Revised Edition. *College Publications. King's College Londond, Strand, London, UK*, 2013.

[Lan02] Serge Lang. Algebra. Revised Third Edition. *Springer-Verlag Berlin Heidelberg*, 2002.

[MS70] Donald A. Martin, Robert M. Solovay. Internal cohen extensions. *Annals of Mathematical Logic*, 2(2):143-178, 1970.

[Mag76] Menachem Magidor. How large is the first strongly compact cardinal? or A study on identity. *Annals of Mathematical Logic*, 10(1):33-57, 1976.

[Men97] Elliott Mendelson. Introduction to Mathematical Logic. Fourth Edition. *Chapman and Hall*, 1997.

[ST71] Robert M. Solovay, Stanley Tennenbaum. Iterated Cohen Extensions and Souslin's Problem. *Annals of Mathematics*, 94(2):201-245, 1971.