# A Scaling of the photonic operators

The scaling properties of $a$, $a^\dagger$ can be derived from the quantization of the field operators in the Coulomb gauge. However, we found insightful to give a more intuitive reasoning. Consider the electromagnetic energy

$$\frac{1}{2}\int dV\epsilon_0 E_0^2 + \frac{1}{2}\int dV\frac{B_0^2}{\mu_0} = e\phi, \tag{113}$$

which is an intensive quantity. This implies that, for instance, $B_0$ scales as $1/\sqrt{V}$. In the thermodynamic limit, we take $N \to \infty$, $V \to \infty$ such that $N/V = \rho$ is well defined. The magnetic field operator $B = B_0\left(a + a^\dagger\right)$ cannot scale as $N$, it is again an intensive quantity, so $a$, $a^\dagger$ must scale as $\sqrt{N}$ to counter the scaling of $B_0$.

# B Bounds for the partition function $Z$

## B.1 *À la* Hepp and Lieb

Let us prove the bounds in Eq. (8). Recall that

$$\mathcal{H} = \omega_c a^\dagger a + \frac{\omega_z}{2}\sum_j \sigma_j^z + \frac{\lambda}{\sqrt{N}}\sum_j \sigma_j^x\left(a + a^\dagger\right). \tag{114}$$

We will start with the lower bound. For that, we make use of Jensen's inequality for convex funtions, which states that, for a convex function $f$

$$f\left(\sum_i \lambda_i x_i\right) \leq \sum \lambda_i f(x_i). \tag{115}$$

Because the exponential is a convex function, it follows that

$$\begin{aligned} Z =& Tr_S\left(\frac{1}{\pi}\int d^2\alpha\ \langle\alpha| e^{-\beta\mathcal{H}}|\alpha\rangle\right) \geq \\ & Tr_S\left(\frac{1}{\pi}\int d^2\alpha\ e^{-\beta\langle\alpha|\mathcal{H}|\alpha\rangle}\right) = Tr_S\left(\frac{1}{\pi}\int d^2\alpha\ e^{-\beta\mathcal{H}(\alpha)}\right) = \bar{Z}. \end{aligned} \tag{116}$$

The upper bound requires some more algebra. We will make use of the Golden-Thompson inequality: let $A$ and $B$ be Hermitian operators, then

$$Tr\left((AB)^n\right) \leq Tr\left(A^n B^n\right). \tag{117}$$

We are concerned with obtaining an upper bound for $Z$, which we can express as

$$Z = Tr\left(e^{-\beta\mathcal{H}}\right) = Tr\left(\left(1 - \frac{\beta}{m}\mathcal{H}\right)^m\right), \tag{118}$$

for $m \to \infty$. Before advancing further, it is convenient to express $a^\dagger a = aa^\dagger - 1$ in terms of a sum of projectors of coherent states, using the closure relation $\pi^{-1}\int d^2\alpha\,|\alpha\rangle\langle\alpha|$,

$$a^\dagger a = \frac{1}{\pi}\int d^2\alpha\left(a|\alpha\rangle\langle\alpha| a^\dagger - |\alpha\rangle\langle\alpha|\right) = \frac{1}{\pi}\int d^2\alpha\left(|\alpha|^2 - 1\right)|\alpha\rangle\langle\alpha|. \tag{119}$$

We can now write

$$1 - \frac{\beta}{m}\mathcal{H} = 1 - \frac{1}{\pi}\int d^2\alpha\left(\mathcal{H}(\alpha) - \omega_c\right)|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d^2\alpha\left(1 - \frac{\beta}{m}\left(\mathcal{H}(\alpha) - \omega_c\right)\right)|\alpha\rangle\langle\alpha|. \tag{120}$$

Defining the multiplication operator $A$

$$A\left|\alpha\right\rangle = \left(1-\frac{\beta}{m}\left(\mathcal{H}(\alpha)-\omega_c\right)\right)\left|\alpha\right\rangle, \tag{121}$$

we can write the previous expression as

$$1-\frac{\beta}{m}\mathcal{H} = A\frac{1}{\pi}\int d^2\alpha \left|\alpha\right\rangle\left\langle\alpha\right| = AB. \tag{122}$$

At this point we use the Golden-Thompson inequality, noticing that $B^n=\mathbb{I}$

$$Tr\left(\left(1-\frac{\beta}{m}\mathcal{H}\right)^m\right) = Tr\left((AB)^m\right) \leq Tr\left(A^m B^m\right) = Tr\left(A^m\right), \tag{123}$$

which finally gives us the upper bound

$$Z \leq Tr\left(\left(1-\frac{\beta}{m}\left(\mathcal{H}(\alpha)-\omega_c\right)\right)^m\right) = Tr\left(e^{-\beta(\mathcal{H}(\alpha)-\omega_c)}\right) = e^{\beta\omega_c}\bar{Z}. \tag{124}$$

### B.2 Including the $A^2$ term in the Coulomb gauge.

As we saw in App. B.1 the quadratic terms of bosonic operators in $\mathcal{H}$ generate extra summands when writing $\mathcal{H}$ as an integral of projectors of coherent states. In that case, the only quadratic term in $\mathcal{H}$ was $\omega_c$. In the Coulomb gauge if we respect gauge invariance we must include the $A^2$ term, which we can write as $\Delta\left(a+a^\dagger\right)^2$. Now $\left(a+a^\dagger\right)^2 = a^2+\left(a^\dagger\right)^2+aa^\dagger+a^\dagger a$, so

$$\left\langle\alpha\right|\left(a+a^\dagger\right)^2\left|\alpha\right\rangle = \alpha^2+(\alpha^*)^2+2|\alpha|^2+1. \tag{125}$$

Consequently

$$\begin{aligned}\left(a+a^\dagger\right)^2 &= \frac{1}{\pi}\int d^2\alpha\ \left(\alpha^2+(\alpha^*)^2+2|\alpha|^2-1\right)\left|\alpha\right\rangle\left\langle\alpha\right| \\ &= \frac{1}{\pi}\int d^2\alpha\ \left(\left\langle\alpha\right|\left(a+a^\dagger\right)^2\left|\alpha\right\rangle-2\right)\left|\alpha\right\rangle\left\langle\alpha\right|.\end{aligned} \tag{126}$$

So in this case, the bounds for $Z$ read

$$\bar{Z} \leq Z \leq e^{\beta(\omega_c+2\Delta)}\bar{Z}. \tag{127}$$

Because $\Delta$ is a finite quantity, as is $\omega_c$, the substitution of $Z$ by $\bar{Z}$ is still exact in the thermodynamic limit. In the sense that they yield the same free energy.

### B.3 In the dipole gauge and magnetic QED

We are now concerned with the applicability of the bounds in Eq. (8) to the Hamiltonians of the dipole gauge (33) and magnetic QED, which as we showed, takes the form of the Dicke model (4). The latter can be dismissed quickly, since it takes the form of a pure Dicke model the bounds proven in App. B.1 apply here as well. For the former, we must simply note that the only quadratic term in bosonic operators that it contains is the photonic term $\omega_c a^\dagger a$, so from the perspective of the $Z$ bounds, the Hamiltonian behaves exactly like the Dicke model (4), and the same bounds hold for $Z$ in this case.

### B.4 For a multimode cavity

When considering a multimode cavity, the photonic term of the Hamiltonian (99) takes the form

$$\sum_{l}^{M} \omega_l a_l^\dagger a_l. \tag{128}$$

This is the only quadratic term in the Hamiltonian, so the derivation is analogous to the one in App. B.1, with the exception that we now obtain

$$\bar{Z} \leq Z \leq e^{\beta \sum_l^M \omega_l} \bar{Z}. \tag{129}$$

So if the number of modes $M$ is finite, or alternatively, if $M \to \infty$ but $\sum_l^M \omega_l$ is kept finite, we can exactly substitute $Z$ by $\bar{Z}$ in the thermodynamic limit $N \to \infty$.

# C Computing $\alpha$ near the critical point

If we recover the minimum condition for $\alpha \neq 0$ we have $\omega_c E = 2\lambda^2 \tanh \beta E$. Close to the transition, for $\alpha \ll \frac{\omega_z \sqrt{N}}{4\lambda}$, we can expand $E$ in powers of $\alpha$

$$E = \frac{\omega_z}{2}\sqrt{1 + \frac{\lambda^2\alpha^2}{\omega_z^2 N}} \approx \frac{\omega_z}{2}\left(1 + \frac{8\lambda^2\alpha^2}{\omega_z^2 N}\right) = \frac{\omega_z}{2} + \frac{4\lambda^2\alpha^2}{\omega_z N}. \tag{130}$$

We can do the same for $\tanh \beta E$, resulting in

$$\tanh \beta E \approx \tanh \frac{\beta\omega_z}{2} + \beta \frac{4\lambda^2\alpha^2}{\omega_z N} \operatorname{sech}^2 \frac{\beta\omega_z}{2}. \tag{131}$$

We can now return no the transcendental equation, which, after these approximations becomes solvable analytically

$$\begin{aligned} &\frac{\omega_c\omega_z}{2} + \frac{4\omega_c\lambda^2\alpha^2}{\omega_z N} = 2\lambda^2 \tanh \frac{\beta\omega_z}{2} + \beta \frac{8\lambda^4\alpha^2}{\omega_z N} \operatorname{sech}^2 \frac{\beta\omega_z}{2} \implies \\ &\frac{\alpha^2}{N}\left(\beta \frac{8\lambda^4}{\omega_z} \operatorname{sech} \frac{\beta\omega_z}{2} - \frac{4\omega_c\lambda^2}{\omega_z}\right) = 2 \tanh \frac{\beta\omega_z}{2} \left(\lambda_c^2 - \lambda^2\right). \end{aligned} \tag{132}$$

so

$$\alpha = \sqrt{N A(\lambda)\left(\lambda^2 - \lambda_c^2\right)}. \tag{133}$$

Where

$$A(\lambda) = \frac{2\tanh \frac{\beta\omega_z}{2}}{\frac{4\omega_c\lambda^2}{\omega_z} - \beta \frac{8\lambda^4}{\omega_z} \operatorname{sech} \frac{\beta\omega_z}{2}}. \tag{134}$$

# D Commutativity in the thermodynamic limit

### D.1 In the Coulomb gauge

Consider Hamiltonian (22), we can express it as

$$\mathcal{H} = \mathcal{H}_m + \mathcal{H}_{ph} + \mathcal{H}_{int}, \tag{135}$$

with

$$\mathcal{H}_m = \sum_i \frac{\boldsymbol{p}^2}{2m} + \sum_i V(\boldsymbol{r}_i) + \frac{1}{2}\sum_{i\neq j} v(\boldsymbol{r}_i - \boldsymbol{r}_j), \tag{136}$$

$$\mathcal{H}_{ph} = \hbar\omega_c a^\dagger a + \sum_i \frac{e^2\boldsymbol{A}_0^2}{2mc^2}\left(a + a^\dagger\right)^2, \tag{137}$$

$$\mathcal{H}_{int} = \sum_i \frac{e}{mc}\boldsymbol{p}_i\boldsymbol{A}_0\left(a + a^\dagger\right). \tag{138}$$

These Hamiltonians scale as $\sim N$, so only their intensive analogues $\mathcal{H}_m/N$, $\mathcal{H}_{ph}/N$, $\mathcal{H}_{int}/N$ are well defined in the thermodynamic limit $N \to \infty$. Let us show that, in the limit $N \to \infty$

$$[\frac{\mathcal{H}_m}{N}, \frac{\mathcal{H}_{int}}{N}] = [\frac{\mathcal{H}_{ph}}{N}, \frac{\mathcal{H}_{int}}{N}] = 0. \tag{139}$$

Consider

$$[\frac{\mathcal{H}_m}{N}, \frac{\mathcal{H}_{int}}{N}] = [\sum_i V(\boldsymbol{r}_i) + \frac{1}{2}\sum_{i\neq j} v(\boldsymbol{r}_i - \boldsymbol{r}_j), \sum_i \frac{e}{mc}\boldsymbol{p}_i\boldsymbol{A}_0\left(a + a^\dagger\right)]\frac{a + a^\dagger}{N^2}, \tag{140}$$

for any function of the position operators $f(\boldsymbol{r}_i)$ we have $[f(\boldsymbol{r}_i), \boldsymbol{p}_j] = \delta_{i,j} i\hbar\nabla_{\boldsymbol{r}_i} f(\boldsymbol{r}_i)$. Introducing the external force $\boldsymbol{F}_i^{\text{ext}} = -\nabla_{\boldsymbol{r}_i} V(\boldsymbol{r}_i)$ and the Coulomb force $\boldsymbol{F}_{i,j}^C = -\nabla_{\boldsymbol{r}_i} v(\boldsymbol{r}_i - \boldsymbol{r}_j)$, we obtain

$$[\frac{\mathcal{H}_m}{N}, \frac{\mathcal{H}_{int}}{N}] = -\frac{i\hbar e\left(a + a^\dagger\right)\boldsymbol{A}_0}{mcN^2}\sum_i \boldsymbol{F}_i^{\text{ext}}. \tag{141}$$

The term containing $\boldsymbol{F}_i^C$ vanishes because $\sum_{i,j}\boldsymbol{F}_{i,j}^C = 0$. Notice that $\left(a + a^\dagger\right)\boldsymbol{A}_0$ is an intensive quantity, and $\sum_i \boldsymbol{F}_i^{\text{ext}}$ scales as $N$, so the commutator scales as $1/N$ and vanishes in the thermodynamic limit.

Consider now

$$[\frac{\mathcal{H}_{ph}}{N}, \frac{\mathcal{H}_{int}}{N}] = \frac{\hbar\omega_c}{N^2}\sum_i \frac{e}{mc}\boldsymbol{p}_i\boldsymbol{A}_0[a^\dagger a, a + a^\dagger] = \frac{\hbar\omega_c}{N^2}\sum_i \frac{e}{mc}\boldsymbol{p}_i\boldsymbol{A}_0\left(a^\dagger - a\right). \tag{142}$$

Now $\sum_i \boldsymbol{p}_i$ scales as $N$, so the commutator again scales as $1/N$, vanishing in the thermodynamic limit.

## D.2 In the dipole gauge

Consider Hamiltonian (33), again it can be expressed as $\mathcal{H}' = \mathcal{H}'_m + \mathcal{H}'_{ph} + \mathcal{H}'_{int}$, with the difference that now

$$\mathcal{H}'_m = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \sum_i V(\boldsymbol{r}_i) + \frac{1}{2}\sum_{i\neq j} v(\boldsymbol{r}_i - \boldsymbol{r}_j) + \frac{\omega_c}{c^2\hbar}(\boldsymbol{d}\boldsymbol{A}_0)^2 \tag{143}$$

$$\mathcal{H}'_{ph} = \hbar\omega_c a^\dagger a, \tag{144}$$

$$\mathcal{H}'_{int} = -i\frac{\omega_c}{c}\boldsymbol{d}\boldsymbol{A}_0\left(a^\dagger - a\right). \tag{145}$$

Now

$$[\frac{\mathcal{H}'_m}{N}, \frac{\mathcal{H}'_{int}}{N}] = -i[\sum_i \frac{\boldsymbol{p}_i^2}{2m}, \boldsymbol{d}\boldsymbol{A}_0]\frac{\omega_c\left(a^\dagger - a\right)}{cN^2} = \frac{\omega_c e\hbar}{cN^2}\sum_i \frac{\boldsymbol{p}_i^2}{m}\boldsymbol{A}_0\left(a^\dagger - a\right). \tag{146}$$

Following the same reasoning as in the previous appendix, this commutator vanishes in the thermodynamic limit. Finally consider

$$[\frac{\mathcal{H}'_{ph}}{N}, \frac{\mathcal{H}'_{int}}{N}] = -i\frac{\hbar\omega_c^2}{cN^2}d\boldsymbol{A}_0[a^\dagger a, a^\dagger - a] = -i\frac{\hbar\omega_c^2}{cN^2}d\boldsymbol{A}_0\left(a^\dagger - a\right). \tag{147}$$

Which again vanishes in the thermodynamic limit.

### D.3 In magnetic QED

We have

$$\mathcal{H}_{int} = \frac{\lambda}{\sqrt{N}} \sum_j \sigma_j^x \,\mathrm{Re}(\alpha) \tag{148}$$

The exact expression of $\mathcal{H}_S$ is not know in general, but so long as it is extensive, its behavior can be captured by the part that we do now

$$\mathcal{H}_S = \frac{\omega_z}{2} \sum_j \sigma^z + \mathcal{H}_C. \tag{149}$$

For that reason, let us compute

$$\frac{1}{N^2}[\frac{\omega_z}{2} \sum_j \sigma_j^z, \frac{\lambda}{\sqrt{N}} \sum_j \sigma_j^x \,\mathrm{Re}(\alpha)] = \frac{1}{N^2} \frac{\omega_z \lambda \,\mathrm{Re}(\alpha)}{2\sqrt{N}} 2i \sum_j \sigma_j^y. \tag{150}$$

$\alpha$ was shown to scales as $N$ in Sec. 1.2, and $\sum_j \sigma_j^y$ scales as $N$, so the commutator scales as $1/N$ and thus vanishes in the thermodynamic limit.

## E Stiffness theorem

### E.1 Zero temperature $T = 0$

Consider a constrained minimization problem

$$E(A) = min \langle \psi_A | \hat{\mathcal{H}} | \psi_A \rangle , \tag{151}$$

where $\psi_A$ indicates $\langle \psi_A | \hat{A} | \psi_A \rangle = A$. Let $|\psi_0\rangle$ be the true ground state of $\mathcal{H}$ such that $\mathcal{H} |\psi_0\rangle = E(0) |\psi_0\rangle$ and $\langle \psi_0 | \hat{A} | \psi_0 \rangle = 0$. In that case, we can expand $E(A)$ in powers of $A$: $E(A) = E(0) + \frac{1}{2}\alpha A^2$, with $\alpha > 0$. We are interested in obtaining an expression for $\alpha$.

Let us consider a tweaked Hamiltonian

$$\hat{\mathcal{H}}_{\mathcal{A}} = \hat{\mathcal{H}} + F_A \hat{A}, \tag{152}$$

where we have coupled the system to a field $F_A$ through the operator $\hat{A}$. The ground state of $\hat{\mathcal{H}}_{\mathcal{A}}$ is $|\psi_A\rangle$. By definition of the static response function -in what follows termed susceptibility- we demand

$$A = \langle \hat{A} \rangle = \langle \hat{A} \rangle_0 + \chi_{AA} F_A \implies F_A = \frac{A}{\chi_{AA}}. \tag{153}$$

It is straightforward to see that

$$E(A) = \langle \psi_A | \hat{\mathcal{H}} | \psi_A \rangle , \tag{154}$$

for, if there existed another state $|\psi'_A\rangle$ with the same expectation value for $\hat{A}$ but lower expectation value for $\hat{\mathcal{H}}$, then we would have

$$\langle \psi'_A | \hat{\mathcal{H}}_{\mathcal{A}} | \psi'_A \rangle < \langle \psi_A | \hat{\mathcal{H}}_{\mathcal{A}} | \psi_A \rangle \tag{155}$$

which is prohibited by the fact that $\psi_A$ is the ground state of $\hat{\mathcal{H}}_{\mathcal{A}}$.

Let us calculate

$$E(A) = \langle \psi_A | \hat{\mathcal{H}}_{\mathcal{A}} | \psi_A \rangle - A F_A. \tag{156}$$

For that purpose, consider now the Hamiltonian

$$\hat{\mathcal{H}}_{\mathcal{A}}(\lambda) = \hat{\mathcal{H}} + \lambda F_A \hat{A}, \tag{157}$$

with $0 \leq \lambda \leq 1$, and $E_A(\lambda)$, $|\psi_A(\lambda)\rangle$ its ground state energy and ground state, respectively. Then, again by definition,

$$A(\lambda) = \langle \psi_A(\lambda)|\hat{A}|\psi_A(\lambda)\rangle = \lambda \chi_{AA} F_A. \tag{158}$$

Making use of the Hellman–Feynman theorem, which we will not prove here, we can write

$$\begin{aligned} E_A(1) = \langle \psi_A|\hat{\mathcal{H}}_{\mathcal{A}}|\psi_A\rangle &= E(0) + \int_0^1 d\lambda \, \langle \psi_A(\lambda)|\frac{\partial \hat{\mathcal{H}}_{\mathcal{A}}}{\partial \lambda}|\psi_A(\lambda)\rangle \\ &= E(0) + \int_0^1 d\lambda \lambda \chi_{AA} F_A^2 \\ &= E(0) + \frac{1}{2}\chi_{AA} F_A^2. \end{aligned} \tag{159}$$

Substituting back into Eq. (156) we obtain

$$E(A) = E(0) - \frac{1}{2}\frac{A^2}{\chi_{AA}}, \tag{160}$$

indicating that $\alpha = -1/\chi_{AA}$.

Notice that in Eq. (153) we have assumed that $\langle \hat{A}\rangle_0 = 0$. If we relax that assumption, the rest of the derivation still holds, in the sense that we can redefine operator $\hat{A}$ as $\hat{A} - \langle \hat{A}\rangle_0$, or equivalently, $A \to A - \langle \hat{A}\rangle_0$. With that, we can generalize our final result for the case of an operator with non-zero expectation value for the ground state.

$$E(A) = E(0) - \frac{1}{2}\frac{\left(A - \langle \hat{A}\rangle_0\right)^2}{\chi_{AA}} \tag{161}$$

## E.2 Finite temperature $T \geq 0$

Let us tackle now the case of non zero temperature. In this case we will assume that we are still concerned with the minimization of the energy (See App. I), but that averages are taken with respect to the density matrix that represents the thermal ensemble of the system as opposed to a single state. We consider a constrained minimization problem, with the energy given by

$$E(A) = Tr(\hat{\mathcal{H}}\rho_A), \tag{162}$$

where $\rho_A$ indicates that $A = \langle \hat{A}\rangle = Tr(\hat{A}\rho_A)$. We now construct the Hamiltonian $\hat{\mathcal{H}}_{\mathcal{A}} = \hat{\mathcal{H}} + F_A\hat{A}$ such that

$$\rho_A = \frac{1}{Z_A} e^{-\beta(\hat{\mathcal{H}} + F_A\hat{A})} \tag{163}$$

By the definition of the susceptibility, we impose

$$A = \langle \hat{A}\rangle = \langle A\rangle_0 + F_A \chi_{AA(T)} \implies F_A = \frac{A}{\chi_{AA}(T)}. \tag{164}$$

With that, we can express

$$E(A) = Tr(\hat{\mathcal{H}}_{\mathcal{A}}\rho_A) - F_A A, \tag{165}$$

which we now wish to compute. For that purpose, consider a parametrized Hamiltonian with parameter $0 \leq \lambda \leq 1$,

$$\hat{\mathcal{H}}_{\mathcal{A}}(\lambda) = \hat{\mathcal{H}} + \lambda F_A \hat{A}. \tag{166}$$

Extending the equations presented thus far, we find

$$A(\lambda) = Tr(\hat{A}\rho_A(\lambda)) = \lambda F_A \chi_{AA}(T) \tag{167}$$

with

$$\rho_A(\lambda) = \frac{1}{Z_A(\lambda)} e^{-\beta \hat{\mathcal{H}}_{\mathcal{A}}(\lambda)}. \tag{168}$$

Applying the Hellman-Feynman theorem again we find

$$\begin{aligned} E_A(1) &= E(0) + \int_0^1 d\lambda Tr(\frac{\partial \hat{\mathcal{H}}_{\mathcal{A}}(\lambda)}{\partial \lambda}) = \\ &= E(0) + \int_0^1 d\lambda \lambda \chi_{AA}(T) F_A^2 = \\ &= E(0) + \frac{1}{2}\chi_{AA}(T) F_A^2 \end{aligned} \tag{169}$$

Substituting back into Eq. (165) we finally obtain

$$E(A) = E(0) - \frac{1}{2}\frac{A^2}{\chi_{AA}(T)}. \tag{170}$$

So the result is analogous to the zero temperature case with the only difference that the susceptibility $\chi_{AA}(T)$ is now a function of $T$. We have assumed that $\langle \hat{A} \rangle_0 = 0$ throughout, the extension is trivial, as it was in the zero temperature case, and yields

$$E(A) = E(0) - \frac{1}{2}\frac{\left(A - \langle \hat{A} \rangle_0\right)^2}{\chi_{AA}(T)}. \tag{171}$$

# F Susceptibility

Consider the Hamiltonian of an originally unperturbed system that is weakly coupled to a field $\lambda$ via the operator $\hat{A}$,

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - \lambda \hat{A}. \tag{172}$$

We define the susceptibility as $\chi_{AA} = \partial_\lambda \langle \hat{A} \rangle$. If we consider the density matrix for the system

$$\rho = \frac{1}{Z} e^{-\beta(\hat{\mathcal{H}}_0 - \lambda \hat{A})}, \tag{173}$$

it follows that

$$\langle \hat{A} \rangle = Tr(\rho \hat{A}) = \partial_{\beta\lambda} Tr(\rho) = \frac{Z'}{Z}. \tag{174}$$

Where $Z = Tr(\rho)$. If we now turn our attention to $\chi_{AA}$ we find

$$\chi_{AA} = \partial_\lambda A = \beta \partial_{\beta\lambda} \langle \hat{A} \rangle = \beta \left( \frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2 \right). \tag{175}$$

We can evaluate this expression by expanding $Z$ in powers of $\xi = \beta\lambda$,

$$Z = Z_0 + \xi Z_1 + \frac{1}{2}\xi^2 Z_2 + \mathcal{O}(\xi^3). \tag{176}$$

Staying at the lowest non trivial order yields

$$\chi_{AA} = \beta\left(\frac{Z_2}{Z_0} - \left(\frac{Z_1}{Z_0}\right)^2\right). \tag{177}$$

We can obtain expressions for the different terms of the power series [27]

$$Z_0 = \sum_m e^{-\beta\epsilon_m} \tag{178}$$

$$Z_1 = \sum_m e^{-\beta\epsilon_m} A_{mm} \tag{179}$$

$$Z_2 = \sum_{m,n} e^{-\beta\epsilon_m} A_{mn}^2 K_{mn}. \tag{180}$$

Where $A_{mn} = \langle\psi_m|\hat{A}|\psi_n\rangle$ and

$$K_{mn} = \frac{e^{\beta\Delta_{mn}} - 1}{\beta\Delta_{mn}}, \tag{181}$$

with $\Delta_{mn} = \epsilon_m - \epsilon_n$.

At $T = 0$, the only contribution to $Z_0$ and $Z_1$ is the $m = 0$ term, so if we assume that $A_{00} = 0$, we have $Z_0 = 1$, $Z_1 = 0$. The susceptibility is then

$$\chi_{AA} = \lim_{\beta\to\infty} \sum_{m,n} A_{mn}^2 \frac{e^{-\beta\epsilon_n} - e^{-\beta\epsilon_m}}{\Delta_{mn}} = 2\sum_{m\neq 0} \frac{A_{m0}^2}{\epsilon_m - \epsilon_0}. \tag{182}$$

Even when $T \neq 0$, we can assume $\langle\hat{A}\rangle \propto Z_1 = 0$ finding

$$\chi_{AA}(T) = \frac{\sum_{m,n} e^{-\beta\epsilon_m} A_{mn}^2 K_{mn}}{\sum_m e^{-\beta\epsilon_m}}. \tag{183}$$

# G TRK sum rule

We are interested in computing

$$\frac{e^2}{c^2}\sum_{n\neq 0} \frac{|\langle\psi_n|\boldsymbol{j}_p\boldsymbol{A}_0|\psi_n\rangle|^2}{\epsilon_n - \epsilon_0}. \tag{184}$$

From canonical commutation we know that $[r_{l\alpha}, p_{k\beta}] = i\hbar\delta_{l,k}\delta_{\alpha,\beta}$, which in turn implies that $-i\hbar e\boldsymbol{j}_p = [\boldsymbol{d}, \mathcal{H}]$. With this, we can write

$$\frac{e^2}{c^2}\sum_{n\neq 0} \frac{|\langle\psi_n|\boldsymbol{j}_p\boldsymbol{A}_0|\psi_n\rangle|^2}{\epsilon_n - \epsilon_0} = \frac{1}{\hbar^2 c^2}\sum_{n\neq 0}(\epsilon_n - \epsilon_0)|\langle\psi_n|\boldsymbol{d}\boldsymbol{A}_0|\psi_n\rangle|^2. \tag{185}$$

On a different note, we can also write

$$[j_{p\alpha}, d_\beta] = i\hbar\sum_l \frac{e}{m}\delta_{\alpha,\beta}. \tag{186}$$

Which allows us to express $\Delta$ as

$$\Delta = \sum_l \frac{e^2 A_0^2}{2mc^2} = \frac{ie}{2\hbar c^2}\langle\psi_n|[\boldsymbol{d}\boldsymbol{A}_0, \boldsymbol{j}_p\boldsymbol{A}_0]|\psi_n\rangle. \tag{187}$$

We can now manipulate the right hand side, by introducing the identity $\mathbb{I} = \sum_n |\psi_n\rangle \langle\psi_n|$ in the right positions to yield

$$
\begin{aligned}
\langle\psi_n| [\boldsymbol{d}\boldsymbol{A}_0, \boldsymbol{j}_p\boldsymbol{A}_0] |\psi_n\rangle &= \sum_n \langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle \langle\psi_0| \boldsymbol{j}_p\boldsymbol{A}_0 |\psi_n\rangle - \sum_n \langle\psi_0| \boldsymbol{j}_p\boldsymbol{A}_0 |\psi_n\rangle \langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle = \\
&= -\frac{2i}{\hbar e} \sum_n \langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle \langle\psi_0| [\boldsymbol{d}\boldsymbol{A}_0, \mathcal{H}] |\psi_n\rangle = \\
&= -\frac{2i}{\hbar e} \sum_n (\epsilon_n - \epsilon_0) |\langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle|^2 = \\
&= -\frac{2i}{\hbar e} \sum_{n\neq 0} (\epsilon_n - \epsilon_0) |\langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle|^2. \qquad (188)
\end{aligned}
$$

Which means that $\Delta$ can be expressed as

$$
\Delta = \frac{1}{\hbar^2 c^2} \sum_{n\neq 0} (\epsilon_n - \epsilon_0) |\langle\psi_n| \boldsymbol{d}\boldsymbol{A}_0 |\psi_0\rangle|^2. \qquad (189)
$$

So finally

$$
\frac{e^2}{c^2} \sum_{n\neq 0} \frac{|\langle\psi_n| \boldsymbol{j}_p\boldsymbol{A}_0 |\psi_n\rangle|^2}{\epsilon_n - \epsilon_0} = \Delta. \qquad (190)
$$

## H From the coulomb gauge to the dipole gauge

We consider a PZW transformation of the form

$$
U = \exp\left[-\frac{i}{c\hbar} F\right]; \qquad \text{with} \qquad F = -\sum_i e\boldsymbol{r}_i \boldsymbol{A} = \boldsymbol{d}\boldsymbol{A}. \qquad (191)
$$

To be applied onto Hamiltonian (19). This transformation is designed to displace the momenta in order to eliminate minimal coupling

$$
U^\dagger p_j U = \boldsymbol{p}_j + \frac{e}{c}\boldsymbol{A} \qquad (192)
$$

But it also displaces the photonic operators, yielding

$$
U^\dagger a U = a - \frac{i}{c\hbar} \boldsymbol{d}\boldsymbol{A}_0, \qquad (193)
$$

$$
U^\dagger a^\dagger U = a^\dagger + \frac{i}{c\hbar} \boldsymbol{d}\boldsymbol{A}_0. \qquad (194)
$$

So the photonic term in the Hamiltonian becomes

$$
U^\dagger \hbar\omega_c U = \hbar\omega_c a^\dagger a - i\frac{\omega_c}{c} \boldsymbol{d}\boldsymbol{A}_0 \left(a^\dagger - a\right) + \frac{\omega_c \left(\boldsymbol{d}\boldsymbol{A}_0\right)}{c^2\hbar}. \qquad (195)
$$

Putting everything together, we get

$$
U^\dagger \mathcal{H} U = \mathcal{H}_m + \hbar\omega_c a^\dagger a - i\frac{\omega_c}{c} \boldsymbol{d}\boldsymbol{A}_0 \left(a^\dagger - a\right) + \frac{\omega_c \left(\boldsymbol{d}\boldsymbol{A}_0\right)}{c^2\hbar}. \qquad (196)
$$

The last term arises from light-matter interaction, but it only depends on matter operators, so we can move it into a reformulated matter Hamiltonian

$$
U^\dagger \mathcal{H} U = \mathcal{H}'_m + \hbar\omega_c a^\dagger a - i\frac{\omega_c}{c} \boldsymbol{d}\boldsymbol{A}_0 \left(a^\dagger - a\right). \qquad (197)
$$

## I Equivalence between minimizing the free energy and the energy at low temperatures

The free energy is defined as $F = E - TS$. Consider now a variation in $F$ at constant temperature

$$\Delta F = \Delta E - T\Delta S. \tag{198}$$

Clearly at $T = 0$, $\Delta F = \Delta E$, so minimizing the free energy is equivalent to minimizing the energy at $T = 0$. At larger temperatures, we can write

$$\frac{\Delta E - \Delta F}{T} = \Delta S. \tag{199}$$

Taking the low temperature limit we have

$$\lim_{T\to 0} = \frac{\Delta E - \Delta F}{T} = \lim_{T\to 0} \Delta S. \tag{200}$$

Applying L'Hopital's rule and Nernst postulate $\lim_{T\to 0}\Delta = 0$, we find

$$\left(\frac{d\Delta E}{dT}\right)_{T=0} - \left(\frac{d\Delta F}{dT}\right)_{T=0} = 0. \tag{201}$$

So not only are $\Delta E$ and $\Delta F$ equal at $T = 0$ but so are their slopes, ensuring that $\Delta E$ and $\Delta F$ are equal in a range of low temperatures, and not only at $T = 0$. This implies that the minimization of $F$ and $E$ are equivalent at low temperatures.

## J No-go theorem for Bamba and colleagues' setup

Bamba and colleagues' proposal for superradiance [21] is based on a particular architecture of circuit QED. The Hamiltonian of the circuit is

$$\mathcal{H} = \frac{q^2}{2C_R} + \frac{\phi^2}{2L_R} + \sum_j^N \left(\frac{\rho_j^2}{2C_J} + \frac{(\psi_j - \phi)^2}{2L_g} + E_J \cos\frac{2\pi\psi_j}{\Phi_0}\right). \tag{202}$$

The details of how the Hamiltonian arises are irrelevant to our cause, it suffices to know that: $q$ and $\phi$ are the operators of the artificial cavity, they play the roles of momentum and position operators respectively; similarly, $\rho_j$ and $\psi_j$ are the operators of the artificial atoms, they play the roles of momentum and position operators, respectively. Accordingly, these operators obey canonical commutation: $[\phi, q] = i\hbar$, $[\psi_i, \rho_j] = i\hbar\delta_{i,j}$. For simplicity, let us consider $\hbar = 1$ in the following. The main difference between this Hamiltonian our light-matter Hamiltonian in the Coulomb gauge (3) resides in the fact that the coupling between the cavity and the atoms is through positions instead of momenta. The authors claim that this is the key ingredient that allows them to dodge previous no-go theorems and achieve superradiance in their model. Let us show that is not the case by transforming the Hamiltonian into one with "minimal coupling" form, i.e. one in which the coupling is through the momenta. In order to do so, we must first group the terms containing $\phi$. We can write

$$\frac{\phi^2}{2L_R} + \sum_j^N \frac{(\psi_j - \phi)^2}{2L_g} = c^2\left(\phi - \frac{b}{c}\sum_j \psi_j\right)^2 - b^2\left(\sum_j \psi_j\right)^2 + \frac{1}{2L_g}\sum_j \psi_j^2. \tag{203}$$

Where

$$c^2 = \frac{1}{2L_R} + \frac{N}{2L_g}; \qquad 2ab = \frac{1}{L_g}. \tag{204}$$

Then, to decouple the position operators of cavity and atoms: $\phi - \frac{b}{c}\sum_j \psi_j$, we apply the unitary transformation

$$U = e^{-i\frac{b}{c}q\sum_j \psi_j}. \tag{205}$$

Notice that

$$U^\dagger \phi U = \phi + \frac{b}{c}\sum_j \psi_j, \tag{206}$$

$$U^\dagger \rho_j U = \rho_j - \frac{b}{c}q. \tag{207}$$

So the resulting transformed Hamiltonian reads

$$U^\dagger \mathcal{H} U = \frac{q^2}{2C_R} + c^2\phi^2 + \sum_j \frac{\left(\rho_j - \frac{b}{c}q\right)^2}{2C_J} + \sum_j E_J \cos\frac{2\pi\psi_j}{\Phi_0} + \frac{1}{2L_g}\sum_j \psi_j^2 - b^2\left(\sum_j \psi_j\right)^2 \tag{208}$$

We can group together the last three terms to form

$$\mathcal{H}' = \frac{q^2}{2C_R} + c^2\phi^2 + \sum_j \frac{\left(\rho_j - \frac{b}{c}q\right)^2}{2C_J} + \sum_{ij} V(\psi_i, \psi_j). \tag{209}$$

We can define bosonic creation and destruction operators to diagonalize the cavity, by defining a frequency $\omega_c$ such that $c^2 = \frac{1}{2}C_R\omega_c^2$, we can write

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{C_R\omega_c}\phi + i\frac{q}{\sqrt{C_R\omega_c}}\right), \tag{210}$$

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{C_R\omega_c}\phi - i\frac{q}{\sqrt{C_R\omega_c}}\right). \tag{211}$$

Thus we have

$$\mathcal{H}' = \omega_c a^\dagger a + \sum_j \frac{\left(\rho_j - \frac{b}{c}q\right)^2}{2C_J} + \sum_{ij} V(\psi_i, \psi_j), \tag{212}$$

with

$$q = -i\sqrt{\frac{C_R\omega_c}{2}}\left(a - a^\dagger\right). \tag{213}$$

We have arrived at a Hamiltonian in which the coupling between the atoms and the cavity appears in the form of *minimal coupling*. The only difference with our original Hamiltonian in the Coulomb gauge (3) is that here the matter momenta are coupled to the light momenta, whereas in the Pauli equation the coupling is of the form $p_j - \frac{e}{c}A$ and $A$ plays the role of a position operator, this is also evident from its expression in terms of bosonic operators which is of the form $A_0\left(a + a^\dagger\right)$. This difference, however, is anecdotal, the no-go theorem proven in Sec. 2 applies here in full force, prohibiting superradiance and thus contradicting the authors.

How, then, where the authors of [21] able to show that a phase transition was present in the model? The answer resides in Eq. (208), the last term of which is a direct atom-atom interaction. This direct interaction was hidden in the original expression of Hamiltonian (202) and it is the cause of the transition, which does not originate from the coupling between atoms and cavity.

## K Computing the free energy in the mean field effective model

From Hamiltonian (84) we can write the partition function as

$$Z = e^{-\beta \frac{\lambda'^2 m_x^2}{\omega_c} N} Tr\left(e^{-\beta h}\right)^N, \tag{214}$$

with

$$h = \frac{\omega_z}{2}\sigma_j^z - \frac{2\lambda'^2 m_x}{\omega_c}\sigma_j^x = \begin{pmatrix} \frac{\omega_z}{2} & -\frac{2\lambda'^2 m_x}{\omega_c} \\ -\frac{2\lambda'^2 m_x}{\omega_c} & -\frac{\omega_z}{2} \end{pmatrix}. \tag{215}$$

Diagonalizing $h$ we obtain the eigenvalues $\pm E$, where

$$E = \sqrt{\frac{\omega_z^2}{4} + \frac{4\lambda'^4 m_x^2}{\omega_c^2}}. \tag{216}$$

## L Computing the free energy in the spin $S$ Dicke model

From Hamiltonian (94) we can write the partition function as

$$Z = e^{-\beta \frac{4\lambda^2 m_x^2}{\omega_c} N} Tr\left(e^{-\beta h}\right)^N, \tag{217}$$

with

$$h = \omega_z S^z - \frac{8\lambda^2}{\omega_c} m_x S^x. \tag{218}$$

We can diagonalize $h$ by rotating the spin operators. Consider an arbitrary rotation

$$S^x = S'^x \cos\phi + S'^z \sin\phi, \tag{219}$$

$$S^z = -S'^x \sin\phi + S'^z \cos\phi. \tag{220}$$

So

$$h = -S'^x\left(\omega_z \sin\phi + \frac{8\lambda^2}{\omega_c} m_x \cos\phi\right) + S'^z\left(\omega_z \cos\phi - \frac{8\lambda^2}{\omega_c} m_x \sin\phi\right). \tag{221}$$

By imposing $0 = \omega_z \sin\phi + \frac{8\lambda^2}{\omega_c} m_x \cos\phi$, we get

$$\tan\phi = -\frac{8\lambda^2 m_x}{\omega_c \omega_z}. \tag{222}$$

With this, we obtain

$$h = S'^z \sqrt{\omega_z^2 + \left(\frac{8\lambda^2 m_x}{\omega_c}\right)^2}. \tag{223}$$

So

$$Tr\left(e^{-\beta h}\right) = \sum_{-S}^{S} e^{-\beta n E} = \ldots = \frac{\sinh\left(\beta \frac{E}{2}(2S+1)\right)}{\sinh\left(\beta \frac{E}{2}\right)}. \tag{224}$$

So we finally get

$$\frac{F}{N} = \frac{4\lambda^2 m_x^2}{\omega_c} - k_b T \ln\left[\frac{\sinh\left(\beta \frac{E}{2}(2S+1)\right)}{\sinh\left(\beta \frac{E}{2}\right)}\right]. \tag{225}$$

# M Transmission

## M.1 Susceptibility in the subradiant phase

Let us solve the following system of equations for $\chi_\alpha$

$$\begin{cases} -i\omega\chi_z = \frac{4\lambda}{\sqrt{N}}\left(\chi_y\alpha_0 + \langle S_y\rangle_0\,\chi_\alpha)\right) - \gamma\chi_z, \\ -i\omega\chi_x = -\omega_z\chi_y - \gamma\chi_x, \\ -i\omega\chi_y = \omega_z\chi_x - 4\frac{\lambda}{\sqrt{N}}\left(\chi_z\alpha_0 + \langle S_z\rangle_0\,\chi_\alpha\right) - \gamma\chi_y, \\ -i\omega\chi_\alpha = -i\omega_c\chi_\alpha - i\frac{\lambda}{\sqrt{N}}\chi_x - i - \kappa\chi_\alpha. \end{cases} \tag{226}$$

In the subradiant phase, we have $\langle S_x\rangle_0 = -N$ and $\langle S_x\rangle_0 = \langle S_y\rangle_0 = \alpha_0 = 0$. This implies that $\chi_z = 0$ and $\chi_x = \frac{\omega_z}{i\omega-\gamma}\chi_y$, which simplifies the system to

$$\begin{cases} -i\omega_z\chi_y = \frac{\omega_z^2}{i\omega-\gamma}\chi_y + 4\lambda\sqrt{N}\chi_\alpha - \gamma\chi_y, \\ -i\omega\chi_\alpha = -i\omega_c\chi_\alpha - i\frac{\lambda}{\sqrt{N}}\frac{\omega_z}{i\omega-\gamma}y - i - \kappa\chi_\alpha. \end{cases} \tag{227}$$

We can solve for $\chi_y$ in the upper equation to obtain

$$\chi_y = \frac{4\lambda\sqrt{N}\alpha}{\frac{i\omega_z^2}{\omega+i\gamma} + \gamma - i\omega}. \tag{228}$$

Introducing it in the lower equation, we can finally solve for $\chi_\alpha$, obtaining

$$\chi_\alpha = \frac{(\omega+i\gamma)^2 - \omega_z^2}{(\omega-\omega_c)\left((\omega+i\gamma)^2 - \omega_z^2\right) - 4\lambda^2\omega_z + i\kappa\left((\omega+i\gamma)^2 - \omega_z^2\right)}. \tag{229}$$

## M.2 Susceptibility in the superradiant phase

In contrast, for the superradiant phase we have

$$\langle S_z\rangle_0 = -N\frac{\omega_z\omega_c}{4\lambda^2}; \quad \langle S_x\rangle_0 = N\sqrt{1-\frac{\omega_z\omega_c}{4\lambda^2}}; \quad \langle S_y\rangle_0 = 0; \quad \alpha_0 = \frac{\lambda\sqrt{N}}{\omega_c}\sqrt{1-\frac{\omega_z\omega_c}{4\lambda^2}}. \tag{230}$$

We can quickly solve the first two equations, yielding $\chi_z = -\frac{1}{i\omega-\gamma}\frac{4\lambda}{\sqrt{N}}\chi_y\alpha_0$ and $\chi_x = \frac{\omega_z}{i\omega-\gamma}\chi_y$, substituting in the other two equations, we arrive at

$$\begin{cases} -i\omega\chi_y = -i\frac{\omega_z^2}{\omega+i\gamma}\chi_y - \frac{4\lambda}{\sqrt{N}}\left(\frac{i}{\omega+i\gamma}\frac{4\lambda}{\sqrt{N}}\chi_y\alpha_0^2 + \langle S_z\rangle_0\,\chi_\alpha\right) - \gamma\chi_y, \\ -i\omega\chi_\alpha = -i\omega_c\chi_\alpha - \frac{\lambda\omega_z}{(\omega+i\gamma)\sqrt{N}}\chi_y - i - \kappa\chi_\alpha. \end{cases} \tag{231}$$

We can solve for $\chi_y$ in the upper equation to obtain

$$\chi_y = \frac{i\frac{4\lambda}{\sqrt{N}}\langle S_z\rangle_0\,\chi_\alpha(\omega+i\gamma)}{\omega_z^2 + \frac{16\lambda^2\alpha_0^2}{N} - (\omega+i\gamma)^2}. \tag{232}$$

Introducing it in the lower equation, we can finally solve for $\chi_\alpha$, obtaining

$$\chi_\alpha = \frac{(\omega+i\gamma)^2 - \frac{16\lambda^4}{\omega_c^2}}{(\omega-\omega_c)\left((\omega+i\gamma)^2 - \frac{16\lambda^4}{\omega_c^2}\right) - \omega_z^2\omega_c + i\kappa\left((\omega+i\gamma)^2 - \frac{16\lambda^4}{\omega_c^2}\right)}. \tag{233}$$