OPERATOR INEQUALITIES, FUNCTIONAL MODELS AND ERGODICITY

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Abstract. We discuss when an operator, subject to a rather general inequality in hereditary form, admits a unitarily equivalent functional model of Agler type in the reproducing kernel Hilbert space associated to the inequality. To the contrary to the previous work, the kernel need not be of Nevanlinna-Pick type. We derive some consequences concerning the ergodic behavior of the operator.

1. Introduction

1.1. **Motivation.** Let $\alpha(t)$ be a function representable by the power series $\sum_{n=0}^{\infty} \alpha_n t^n$ in $\mathbb{D} := \{|t| < 1\}$, where the coefficients α_n are real numbers, and let $T \in L(H)$ be a bounded linear operator on a Hilbert[1] space H. Put

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n, \tag{1.1}$$

where the series is assumed to converge in the strong operator topology SOT in $L(H)$. When α is a polynomial, the series above is just a finite sum, and there is no convergence problem. In particular, when $\alpha(t) = 1 - t$, the right hand side of (1.1) is $I - T^*T$, so $T \in L(H)$ is a contraction if and only if $(1 - t)(T^*, T) \geq 0$. In the 1960's Sz.-Nagy and Foias developed a beautiful spectral theory of contractions (see [62]) based on the construction of their functional model.

In his landmark paper [5], Agler showed that if T has spectrum $\sigma(T)$ contained in the unit disc $\mathbb{D}$ and $\alpha(T^*, T) \geq 0$, then it is natural to model T by parts of $B \otimes I_{\mathcal{E}}$, where B is a suitable weighted backward shift and $I_{\mathcal{E}}$ is the identity operator on some auxiliary Hilbert space $\mathcal{E}$. (By *a part of an operator* we mean its restriction to an invariant subspace.) More generally, when $\sigma(T) \subset \overline{\mathbb{D}}$, it has been found in various particular cases that instead of $B \otimes I_{\mathcal{E}}$ one should consider operators of the form $(B \otimes I_{\mathcal{E}}) \oplus S$, where S is an isometry or a unitary operator. This representation is called *a coanalytic model*. As Agler proved in [6], it holds, in particular, for *m-hypercontractions*, i.e., operators $T \in L(H)$ such that $(1 - t)^j(T^*, T) \geq 0$ for $j = 1, \dots, m$. Agler's theorem was generalized in [46] by Müller and Vasilescu to tuples

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[1]All Hilbert spaces will be assumed to be separable

of operators. The first results on Agler model techniques are exposed in the book [7] by Agler and McCarthy. In [48], Olofsson obtained operator formulas for wandering subspaces, relevant in the models of m-hypercontractions. His results were generalized by Eschmeier in [31] to tuples of commuting operators, and by Ball and Bolotnikov in [10] to what they call β-*hypercontractions*.

Müller studied the case where $\alpha = p$ is a polynomial in [45]. He considers the class $\mathcal{C}(p)$ of operators $T \in L(H)$ such that $p(T^*, T) \geq 0$. He proves that any contraction $T \in \mathcal{C}(p)$ has a coanalytic model whenever $p(1) = 0, 1/p(t)$ is analytic in $\mathbb{D}$, and $1/p(\bar{w}z)$ is a reproducing kernel. This last condition is equivalent to the fact that all Taylor coefficients of $1/p(t)$ at the origin are positive. Müller also considers some operator inequalities for T with infinitely many terms, with the same property of positivity. This permits him to show that any operator T is unitarily equivalent to a part of a backward weighted shift with the same spectral radius (see [45, Corollary 2.3]).

In [50], Olofsson deals with the case where α is not a polynomial. His assumptions are that α is analytic on $\mathbb{D}$, does not vanish on $\mathbb{D}$, and $1/\alpha$ has positive Taylor coefficients at the origin. Under this setting, he studies contractions T on H such that $\alpha(rT^*, rT) \geq 0$ for every $r \in [0, 1)$. With more assumptions, he obtains the coanalytic model for this class of operators.

In [11], the last two authors considered functions α in the Wiener algebra A_W of analytic functions in the unit disc with summable sequence of Taylor coefficients, subject to certain conditions. It was assumed that the series $\sum \alpha_n T^{*n} T^n$ converges in norm. The operators studied there turn out to be similar to contractions (see [11, Theorem I]). This will no longer be true in the setting of the present paper (see Example 7.3).

In [11], an explicit model in the spirit of Sz.-Nagy and Foias model was constructed for the class of operators considered there. The roles of the defect operator and the defect space were played by

$$D := (\alpha(T^*, T))^{1/2}, \quad \mathfrak{D} := \overline{DH}, \tag{1.2}$$

where the non-negative square root is taken.

1.2. Our setting.

Here the operator D and the space $\mathfrak{D}$, defined by (1.2) whenever $\alpha(T^*, T) \geq 0$, will also play an important role. Recall that now we consider the convergence of (1.1) in SOT. As it will be seen from Example 7.3, this is the appropriate convergence in this context.

Our assumptions are the following.

Hypotheses 1.1. Suppose α is a function in A_W which does not vanish on $\mathbb{D}$. We put

$$k(t) = 1/\alpha(t) = \sum_{n=0}^{\infty} k_n t^n \qquad t \in \mathbb{D},$$

with $\alpha_0 = k_0 = 1$, and assume that $k_n > 0$ for every $n \geq 1$.

Under Hypotheses 1.1, we denote by $\mathcal{H}_k$ the weighted Hilbert space of power series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ with finite norm

$$\|f\|_{\mathcal{H}_k} := \left(\sum_{n=0}^{\infty} |f_n|^2 k_n \right)^{1/2}.$$

Let B_k be the *backward shift* on $\mathcal{H}_k$, defined by

$$B_k f(t) = \frac{f(t) - f(0)}{t}. \tag{1.3}$$

Definition 1.2. Fix a function α satisfying Hypotheses 1.1, and let T be an operator in $L(H)$. We say that T is *α-modelable* if T is unitarily equivalent to a part of an operator of the form $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where S is an isometry.

We remark that $B_k \otimes I_{\mathcal{E}}$ acts on the Hilbert space $\mathcal{H}_k \otimes \mathcal{E}$, which can be identified with the weighted Hilbert space of $\mathcal{E}$-valued power series $f(t) = \sum_{n=0}^{\infty} f_n t^n$ with norm given by

$$\|f\|_{\mathcal{H}_k \otimes \mathcal{E}} = \left(\sum_{n=0}^{\infty} \|f_n\|_{\mathcal{E}}^2 \, k_n \right)^{1/2}.$$

It acts according to the same formula (1.3).

It is natural to pose the following question.

Question 1.3. Given a function α satisfying Hypotheses 1.1, give a good sufficient condition for an operator $T \in L(H)$ to be α-modelable.

One of the strongest results in this direction is contained in the recent papers by Bickel, Hartz and McCarthy [17] and by Clouâtre and Hartz [21]. It is stated for spherically symmetric tuples of operators. For the case of a single operator, their result can be formulated as follows.

Theorem 1.4 ([21, Theorem 1.3])**.** *Let α be a function with $\alpha_0 = 1$ and $\alpha_n \le 0$ for all $n \ge 1$. Suppose that $k = 1/\alpha$ has radius of convergence 1, $k_n > 0$ for every $n \ge 0$ and*

$$\lim_{n\to\infty} \frac{k_n}{k_{n+1}} = 1. \tag{1.4}$$

Then B_k is bounded, and a Hilbert space operator T is α-modelable if and only if $\alpha(T^*, T) \ge 0$.

It is easy to see that the hypotheses of Theorem 1.4 imply Hypotheses 1.1. This theorem concerns the *Nevanlinna-Pick case*, that is, when $\alpha_0 = 1$ and $\alpha_n \le 0$ for $n \ge 1$. Alternatively, we say that k is a *Nevanlinna-Pick kernel.* In the recent work [22], Clouâtre, Hartz and Schillo establish a Beurling–Lax–Halmos theorem for reproducing kernel Hilbert spaces in the Nevanlinna-Pick context. We refer the reader to [26, 49, 56, 57] for more results in the Nevanlinna-Pick case. In the recent preprint [32], Eschmeier and Toth extend previous results by Eschmeier [31] to all complete Nevanlinna-Pick kernels, in the context of operator tuples.

1.3. **Main results.** The following result gives a new answer to Question 1.3.

Theorem 1.5. *Assume Hypotheses 1.1. If $k \in A_W$, and its Taylor coefficients $\{k_n\}$ satisfy $k_n^{1/n} \to 1, \sup k_n/k_{n+1} < \infty$ and*

$$\lim_{m\to\infty} \sup_{n\geq 2m} \sum_{m\leq j\leq n/2} \frac{k_j k_{n-j}}{k_n} = 0, \tag{1.5}$$

then B_k is bounded, and the operator $T \in L(H)$ is a part of $B_k \otimes I_{\mathcal{E}}$ (for some Hilbert space $\mathcal{E}$) if and only if both $\sum |\alpha_n| T^{*n}T^n$ and $\sum k_n T^{*n}T^n$ converge in SOT and $\alpha(T^*, T) \geq 0$. Moreover, in this case one can take $\mathcal{E} = \mathfrak{D}$.

As it will be seen later, the SOT-convergence of $\sum |\alpha_n| T^{*n}T^n$ implies the the SOT-convergence of $\sum \alpha_n T^{*n}T^n$.

Notice that in Theorem 1.5, the isometric part S is unnecessary (see Theorem 1.12 (ii) below for more information). This theorem shows that T is α-modelable in many cases when k is not a Nevanlinna-Pick kernel, and so Theorem 1.4 does not apply. Not much about these kernels has been known previously. Given an integer $N \geq 2$, there are examples of functions k satisfying the hypotheses of Theorem 1.5 with whatever prescribed signs of the coefficients $\alpha_2, \dots, \alpha_N$ (see Example 5.1). Note that $\alpha_1 = -k_1$ is always negative.

Remark 1.6. Suppose that $k \in A_W$ and the sequence

$$\left\{ \frac{k_n}{k_{n+1}} \left(1 + \frac{1}{n+1}\right)^a \right\}$$

is increasing for some $a > 1$. Then (1.5) holds. This is close to [60, Proposition 34]. Indeed, put $k_j^* := (j+1)^{-a}$, and define $\rho_j := k_j/k_j^*$. Then our condition reduces to the condition $\rho_{n+2}/\rho_{n+1} \geq \rho_{n+1}/\rho_n$, for all n, which implies that $\rho_j \rho_{n-j}/\rho_n \leq C$, for $0 \leq j \leq n$. Since

$$\frac{k_j k_{n-j}}{k_n} = \frac{\rho_j \rho_{n-j}}{\rho_n} \frac{k_j^* k_{n-j}^*}{k_n^*}$$

and $\{k_n^*\}$ satisfies (1.5), it follows that $\{k_n\}$ also satisfies (1.5).

Hence, for sufficiently regular sequences $\{k_n\}$, the condition (1.5) is rather close to the condition $\sum k_n < \infty$. It can be added that, in fact, in Theorem 1.5 $\{k_n\}$ need not be regular; moreover, the quotients k_n/k_{n+1} need not converge (see Remark 5.2).

The techniques employed in the proof are different from [21]. We use, basically, a combination of Müller's arguments in [45] and Banach algebras techniques.

The above theorems open the question of describing invariant subspaces of $B_k \otimes I_{\mathcal{E}}$ and of constructing a functional model of operators under the study, which certainly would be interesting. We do not address this question in this paper.

Given an operator $C : H \to \mathcal{E}$, where $\mathcal{E}$ is an auxiliary Hilbert space, we define

$$V_C x(z) = C(I_H - zT)^{-1}x, \quad x \in H, \quad z \in \mathbb{D}. \tag{1.6}$$

The next result shows that whenever T is α-modelable, the operator $V_D : H \to \mathcal{H}_k \otimes \mathfrak{D}$ is a contraction, and we can give an explicit model for T (that is, give explicitly $\mathcal{E}$, S and the transform which sends the initial space into the model space). First we need to state one more technical hypothesis, whose meaning will be clear later.

Hypotheses 1.7. Let α be a function satisfying Hypothesis 1.1. Put

$$\beta(t) = \sum_{n\geq 0} \beta_n t^n, \quad \text{where } \beta_n = |\alpha_n|, \tag{1.7}$$

and $\gamma(t) = \beta(t)k(t)$. We assume that $k_n/k_{n+1} \leq C'$ and $\gamma_n \leq C'' k_n$ for all $n \geq 0$.

The condition $k_n/k_{n+1} \leq C'$ is equivalent to boundedness of B_k. If it holds, then the second condition is satisfied whenever there is some N such that either $\alpha_n \geq 0$ for $n \geq N$, or $\alpha_n \leq 0$ for $n \geq N$.

Theorem 1.8 (Explicit model)**.** *Assume Hypotheses 1.1 and 1.7. Let T be α-modelable. Then $\alpha(T^*, T) \geq 0$, V_D is a contraction, and hence we can define*

$$W = (I_H - V_D^* V_D)^{1/2}, \quad \mathcal{W} = \overline{WH}.$$

Moreover, $S : \mathcal{W} \to \mathcal{W}$, given by $SWx := WTx$, is an isometry and the operator

$$(V_D, W) : H \to (\mathcal{H}_k \otimes \mathfrak{D}) \oplus \mathcal{W}, \qquad (V_D, W)h = (V_D h, Wh)$$

provides a model of T, in the sense that (V_D, W) is isometric and

$$((B_k \otimes I_{\mathfrak{D}}) \oplus S) \cdot (V_D, W) = (V_D, W) \cdot T.$$

Remark 1.9. Suppose α satisfies the above two hypotheses, and suppose that T is an α-modelable operator, which is given already by its model without the isometric part. That is, there is an invariant subspace L of an operator $B_k \otimes I_{\mathcal{E}}$, acting on $\mathcal{H}_k \otimes \mathcal{E}$, such that T is the restriction of this operator to L. Then $\mathfrak{D} = \mathcal{E}$ (identified with the constant functions in $\mathcal{H}_k \otimes \mathcal{E}$), and V_D is the identity operator on L. This follows from Corollary 2.13 below.

Similarly, in the general case, if T is a part of an operator $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where S is an isometry, there is a unitary operator u such that the transform (V_D, uW) is just the identity.

If it is known that T is α-modelable, one can ask about the uniqueness of the model. For answering this question, we need the following definitions.

Definition 1.10. Let $\mathcal{L}$ be an invariant subspace of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S : \mathcal{W} \to \mathcal{W}$ is an isometry. We will say that the corresponding model operator

$$\big((B_k \otimes I_{\mathcal{E}}) \oplus S\big)|\mathcal{L}$$

is *minimal* if the following two conditions hold.

(i) $\mathcal{L}$ is not contained in $(\mathcal{H}_k \otimes \mathcal{E}') \oplus \mathcal{W}$ for any $\mathcal{E}' \subsetneqq \mathcal{E}$.

(ii) $\mathcal{L}$ is not contained in $(\mathcal{H}_k \otimes \mathcal{E}) \oplus \mathcal{W}'$ for any $\mathcal{W}' \subsetneq \mathcal{W}$ invariant by S.

In Remark 3.5 we show that the explicit model obtained in Theorem 1.8 is indeed minimal.

Note that under Hypotheses 1.1, α is defined on the closed unit disc $\overline{\mathbb{D}}$ and does not vanish on the interval $[0,1)$. Since $\alpha(0) = \alpha_0 = 1$, we obtain that $\alpha(1) \geq 0$. We distinguish the following two cases. This distinction appears already in [21, Subsection 2.3] for the Nevanlinna-Pick case.

Definition 1.11. Suppose that α meets Hypotheses 1.1. We will say that α is of *critical type* (or, alternatively, that we have the *critical case*) if $\alpha(1) = 0$. If $\alpha(1) > 0$, we will say that α is of *subcritical type* (or, alternatively, that we have the *subcritical case*).

Theorem 1.12 (Uniqueness of the minimal model)**.** *Suppose that α meets Hypotheses 1.1 and 1.7. Let T be an α-modelable operator.*

(i) *In the critical case, the minimal model of T is unique. More precisely, the pair of transforms (V_D, W_0), where $W_0 = (I - V_D^* V_D) : H \to \mathcal{W}_0$ and $\mathcal{W}_0 := \overline{\operatorname{Ran}}(I - V_D^* V_D)$, gives rise to a minimal model, and any minimal model is provided by (V_C, W), where $C = vD$, $W = wW_0 : H \to \mathcal{W}$ and v, w are unitary isomorphisms.*

(ii) *In the subcritical case, the minimal model of T is not unique, in general. However, there always exists a minimal model given by $V = V_D$, in the sense that $V_D : H \to \mathcal{H}_k \otimes I_{\mathfrak{D}}$ is an isometry such that $(B_k \otimes I_D)V_D = V_D T$. Note that in this case the isometry S is absent.*

We remark that there are other works that give answers to the above Question 1.3. In particular, Pott [52] gave a model for operators satisfying two inequalities $(1-p)(T^*, T) \geq 0$ and $(1-p)^m(T^*, T) \geq 0$, where p is a polynomial with nonnegative coefficients, $m \geq 1$ and $p(0) = 0$ (this class is a generalization of m-hypercontractions). In fact, she treats tuples of operators. In [10], Ball and Bolotnikov consider a function $\alpha(t)$ in the Wiener algebra such that $k = 1/\alpha$ has positive coefficients satisfying $0 < \varepsilon \leq k_n/k_{n+1} \leq 1$ for all n (so that B_k is a contraction). They show that an operator T is α-modelable, with absent isometric part, if and only if if $\alpha(T^*, T) \geq 0$ as well as infinitely many additional inequalities hold (T is β-hypercontractive, where $\beta_n = 1/k_n$), and T is what they call β-stable. See [10], Theorem 4.3. In [10], Theorem 7.2, Ball and Bolotnikov give a model of T in terms of their generalization of the characteristic function, which is an infinite family of operator-valued functions.

Whereas these authors treat both subcritical and critical cases, Theorem 1.5 only concerns the subcritical case (because of the condition $k \in A_W$).

1.4. **Consequences of the model.** If an operator T is α-modelable, it is natural to study what consequences can be derived from the model. Here we obtain two types of consequences:

(1) when the defect operator D has finite rank (that is, $\dim \mathfrak{D} < \infty$), and
(2) ergodic consequences when $\alpha(t) = (1-t)^a$ with $0 < a < 1$.

We will use the space $\mathcal{R}_k = \mathcal{H}_{\tilde{k}}$, where $\tilde{k}_n = 1/k_n$. It is easy to see that it is the reproducing kernel Hilbert space, corresponding to the positive definite kernel $k(z,w) := k(\bar{w}z)$. The pairing $\langle f, g\rangle = \sum f_n \bar{g}_n$ ($f \in \mathcal{H}_k, g \in \mathcal{R}_k$) makes $\mathcal{R}_k$ naturally dual to $\mathcal{H}_\varkappa$. In this interpretation, the adjoint operator to B_k is the operator $g(z) \mapsto zg(z)$, acting on $\mathcal{R}_k$.

If α is of subcritical type, we have the following result related to the Carleson condition.

Theorem 1.13. *Let T be an operator similar to a part of $B_k \otimes I_{\mathfrak{D}}$, acting on the space $\mathcal{H}_k \otimes \mathfrak{D}$, where $\mathcal{R}_k$ is a Banach algebra and $\mathfrak{D}$ is finite dimensional. Suppose that*

$$\lim_{n\to\infty} \left(\inf_{j\ge 0} \frac{k_j}{k_{n+j}} \right)^{1/n} = \lim_{n\to\infty} k_n^{1/n} = 1,$$

and also that

$$\sum_{n=N}^{\infty} k_n \le C N^{-\varepsilon} \qquad \forall N \ge 0, \tag{1.8}$$

for some positive constants C and ε which do not depend on N. Suppose that the spectrum $\sigma(T)$ does not cover $\mathbb{D}$. Put

$$E := (\overline{\sigma(T) \cap \mathbb{D}}) \cap \mathbb{T}$$

and let $\{l_\nu\}$ denote the lengths of the finite complementary intervals of E (in $\mathbb{T}$). Then the Lebesgue measure of E is 0, and the Carleson condition holds:

$$\sum_\nu l_\nu \log \frac{2\pi}{l_\nu} < \infty.$$

Some of the arguments employed in the proof of this theorem are related with the so-called index of an invariant subspace of $\mathcal{R}_k \otimes \mathcal{E}$; see Section 6 for more details.

In the critical case, an important family of functions α are those of the form $\alpha(t) := (1-t)^a$, for $a > 0$. Note that they satisfy Hypotheses 1.1. When $a = m$ is a positive integer, it is said that $T \in L(H)$ is an *m-contraction* if $(1-t)^m(T^*, T) \ge 0$, and that T is an *m-isometry* if $(1-t)^m(T^*, T) = 0$. The papers [13, 15, 16, 35, 55] (among others) study m-isometries. The paper [43] is dedicated to a profound study of 2-isometries. In [36], Gu treats a more general class of (m, p)-isometries on Banach spaces, and in [37], he discusses m-isometric tuples of operators on a Hilbert space. In [20], Chavan and Sholapurkar study another interesting class of operators: T is a *joint complete hyperexpansion of order m* if $(1-t)^n(T^*, T) \le 0$ for every integer $n \ge m$. That work, in fact, is devoted to tuples of commuting operators.

Here we introduce the case when the exponent a is not an integer. The definitions of a-contraction and a-isometries are the natural ones: we say that T is an *a-contraction* if $(1-t)^a(T^*, T) \ge 0$, and T is an *a-isometry* if $(1-t)^a(T^*, T) = 0$.

Note that $\alpha(t) := (1-t)^a$ is of Nevanlinna-Pick type when $0 < a < 1$. In this case, with the help of the model given by Theorem 1.4, we will get the following two ergodic results.

Theorem 1.14. *If T is an a-contraction, with $0 < a < 1$, then T is quadratically (C,b)-bounded for any $b > 1 - a$.*

That T is *quadratically (C,b)-bounded* (where the letter C stands for Cesàro) means that there exists a constant $c > 0$ such that

$$\sup_{n\geq 0} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 \leq c\|x\|^2 \qquad (\forall x \in H),$$

where the numbers $k^{-s}(n)$, called *Cesàro numbers*, are defined by

$$(1-t)^s =: \sum_{n=0}^{\infty} k^{-s}(n)t^n.$$

As we will show (see Example 7.3), for any $a \in (0,1)$, the class of a-contractions on H is strictly larger than the class of contractions. It is obvious that any contraction is quadratically (C,b)-bounded (due to the equality $\sum_{j=0}^{n} k^b(n-j) = k^{b+1}(n)$ for any $b > 0$). The meaning of the above fact is that some ergodic properties of contractions still hold true for a-contractions.

Theorem 1.15. *Let T be an a-contraction with $0 < a < 1$ and let $b > 1 - a$. Then the following statements are equivalent.*

(i) *The isometry S does not appear in the $(1-t)^a$-model of T.*

(ii) *For every $x \in H$,*

$$\exists \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j)\|T^j x\|^2 = 0. \tag{1.9}$$

(iii) *For every $x \in H$,*

$$\liminf_{n\to\infty} \|T^n x\| = 0.$$

Remark 1.16. For any $a \in (0,1)$, there are a-contractions which are not contractions. This follows from Theorem 7.2 below. The same holds for $a > 1$. Indeed, if $m < a \leq m+1$, where m is an integer, then it is easy to get (see our forthcoming paper [1]) that any $(m+1)$-isometry T is also an a-isometry, which means that $(1-t)^a(T^*,T) = 0$. There are $(m+1)$-isometries that are not contractions, and each of them is an example of this type.

1.5. **Contents.** The paper is organized as follows. In Section 2 we introduce two families of operators in $L(H)$ depending on a fixed function $\alpha(t) = \sum_{n\geq 0} \alpha_n t^n$: Adm_α^w and $\mathcal{C}_\alpha^w$. Essentially, Adm_α^w is the family of operators T for which we can define $\alpha(T^*,T)$, and its subfamily $\mathcal{C}_\alpha^w$ consists of those T for which $\alpha(T^*,T) \geq 0$. We use the superscript notation "w" in Adm_α^w and $\mathcal{C}_\alpha^w$ to make it easier to compare the results from [11] and from the present paper. Notice that in [11], only the convergence of

We obtain some interesting properties of these families and characterize the membership of backward and forward weighted shifts to them. In Section 3, we prove Theorems 1.8 and 1.12.

The proof of Theorem 1.5 is given in Section 4. In Section 5 we study the scope of Theorem 1.5. There we present examples satisfying the hypothesis of Theorem 1.5, where Theorem 1.4 does not apply. In Section 6 we prove Theorem 1.13. The proofs of Theorems 1.14 and 1.15 are given in Section 7.

In our forthcoming paper [1], we will study models up to similarity (instead of unitary equivalence). There we will consider functions α that may have zeroes in $\mathbb{D}$. We will prove that under certain hypotheses, any operator in $\mathcal{C}^w_\alpha$ is similar to an a-contraction if $\alpha(t)$ "behaves like" $(1-t)^a$ in a neighborhood of 1. We will also study a-contractions in more detail.

2. Preliminaries on classes defined by operator inequalities

In this section we introduce the operator classes Adm^w_α and $\mathcal{C}^w_\alpha$ associated to a function $\alpha(t)=\sum_{n\geq 0}\alpha_n t^n$, with $\alpha_n\in\mathbb{R}$. After studying them, we analyze why Hypotheses 1.1 are natural. Finally, at the end of the section we discuss the membership of weighted shifts in the classes Adm^w_α and $\mathcal{C}^w_\alpha$.

2.1. The classes Adm^w_α and $\mathcal{C}^w_\alpha$.

Before entering into the definitions and basic properties of these classes, let us mention the following well known result that will be used repeatedly.

Lemma 2.1 (see [38, Problem 120]). *If an increasing sequence $\{A_n\}$ of selfadjoint Hilbert space operators satisfies $A_n\leq CI$ for all n, where C is a constant, then $\{A_n\}$ converges in the strong operator topology.*

Definition 2.2. Given a function $\alpha(t)=\sum_{n\geq 0}\alpha_n t^n$ with $\alpha_n\in\mathbb{R}$, we put

$$\mathrm{Adm}^w_\alpha := \Big\{T\in L(H) : \sum_{n=0}^{\infty}|\alpha_n|\,\|T^n x\|^2<\infty \text{ for every } x\in H\Big\}. \tag{2.1}$$

Note that this class of operators is not affected if we change the signs of some coefficients α_n's.

If X and Y are two quantities (typically non-negative), then $X\lesssim Y$ (or $Y\gtrsim X$) will mean that $X\leq CY$ for some absolute constant $C>0$. If the constant C depends on some parameter p, then we write $X\lesssim_p Y$. We write $X\asymp Y$ when both $X\lesssim Y$ and $Y\lesssim X$.

Proposition 2.3. *The following statements are equivalent.*

(i) $T\in\mathrm{Adm}^w_\alpha$.
(ii) $\sum_{n=0}^\infty|\alpha_n|\,\|T^n x\|^2\lesssim\|x\|^2$ *for every* $x\in H$.
(iii) *The series* $\sum_{n=0}^\infty|\alpha_n|T^{*n}T^n$ *converges in the strong operator topology in* $L(H)$.

Proof. Suppose that (i) is true. Note that for every $x, y \in H$ and $M > N$ we have

$$\left|\sum_{n=N+1}^{M} |\alpha_n| \langle T^n x, T^n y\rangle\right| \leq \sum_{n=N+1}^{M} |\alpha_n| \, \|T^n x\| \, \|T^n y\|$$
$$\leq \frac{1}{2}\left\{\sum_{n=N+1}^{M} |\alpha_n| \, \|T^n x\|^2 + \sum_{n=N+1}^{M} |\alpha_n| \, \|T^n y\|^2\right\} \to 0$$

as N and M go to infinity. Therefore

$$\sum_{n=0}^{\infty} |\alpha_n| \langle T^n x, T^n y\rangle \qquad \text{converges (in } \mathbb{C}), \tag{2.2}$$

for every $x, y \in H$. Put

$$A_N := \sum_{n=0}^{N} |\alpha_n| T^{*n} T^n \in L(H) \tag{2.3}$$

for every non-negative integer N. Fix $x \in H$. By (2.2) we know that $\langle A_N x, y\rangle$ converges for every $y \in H$. This means that the sequence $\{A_N x\} \subset H$ is weakly convergent. Then $\sup_N \|A_N x\| < \infty$ for any $x \in H$ and therefore $\sup_N \|A_N\| < \infty$. Hence (ii) follows with absolute constant $\sup_N \|A_N\|$.

Now suppose we have (ii). This means that the operators A_N given by (2.3) are uniformly bounded from above. So we can apply Lemma 2.1 to obtain (iii).

Finally, it is immediate that (iii) implies (i). This completes the proof. ☐

Corollary 2.4. *If $T \in \mathrm{Adm}_\alpha^w$, then the series*

$$\alpha(T^*, T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n$$

converges in the strong operator topology in $L(H)$.

Proof. Let $T \in \mathrm{Adm}_\alpha^w$. By Proposition 2.3, the series $\sum |\alpha_n| T^{*n} T^n$ converges in SOT. Put

$$\alpha_n^+ := \begin{cases} \alpha_n & \text{if } \alpha_n \geq 0 \\ 0 & \text{if } \alpha_n < 0 \end{cases}, \qquad \alpha_n^- := \begin{cases} 0 & \text{if } \alpha_n \geq 0 \\ -\alpha_n & \text{if } \alpha_n < 0 \end{cases}.$$

Hence

$$\sum_{n=0}^{N} \alpha_n T^{*n} T^n = \sum_{n=0}^{N} \alpha_n^+ T^{*n} T^n - \sum_{n=0}^{N} \alpha_n^- T^{*n} T^n. \tag{2.4}$$

It is immediate (using again Lemma 2.1) that both sums on the right hand side of (2.4) have limits in SOT as $N \to \infty$, and therefore the corollary follows. ☐

This corollary allows us to introduce the following class of operators in $L(H)$, also depending on α.

Definition 2.5. Let

$$\mathcal{C}_\alpha^w := \{T \in \mathrm{Adm}_\alpha^w \,:\, \alpha(T^*, T) \geq 0\}.$$

Sometimes, by abuse of notation, we will simply write $\alpha(T^*, T) \geq 0$ instead of $T \in \mathcal{C}_\alpha^w$. In particular, this means that $T \in \mathrm{Adm}_\alpha^w$.

Proposition 2.6.

(i) *If $T \in \mathcal{C}_\alpha^w$, then any part of T also belongs to $\mathcal{C}_\alpha^w$.*
(ii) *If $T_1, T_2 \in \mathcal{C}_\alpha^w$, then $T_1 \oplus T_2 \in \mathcal{C}_\alpha^w$.*
(iii) *If $T \in \mathcal{C}_\alpha^w$, then $T \otimes I_\mathcal{E}$ (where $I_\mathcal{E}$ is the identity operator on some Hilbert space $\mathcal{E}$) also belongs to $\mathcal{C}_\alpha^w$.*

Proof. Note that an operator $T \in L(H)$ belongs to $\mathcal{C}_\alpha^w$ if and only if

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|T^n x\|^2 < \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} \alpha_n \, \|T^n x\|^2 \geq 0$$

for every $x \in H$. Then (i) and (ii) are immediate. For (iii), observe that if $d := \dim \mathcal{E} \leq \infty$, then the orthogonal sum of d copies of an operator in $\mathcal{C}_\alpha^w$ is clearly in $\mathcal{C}_\alpha^w$ (by the Pythagoras Theorem). □

The following proposition will serve us to discuss why our Hypotheses 1.1 are natural.

Proposition 2.7. *Let $T \in \mathcal{C}_\alpha^w$. If $\alpha \notin A_W$, then $\sigma(T) \subset \mathbb{D}$.*

Proof. Let $T \in \mathcal{C}_\alpha^w$, where $\alpha \notin A_W$ (that is, $\sum |\alpha_n| = \infty$). By Proposition 2.3 we know that there exists a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|T^n x\|^2 \leq C \tag{2.5}$$

for every $x \in H$ with $\|x\| = 1$.

Suppose that T has spectral radius $\rho(T) \geq 1$. Let λ be any point of $\sigma(T)$ such that $|\lambda| = \rho(T)$. Then λ belongs to the boundary of the spectrum of T and therefore it belongs to the approximate point spectrum. Put $R := |\lambda|^2 = \rho(T)^2 \geq 1$. Fix an integer N sufficiently large so that

$$\sum_{n=0}^{N} |\alpha_n| > C + 1.$$

Now, choose a unit approximate eigenvector $h \in H$ corresponding to λ such that $\|Th - \lambda h\|$ is sufficiently small, so that

$$\Big| \|T^m h\|^2 - |\lambda|^{2m} \Big| < \left(\sum_{n=0}^{N} |\alpha_n| \right)^{-1}, \qquad m = 0, 1, \ldots, N.$$

Then

$$\left|\sum_{n=0}^{N}|\alpha_n|R^n-\sum_{n=0}^{N}|\alpha_n|\,\|T^nh\|^2\right|\le\sum_{n=0}^{N}|\alpha_n|\left|R^n-\|T^nh\|^2\right|<1,$$

and therefore

$$\sum_{n=0}^{N}|\alpha_n|\,\|T^nh\|^2\ge\left(\sum_{n=0}^{N}|\alpha_n|R^n\right)-1\ge\left(\sum_{n=0}^{N}|\alpha_n|\right)-1>C.$$

But this contradicts (2.5). Hence $\rho(T)$ must be strictly less that 1, that is, $\sigma(T)\subset\mathbb{D}$, as we wanted to prove. □

The next result follows immediately imitating the above proof. We denote by $r(\alpha)$ the radius of convergence of the series for α.

Proposition 2.8. *If $T\in\mathcal{C}_\alpha^w$, then $\rho(T)^2\le r(\alpha)$.*

One can compare the above two propositions with [61, Corollary 22], which concerns the case when T satisfies an equality $\alpha(T^*,T)=0$.

2.2. **Analysis of the Hypotheses 1.1.** Observe that Hypotheses 1.1 do not restrict to the Nevanlinna-Pick case. Let us explain briefly why these hypothesis are natural.

First of all, the assumption that α belongs to A_W is natural due to Proposition 2.7. To assure that $k=1/\alpha$ is analytic in $\mathbb{D}$, we need that α do not vanish in $\mathbb{D}$. In order to guarantee that we can obtain a reproducing kernel Hilbert space $\mathcal{R}_k$ of analytic functions, we need to assume that $k_n>0$ for every $n\ge0$. (See Remark 2.9 below.) The assumption $k_0=1$ is just a normalization of the coefficients. Finally, note that in Theorem 1.5, which is our new source of examples when compared with Theorem 1.4, we need that $k\in A_W$. However, this assumption excludes automatically the critical case (when $\alpha(1)=0$). Therefore, it is natural to just make the assumption that k is analytic in $\mathbb{D}$, so we can still consider both cases: critical and subcritical.

As we already mentioned in the Introduction, in [1] we will drop the assumption that α does not vanish on $\mathbb{D}$.

2.3. **The weighted shifts $B_\varkappa$ and $F_\varkappa$.** Given a sequence of positive numbers $\{\varkappa_n:n\ge0\}$, we denote by $\mathcal{H}_\varkappa$ the corresponding weighted Hilbert space of power series $f(t)=\sum_{n=0}^\infty f_nt^n$ with the norm

$$\|f\|_{\mathcal{H}_\varkappa}:=\left(\sum_{n=0}^{\infty}|f_n|^2\varkappa_n\right)^{1/2}.$$

Obviously, the monomials $e_n(t):=t^n$, for $n\ge0$, form an orthogonal basis on $\mathcal{H}_\varkappa$, and

$$\|e_n\|^2_{\mathcal{H}_\varkappa}=\varkappa_n. \tag{2.6}$$

The backward and forward shifts $B_\varkappa$ and $F_\varkappa$ on $\mathcal{H}_\varkappa$ are defined by

$$B_\varkappa f(t) := \frac{f(t)-f(0)}{t} \quad \text{and} \quad F_\varkappa f(t) := tf(t) \qquad (\forall f \in \mathcal{H}_\varkappa), \tag{2.7}$$

or equivalently

$$B_\varkappa e_n := \begin{cases} e_{n-1}, & \text{if } n \geq 1 \\ 0, & \text{if } n = 0 \end{cases} \quad \text{and} \quad F_\varkappa e_n := e_{n+1} \qquad (\forall n \geq 0). \tag{2.8}$$

It is immediate that $\|B_\varkappa\|^2 = \sup_{n\geq 0} \varkappa_n/\varkappa_{n+1}$. Hence $B_\varkappa$ is bounded if and only if

$$\frac{\varkappa_n}{\varkappa_{n+1}} \leq C \qquad (\forall n \geq 0), \tag{2.9}$$

for a constant $C > 0$. Analogously, $\|F_\varkappa\|^2 = \sup_{n\geq 0} \varkappa_{n+1}/\varkappa_n$, and therefore $F_\varkappa$ is bounded if and only if

$$0 < c \leq \frac{\varkappa_n}{\varkappa_{n+1}} \qquad (\forall n \geq 0), \tag{2.10}$$

for some constant c.

Remark 2.9. At the beginning of Subsection 1.4 we discussed the duality of the spaces $\mathcal{H}_k$ and $\mathcal{R}_k$, where $k(t)$ was, as usual, the function $1/\alpha(t)$. Of course, if we replace k with any other function $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$ (where $\varkappa_n > 0$), the same duality will hold for the spaces $\mathcal{H}_\varkappa$ and $\mathcal{R}_\varkappa$.

Notation 2.10. Let us mention here a convenient notation that will be used in Section 7. When $\{\varkappa_n\}$ is precisely the sequence of Taylor coefficients of the function $(1-t)^{-s}$ for some $s > 0$, that is,

$$\varkappa_0 = 1 \quad \text{and} \quad \varkappa_n = \frac{s(s+1)\cdots(s+n-1)}{n!} \quad \text{for } n \geq 1,$$

we denote the space $\mathcal{H}_\varkappa$ by $\mathcal{H}_s$, emphasizing the exponent s. In the same way we use B_s and F_s.

Lemma 2.11. *Let T be one of the operators $B_\varkappa$ or $F_\varkappa$, for some $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$. Suppose that T is bounded (i.e., assume (2.9) or (2.10), respectively). Then:*

(i) $T \in \mathrm{Adm}_\alpha^w$ *if and only if*

$$\sup_{m\geq 0} \left\{ \sum_{n=0}^{\infty} |\alpha_n| \frac{\|T^n e_m\|^2}{\|e_m\|^2} \right\} < \infty. \tag{2.11}$$

(ii) *Suppose that $T \in \mathrm{Adm}_\alpha^w$. Then $T \in \mathcal{C}_\alpha^w$ if and only if*

$$\sum_{n=0}^{\infty} \alpha_n \|T^n e_m\|^2 \geq 0 \qquad (\forall m \geq 0). \tag{2.12}$$

Proof. (i) Let $T \in \mathrm{Adm}_\alpha^w$. By Proposition 2.3 (ii) we have

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|T^n f\|^2 \lesssim \|f\|^2 ,$$

for every function $f \in \mathcal{H}_\varkappa$. Taking the vectors of the basis $f = e_m$ we obtain (2.11).

Conversely, let us assume now (2.11). Fix a function $f \in \mathcal{H}_\varkappa$. Then

$$T^n f = \sum_{m=0}^{\infty} f_m T^n e_m \qquad (\forall n \geq 0),$$

where the series is orthogonal. Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} |\alpha_n| \, \|T^n f\|^2 &= \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \, \|T^n e_m\|^2 = \sum_{n=0}^{\infty} |\alpha_n| \sum_{m=0}^{\infty} |f_m|^2 \, \|e_m\|^2 \, \frac{\|T^n e_m\|^2}{\|e_m\|^2} \\ &= \sum_{m=0}^{\infty} |f_m|^2 \, \|e_m\|^2 \sum_{n=0}^{\infty} |\alpha_n| \frac{\|T^n e_m\|^2}{\|e_m\|^2} \lesssim \sum_{m=0}^{\infty} |f_m|^2 \, \|e_m\|^2 < \infty, \end{aligned} \tag{2.13}$$

where (2.11) allows us to justify the change of the summation indexes in the last equality. Hence $T \in \mathrm{Adm}_\alpha^w$.

(ii) Let $T \in \mathrm{Adm}_\alpha^w$. If $T \in \mathcal{C}_\alpha^w$, then obviously (2.12) follows. For the converse implication, note that similarly to (2.13) we get

$$\sum_{n=0}^{\infty} \alpha_n \, \|T^n f\|^2 = \sum_{m=0}^{\infty} |f_m|^2 \, \|e_m\|^2 \sum_{n=0}^{\infty} \alpha_n \frac{\|T^n e_m\|^2}{\|e_m\|^2},$$

so (2.12) implies that $T \in \mathcal{C}_\alpha^w$. □

Writing down this lemma for $B_\varkappa$ and $F_\varkappa$ separately, we immediately get the next two results.

Theorem 2.12. *Let $\varkappa(t) = \sum_{n\geq 0} \varkappa_n t^n$, such that the coefficients $\{\varkappa_n\}$ satisfy (2.9). Set $\beta(t) = \sum_{n\geq 0} \beta_n t^n$ with $\beta_n = |\alpha_n|$. Put $\gamma(t) = \beta(t)\varkappa(t)$. Then:*

(i) *$B_\varkappa \in \mathrm{Adm}_\alpha^w$ if and only if*

$$\sup_{m\geq 0} \left\{ \frac{\gamma_m}{\varkappa_m} \right\} < \infty.$$

(ii) *Suppose that $B_\varkappa \in \mathrm{Adm}_\alpha^w$. Then $B_\varkappa \in \mathcal{C}_\alpha^w$ if and only if all the Taylor coefficients of $\alpha(t)\varkappa(t)$ are non-negative.*

The next statement explains the meaning of Hypotheses 1.7.

Corollary 2.13. *Suppose that Hypotheses 1.7 hold, and let $T = B_k \otimes I_{\mathcal{E}}$. Then*

(i) *$T \in \mathcal{C}_\alpha^w$;*
(ii) *$\alpha(T^*, T) f = f_0$, $f = \sum f_n z^n \in \mathcal{H}_k \otimes \mathcal{E}$;*
(iii) *The corresponding operator $V_D : \mathcal{H}_k \otimes \mathcal{E} \to \mathcal{H}_k \otimes \mathcal{E}$ is the identity operator.*

Indeed, consider first the case when $T = B_k$. Hypotheses 1.7 imply that $B_k \in \mathrm{Adm}^w_\alpha$. Equality $\alpha(t)k(t) = 1$ gives that $\sum_{n=0}^{\infty} \alpha_n \|T^n e_0\|^2 = 1$ and $\sum_{n=0}^{\infty} \alpha_n \|T^n e_m\|^2 = 0$ if $m \geq 1$. This implies (ii) for this case. In particular, $\alpha(T^*, T) \geq 0$ (that is, $B_k \in \mathcal{C}^w_\alpha$), and (iii) follows. Finally, the operator $T = B_k \otimes I_{\mathcal{E}}$ can be seen as an orthogonal sum of $\dim \mathcal{E}$ copies of B_k, which gives the general case.

Before restating Lemma 2.11 for the forward shift $F_\varkappa$, we need to introduce some notation.

Notation 2.14. Given a sequence of real numbers $\Lambda = \{\Lambda_m\}_{m \geq 0}$, we denote by $\nabla\Lambda$ the sequence whose m-th term, for $m \geq 0$, is given by $(\nabla\Lambda)_m = \Lambda_{m+1}$. In general, if $\beta(t) = \sum \beta_n t^n$ is an analytic function, we denote by $\beta(\nabla)\Lambda$ the sequence whose m-th term is given by

$$(\beta(\nabla)\Lambda)_m = \beta(\nabla)\Lambda_m := \sum_{n=0}^{\infty} \beta_n \Lambda_{m+n},$$

whenever the series on the right hand side converges for every $m \geq 0$.

Theorem 2.15. *Let $\varkappa(t) = \sum_{n \geq 0} \varkappa_n t^n$ be a function such that the coefficients $\{\varkappa_n\}$ satisfy* (2.10)*. Set $\beta(t) = \sum_{n \geq 0} \beta_n t^n$ with $\beta_n = |\alpha_n|$. Then:*

(i) *$F_\varkappa \in \mathrm{Adm}^w_\alpha$ if and only if*

$$\sup_{m \geq 0} \left\{ \frac{\beta(\nabla)\varkappa_m}{\varkappa_m} \right\} < \infty.$$

(ii) *Suppose that $F_\varkappa \in \mathrm{Adm}^w_\alpha$. Then $F_\varkappa \in \mathcal{C}^w_\alpha$ if and only if $\alpha(\nabla)\varkappa_m \geq 0$ for every $m \geq 0$.*

3. EXPLICIT MODEL AND ITS UNIQUENESS

In this section we prove Theorems 1.8 and 1.12. Let us start by proving that the operator V_D is a contraction in the Nevanlinna-Pick case. Notice that in the following theorem, we do not require that α belongs to A_W.

Theorem 3.1. *Let $\alpha(t) := \sum_{n \geq 0} \alpha_n t^n$, with $\alpha_0 = 1$ and $\alpha_n \leq 0$ for $n \geq 1$. If $T \in L(H)$ satisfies $\alpha(T^*, T) \geq 0$, then the operator V_D is a contraction.*

Proof. Recall that $D^2 = \alpha(T^*, T)$. Therefore

$$\|Dx\|^2 = \sum_{m=0}^{\infty} \alpha_m \|T^m x\|^2$$

for every $x \in H$. Hence

$$\|DT^n x\|^2 = \sum_{m=0}^{\infty} \alpha_m \left\|T^{m+n} x\right\|^2$$

for every $x \in H$ and every non-negative integer n. Fix a positive integer N. Then

$$\sum_{n=0}^{N} k_n \|DT^n x\|^2 = \sum_{n=0}^{N} k_n \sum_{m=0}^{\infty} \alpha_m \left\|T^{m+n}x\right\|^2$$
$$= \sum_{j=0}^{\infty} \left(\sum_{n+m=j,\, n\le N} k_n \alpha_m \right) \left\|T^j x\right\|^2 =: \sum_{j=0}^{\infty} \tau_j \left\|T^j x\right\|^2 .$$

Since $\alpha k = 1$ we get $\tau_0 = 1$ and $\tau_1 = \cdots = \tau_N = 0$. Moreover,

$$\tau_{N+i} = k_0 \alpha_{N+i} + \cdots + k_N \alpha_i < 0$$

for every $i \geq 1$, because all the α_j's above are negative or zero and the k_j's are positive. Therefore

$$\sum_{n=0}^{N} k_n \|DT^n x\|^2 \leq \|x\|^2$$

for every N and hence the series $\sum k_n \|DT^n x\|^2$ converges for every $x \in H$. This gives

$$\|V_D x\|^2 = \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 \leq \|x\|^2 ,$$

as we wanted to prove. □

The following fact is simple and well-known .

Proposition 3.2. *Let $T \in L(H)$ with $\sigma(T) \subset \overline{\mathbb{D}}$, and let $\mathcal{E}$ be a Hilbert space. A bounded transform $V : H \to \mathcal{H}_\varkappa \otimes \mathcal{E}$ satisfies*

$$VT = (B_\varkappa \otimes I_{\mathcal{E}})V \tag{3.1}$$

if and only if there is a bounded linear operator $C : H \to \mathcal{E}$ such that $V = V_C$ (see (1.6)).

Proof. It is well-known (and straightforward) that any bounded transform V_C satisfies (3.1). Conversely, suppose that $VT = (B_\varkappa \otimes I_{\mathcal{E}})V$. Define $a_n(x)$ by

$$Vx(z) := \sum_{n=0}^{\infty} a_n(x) z^n, \qquad x \in H.$$

Then

$$\sum_{n=0}^{\infty} a_n(Tx) z^n = VTx = (B_\varkappa \otimes I_{\mathcal{E}})Vx = \sum_{n=0}^{\infty} a_{n+1}(x) z^n.$$

Therefore $a_{n+1}(x) = a_n(Tx)$. The statement follows, putting $C := a_0$, which has to be a bounded linear operator. □

Proposition 3.3. *Let $C : H \to \mathcal{E}$ be a bounded operator and let $T \in \mathcal{C}_\alpha^w$. Then there exists a bounded operator $W : H \to \mathcal{W}$ such that the operator (V_C, W) is isometric and transforms*

T into a part of the operator $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S \in L(\mathcal{W})$ is an isometry, if and only if the following conditions hold.

(i) *$V_C : H \to \mathcal{H}_k \otimes \mathcal{E}$ is a contraction.*
(ii) *For every $x \in H$,*

$$\|x\|^2 - \|V_C x\|^2 = \|Tx\|^2 - \|V_C Tx\|^2.$$

Proof. Let us suppose first the existence of such operator W. Since (V_C, W) is an isometry, (i) holds. Notice that (ii) is equivalent to proving that $\|Wx\|^2 = \|WTx\|^2$ for every $x \in H$. But this is also immediate since $SWx = WTx$ and S is an isometry.

Conversely, suppose now that (i) and (ii) are true. By (i), we can put $W := (I - V_C^* V_C)^{1/2}$ and $\mathcal{W} := \overline{\operatorname{Ran}} W$. Using (ii) we have

$$\|Wx\|^2 = \|x\|^2 - \|V_C x\|^2 = \|Tx\|^2 - \|V_C Tx\|^2 = \|WTx\|^2 . \tag{3.2}$$

We define

$$S(Wx) := WTx,$$

for every $x \in H$. Note that S is well defined, since $\|SWx\| = \|Wx\|$ by (3.2). Since WH is dense in $\mathcal{W}$, S can be extended to an isometry on $\mathcal{W}$. By the definition of W, we know that (V_C, W) is an isometry and it is immediate that

$$(B_k \otimes I_{\mathcal{D}}) V_C = V_C T \quad \text{and} \quad SW = WT.$$

This completes the converse implication. □

Proposition 3.4. *Let $T \in \mathcal{C}_\alpha^w$. Assume that $C : H \to \mathcal{E}$ and $W : H \to \mathcal{W}$ are any bounded operators such that (V_C, W) is isometric on $(\mathcal{H}_k \otimes \mathcal{E}) \oplus \mathcal{W}$ and transforms T into a part of $(B_k \otimes I_{\mathcal{E}}) \oplus S$, where $S \in B(\mathcal{W})$ is an isometry. Then C and D are related by*

$$\|Dx\|^2 = \|Cx\|^2 + \alpha(1)\|Wx\|^2, \quad \forall x \in H. \tag{3.3}$$

Proof. Since (V_C, W) is isometric, we have

$$\|x\|^2 = \|V_C x\|^2 + \|Wx\|^2 = \sum_{n=0}^{\infty} k_n \|CT^n x\|^2 + \|Wx\|^2 , \tag{3.4}$$

for every $x \in H$. Substituting x by $T^j x$ above and multiplying by α_j, we obtain that

$$\begin{aligned} \alpha_j \left\|T^j x\right\|^2 &= \sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{n+j} x\right\|^2 + \alpha_j \left\|WT^j x\right\|^2 \\ &= \sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{n+j} x\right\|^2 + \alpha_j \|Wx\|^2 , \end{aligned}$$

where we have used that $\|Wx\|^2 = \|WTx\|^2$. Therefore

$$\|Dx\|^2 = \sum_{j=0}^{\infty} \alpha_j \left\|T^j x\right\|^2 = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \alpha_j k_n \left\|CT^{j+n}x\right\|^2 + \left(\sum_{j=0}^{\infty} \alpha_j\right) \|Wx\|^2$$

$$\overset{(\star)}{=} \sum_{m=0}^{\infty} \left(\sum_{j+n=m} \alpha_j k_n\right) \|CT^m x\|^2 + \alpha(1) \|Wx\|^2 .$$

Since $\alpha k = 1$, the only non-vanishing summand in the last series above is for $m = 0$ and we obtain (3.3). Finally, note that the rearrangement in $(\star)$ is correct as

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} |\alpha_j| k_n \left\|CT^{n+j}x\right\|^2 \le \sum_{j=0}^{\infty} |\alpha_j| \left\|T^j x\right\|^2 < \infty,$$

where we have used (3.4) and that $T \in \mathcal{C}_\alpha^w$. □

Recall the definition of the *minimal model* (Definition 1.10).

Remark 3.5. Suppose that T is α-modelable. Then T is unitarily equivalent to $\big((B_k \otimes I_{\mathcal{E}}) \oplus S\big)|\mathcal{L}$, where $\mathcal{L} = \overline{\operatorname{Ran}}\,(V_C, W)$. This model is minimal if and only if

(a) $\overline{\operatorname{Ran}}\, C = \mathcal{E}$; and
(b) $\overline{\operatorname{Ran}}\, W = \mathcal{W}$.

Indeed, in this case, it is easy to see that (a) is equivalent to (i), and (b) is equivalent to (ii) in Definition 1.10.

Proof of Theorem 1.12. Suppose that the hypotheses are satisfied. First we notice that $\alpha(T^*, T) \ge 0$, as it follows from Corollary 2.13 and Proposition 2.6. Therefore D is well-defined.

(i) In the critical case (i.e., $\alpha(1) = 0$), (3.3) gives

$$\|Dx\| = \|Cx\| \qquad \forall x \in H,$$

so there exists a unitary operator v such that $C = vD$. This implies the statement.

(ii) Suppose we are in the subcritical case (i.e., $\alpha(1) > 0$). First, we remark that the model is not unique in general. For instance, take $T = U$ any unitary operator. Using Proposition 2.3 and that $\alpha \in A_W$, we obtain that $T \in \operatorname{Adm}_\alpha^w$. Since

$$\sum_{n=0}^{\infty} \alpha_n \|T^n x\|^2 = \left(\sum_{n=0}^{\infty} \alpha_n\right) \|x\|^2 \qquad \forall x \in H,$$

we get that $\alpha(T^*, T) = \alpha(1) I \ge 0$. Obviously, $T = U$ is a minimal model for T (where $\mathcal{E} = 0$, and $\mathcal{W} = H$). Moreover, if $k = 1/\alpha$ fits (1.5), then Theorem 1.5 (which is proved in the next section, but its proof is completely independent) gives another model for T. (See Example 5.1 and Remark 5.2.)

Now suppose that T is any α-modelable operator and (V_C, W) provides its model. Let us see that there exists a minimal model of T with $V = V_D$ and W absent. Changing x by $T^n x$ in (3.3) we obtain

$$\|DT^n x\|^2 = \|CT^n x\|^2 + \alpha(1)\,\|Wx\|^2\,,$$

where we have used that $\|WTx\| = \|Wx\|$. Therefore

$$\begin{aligned}\|V_D x\|^2 &= \sum_{n=0}^{\infty} k_n\,\|DT^n x\|^2 = \sum_{n=0}^{\infty} k_n\,\|CT^n x\|^2 + k(1)\alpha(1)\,\|Wx\|^2 \\ &= \|V_C x\|^2 + \|Wx\|^2 = \|x\|^2\,,\end{aligned}$$

so $V_D : H \to \mathcal{H}_k \otimes \mathcal{E}$ is an isometry and therefore provides a model of T. The space $\mathcal{L}$ is just $\operatorname{Ran} V_D$ in $\mathcal{H}_k \otimes \mathfrak{D}$ (which is closed). This model is minimal, because $\operatorname{Ran} D$ is dense in $\mathfrak{D}$. (See Remark 3.5.) This gives all statements of (ii). □

Proof of Theorem 1.8. It is an immediate consequence of Theorem 1.12 and Proposition 3.3 (i) that V_D is a contraction. Finally, for proving that (V_D, W) gives a model, we just need to use the same argument employed in the reciprocal implication of Proposition 3.3. □

Results close to Theorems 1.8 and 1.12 appear in Schillo's PhD thesis [58]. He deals with the generality of tuples of commuting operators, but for the case of one operator, the hypotheses needed there are more restrictive than ours.

For example, in [58, Theorem 5.16], the uniqueness of the coextension is proved when T is what he calls a *strong k-contraction.* For one single operator T and using our notations, these are operators such that $\alpha(T^*, T) \geq 0$, the limit

$$\Sigma(T) := I_H - \lim_{N\to\infty} \sum_{n=0}^{N} k_n T^{*n} \alpha(T^*, T) T^n$$

exists (in SOT), $\Sigma(T) \geq 0$, and $\Sigma(T) = T^*\Sigma(T)T$. In [58, Corollary 5.17], he gives an explicit model involving the defect space $\mathfrak{D}_T$. His assumptions are somewhat technical (see [58, Assumption 5.8]). He also assumes the existence of $\alpha(B_k^*, B_k)$, for which [58, Proposition 2.10] says that a sufficient condition is that the coefficients $\{\alpha_n\}$ of the function α have eventually the same sign.

Recall that our Theorem 1.12 (ii) says that for the subcritical case the model is not unique in general. Therefore, since Schillo obtains uniqueness of the coextension, it seems that his assumptions exclude the subcritical case.

Schillo's thesis also contains a result on the description of invariant subspaces of a backward shift, analogous to $B_k \otimes I_{\mathcal{E}}$, in his setting of operator tuples.

Notice that in Theorems 1.8 and 1.12 we are only assuming that T is α-modelable. In particular, we do not impose any restriction about the signs of the Taylor coefficients of the function α.

4. Proof of Theorem 1.5

In this section we prove Theorem 1.5. For that, we need to cite some results concerning Banach algebras.

For any sequence $\omega = \{\omega_n\}_{n=0}^{\infty}$ of positive weights, define the weighted space

$$\ell^\infty(\omega) := \left\{ f(t) = \sum_{n=0}^{\infty} f_n t^n \ : \ \sup_{n\geq 0} |f_n|\omega_n < \infty \right\}.$$

In general, its elements are formal power series. We will also use the separable version of this space:

$$\ell_0^\infty(\omega) := \left\{ f(t) = \sum_{n=0}^{\infty} f_n t^n \ : \ \lim_{n\to\infty} |f_n|\omega_n = 0 \right\}.$$

Proposition 4.1 (see [47])**.** *$\ell^\infty(\omega)$ is a Banach algebra (with respect to the formal multiplication of power series) if and only if*

$$\sup_{n\geq 0} \sum_{j=0}^{n} \frac{\omega_n}{\omega_j \omega_{n-j}} < \infty. \tag{4.1}$$

Theorem 4.2. *Let $\omega_n > 0$ and $\omega_n^{1/n} \to 1$. If $\sup_n \omega_{n+1}/\omega_n < \infty$ and*

$$\lim_{m\to\infty} \sup_{n\geq 2m} \sum_{m\leq j\leq n/2} \frac{\omega_n}{\omega_j \omega_{n-j}} = 0, \tag{4.2}$$

then the following is true.

(i) *$\ell^\infty(\omega)$ is a Banach algebra.*
(ii) *If $f \in \ell^\infty(\omega)$ does not vanish on $\overline{\mathbb{D}}$, then $1/f \in \ell^\infty(\omega)$.*

Proof. The hypotheses imply (4.1), so that (i) follows from Proposition 4.1. To get (ii), we apply the results of the paper [29] by El-Fallah, Nikolski and Zarrabi. We use the notation of this paper. Put $\omega'(n) = \omega(n)/(n+1)$, $A = \ell^\infty(\omega)$ and $A_0 = \ell_0^\infty(\omega)$. The hypotheses imply that A (and hence A_0) is compactly embedded into the multiplier convolution algebra $\mathrm{mult}(\ell^\infty(\omega'))$, see [29, Lemma 3.6.3]. Hence, by [29, Theorem 3.4.1], for any $f \in A_0$, $\delta_1(A_0, \mathfrak{M}(A_0)) = 0$, see [29, Subsection 0.2.3] for the definition of this quantity. This means that for any $\delta > 0$ there is a constant $c_1(\delta) < \infty$ such that the conditions $f \in A_0$, $\|f\|_A = 1$ and $|f| > \delta$ on $\overline{\mathbb{D}}$ imply that $1/f \in A_0$ and $\|1/f\|_A \leq c_1(\delta)$. In particular, (ii) holds for f in A_0. To get (ii) in the general case, suppose that $f \in A$ and $|f| > \delta > 0$ on $\overline{\mathbb{D}}$. Since $f(rt) \in A_0$ for all $r < 1$, we get that the norms of the functions $1/f(rt)$ in A are uniformly bounded by $c_1(\delta)$ for all $r < 1$. When $r \to 1^-$, each Taylor coefficient of $1/f(rt)$ tends to the corresponding Taylor coefficient of $1/f(t)$. It follows that $1/f$ is in A (and $\|1/f\|_A \leq c_1(\delta)$). □

Proof of Theorem 1.5. Put

$$\omega_n := 1/k_n.$$

The first part of Theorem 1.5 (that B_k is bounded) is straightforward. Also, by Theorem 4.2 (i), $\ell^\infty(\omega)$ is an algebra.

First suppose that T is a part of $B_k \otimes I_{\mathcal{E}}$, and let us prove that $B_k \in \mathcal{C}^w_\alpha \cap \mathrm{Adm}^w_k$.

By Theorem 2.12 (i), we know that $B_k \in \mathrm{Adm}^w_k$ if and only if

$$\sum_{j=0}^{m} k_j k_{m-j} \lesssim k_m,$$

which follows from Theorem 4.2 (i) and Proposition 4.1.

Now let us see that $B_k \in \mathcal{C}^w_\alpha$. By Theorem 4.2 (ii), $\alpha = 1/k$ belongs to $\ell^\infty(\omega)$, and therefore $|\alpha_n| \lesssim k_n$. Then, since $B_k \in \mathrm{Adm}^w_k$, we obtain that $B_k \in \mathrm{Adm}^w_\alpha$. Finally, Theorem 2.12 (ii) gives that $B_k \in \mathcal{C}^w_\alpha$ (because $\alpha k = 1$ has non-negative Taylor coefficients). Hence T also is in $\mathcal{C}^w_\alpha \cap \mathrm{Adm}^w_k$.

Conversely, let us assume now that $T \in \mathcal{C}^w_\alpha \cap \mathrm{Adm}^w_k$. We want to prove that T is a part of $B_k \otimes I_{\mathcal{E}}$. We adapt the argument of [45, Theorem 2.2] (where the convergence of the series of operators is in the uniform operator topology).

By Proposition 3.2,

$$(B_k \otimes I_{\mathfrak{D}})V_D = V_D T. \tag{4.3}$$

Moreover,

$$\begin{aligned}\|V_D x\|^2 &= \sum_{n=0}^{\infty} k_n \|DT^n x\|^2 = \sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} \alpha_m \left\|T^{n+m}x\right\|^2 \\ &= \sum_{j=0}^{\infty} \Bigg(\sum_{n+m=j} k_n \alpha_m \Bigg) \left\|T^j x\right\|^2 = \|x\|^2 ,\end{aligned}$$

where we have used that $\sum_{n+m=j} k_n\alpha_m$ is equal to 1 if $j = 0$ and is equal to 0 if $j \geq 1$. The re-arrangement of the series is correct since, using that $T \in \mathrm{Adm}^w_\alpha \cap \mathrm{Adm}^w_k$, we have

$$\sum_{n=0}^{\infty} k_n \sum_{m=0}^{\infty} |\alpha_m| \left\|T^{n+m}x\right\|^2 \lesssim \sum_{n=0}^{\infty} k_n \|T^n x\|^2 \lesssim \|x\|^2$$

and the series converges absolutely.

Hence V_D is an isometry. Joined to (4.3), this proves that T is unitarily equivalent to a part of $B_k \otimes I_{\mathfrak{D}}$. □

Notice that in particular, we showed that the hypotheses of Theorem 1.5 imply Hypotheses 1.7.

5. DISCUSSION OF THEOREM 1.5

In this section we discuss the scope of Theorem 1.5 and give a series of examples where it applies, whereas Theorem 1.4 does not. We also will give a direct proof of a particular case of Theorem 4.2, which does not use the results of [29].

Given an analytic function $f(t) = \sum f_n t^n$, we denote by $[f]_N$ its truncated polynomial of degree N, that is,

$$[f]_N := f_0 + f_1 t + \ldots + f_N t^N.$$

Example 5.1. Let $\sigma_2, \ldots, \sigma_N$ be an arbitrary sequence of signs (that is, a sequence of numbers ± 1). We assert that there are functions α, k meeting all the hypotheses of Theorem 1.5 such that $\operatorname{sign}(\alpha_n) = \sigma_n$, for $n = 2, \ldots, N$. This is in contrast with Theorem 1.4, where the Nevanlinna-Pick condition was assumed: $\alpha_n \le 0$ for $n \ge 2$.

To prove the existence of α and k as above, take a polynomial $\widetilde{\alpha}$ of degree N such that $\widetilde{\alpha}_0 = 1, \widetilde{\alpha}_1 < 0$. For $n = 2, \ldots, N$, we set $\widetilde{\alpha}_n < 0$ if $\sigma_n = -1$ and $\widetilde{\alpha}_n = 0$ if $\sigma_n = 1$. Put $\widetilde{k} := [1/\widetilde{\alpha}]_N$. The formula

$$\widetilde{k}_n = \sum_{\substack{s \ge 1 \\ n_1 + \cdots + n_s = n}} (-1)^s \, \widetilde{\alpha}_{n_1} \cdots \widetilde{\alpha}_{n_s} \tag{5.1}$$

shows that all the coefficients of $\widetilde{k}$ are positive. We also require that neither $\widetilde{\alpha}$ nor the polynomial $\widetilde{k}$ vanish on $\overline{\mathbb{D}}$. It is so if, for instance, $|\alpha_n|$ are sufficiently small for $n = 2, \ldots, N$.

Now perturb the coefficients $\widetilde{\alpha}_j$ that are equal to zero, obtaining a new polynomial $\widehat{\alpha}$ such that

$$\widehat{\alpha}_j := \begin{cases} \varepsilon & \text{if } \sigma_j = 1 \\ \widetilde{\alpha}_j & \text{otherwise} \end{cases} \qquad (2 \le j \le N).$$

By continuity, if $\varepsilon > 0$ is small enough, we can guarantee that the polynomial $\widehat{k} = [1/\widehat{\alpha}]_N$ also has positive Taylor coefficients, and we can also guarantee that $\widehat{k}$ (which is a slight perturbation of $\widetilde{k}$) does not vanish on $\overline{\mathbb{D}}$.

Finally, take as k any function in A_W with real Taylor coefficients such that the first ones are

$$k_0 = \widehat{k}_0 = 1, \quad k_1 = \widehat{k}_1, \quad \ldots, \quad k_N = \widehat{k}_N,$$

and

$$\frac{k_{n-j}}{k_n} \le C_0 \qquad (\forall n \ge 2j),$$

for some constant C_0. For instance, one can put $k_n = An^{-b}$ for $n > N$, with $A > 0$ (small enough) and $b > 1$. Then $k \in A_W$ does not vanish on $\overline{\mathbb{D}}$.

Then obviously k satisfies (1.5) and hence all the hypotheses of Theorem 1.5. The function $\alpha := 1/k$ in A_W has the desired pattern of signs.

Finally, it is important to note that $\alpha_1 = -k_1$ is always negative.

Remark 5.2. It is also easy to see that whenever $\{k_n\}$ satisfies (1.5), any other sequence $\{\tilde{k}_n\}$ with $k_0 = 1$ and $c < \tilde{k}_n / k_n < C$ for $n > 1$, where c, C are positive constants, also satisfies this condition. In particular, if $\{k_n\}$ satisfies (1.5) and $\{\tilde{k}_n\}$ is as above, where C is sufficiently small, then $k(t)$ is invertible in A_W, so that all hypotheses of Theorem 1.5 are fulfilled. So there are many examples of functions $k(t)$ meeting these hypotheses, such that the quotients k_n/k_{n+1} do not converge.

Let us mention now some remarks on Theorem 4.2.

Remark 5.3. It is immediate that the condition

$$\frac{\omega_n}{\omega_j \omega_{n-j}} \le \tau_j \quad (\forall n \ge 2j), \qquad \text{where} \quad \sum_{j=0}^{\infty} \tau_j < \infty, \tag{5.2}$$

implies (4.2) and (4.1) (in particular, $\sup_n \omega_{n+1}/\omega_n < \infty$). Let us give a direct proof of Theorem 4.2 for this particular case.

Statement (i) follows using Proposition 4.1.

(ii) Put $g := 1/f$. Suppose that $g \notin \ell^\infty(\omega)$. This means that

$$\sup_{n\ge 0} |g_n|\omega_n = \infty.$$

Hence, it is clear that there exists a sequence $\{\rho_n^0\}$ in $[0,1]$ such that $\rho_n^0 \to 0$ (slowly) and

$$\sup_{n\ge 0} |g_n|\omega_n \rho_n^0 = \infty. \tag{5.3}$$

Claim. *There exists a sequence* $\{\rho_n\}$ *with*

$$\rho_n^0 \le \rho_n \le 1 \quad \text{and} \quad \rho_n \to 0 \tag{5.4}$$

such that $\widetilde{\omega}_n := \rho_n \omega_n$ *defines a Banach algebra* $\ell^\infty(\widetilde{\omega})$.

Indeed, since $\sum \tau_j < \infty$, there exists a sequence of positive numbers $\{c_j\}$ such that $c_j \nearrow \infty$ and still $\sum c_j \tau_j < \infty$. Take any sequence $\{\rho_n\}$ that decreases, tends to zero, and satisfies $\rho_n \ge \max(\rho_n^0, 1/c_n)$. Then, for $\widetilde{\omega}_n := \rho_n \omega_n$ we have

$$\frac{\widetilde{\omega}_n}{\widetilde{\omega}_j \widetilde{\omega}_{n-j}} = \frac{\omega_n}{\omega_j \omega_{n-j}} \frac{\rho_n}{\rho_j \rho_{n-j}} \le \frac{\omega_n}{\omega_j \omega_{n-j}} \frac{1}{\rho_j} \le \tau_j c_j \qquad (\forall n \ge 2j).$$

Since $\sum \tau_j c_j < \infty$, Proposition 4.1 implies that $\ell^\infty(\widetilde{\omega})$ is a Banach algebra, and the proof of the claim is completed.

Now fix $\{\widetilde{\omega}_n\}$ as in the claim. We may assume that $(\rho_n^0)^{1/n} \to 1$ and therefore $(\rho_n)^{1/n} \to 1$. Since the polynomials are dense in the Banach algebra $\ell_0^\infty(\widetilde{\omega})$, any complex homomorphism χ on $\ell_0^\infty(\widetilde{\omega})$ is determined by its value on the power series t. So the map $\chi \mapsto \chi(t)$ is injective and continuous from the spectrum (the maximal ideal space) of $\ell_0^\infty(\widetilde{\omega})$ to $\mathbb{C}$. Since $\widetilde{\omega}_n^{1/n} \to 1$, its image contains $\mathbb{D}$ and is contained in $\overline{\mathbb{D}}$. Hence the spectrum of $\ell_0^\infty(\widetilde{\omega})$ is exactly the set $\{\chi_\lambda : \lambda \in \overline{\mathbb{D}}\}$, where $\chi_\lambda(f) = f(\lambda)$. (We borrow this argument from [29].) As

$$f_n \widetilde{\omega}_n = (f_n \omega_n)\rho_n \to 0,$$

we have $f \in \ell_0^\infty(\widetilde{\omega})$. Then, using the Gelfand theory (see, for instance, [54, Chapter 10]), we get that $g = 1/f \in \ell_0^\infty(\widetilde{\omega})$, which contradicts (5.3). Therefore, the assumption $g \notin \ell^\infty(\omega)$ is false, as we wanted to prove.

Remark 5.4. Notice that the above characterization of the spectrum of the algebra $\ell_0^\infty(\widetilde{\omega})$ (see the above Remark 5.3) implies the following fact: the conditions (4.1) and $\omega_n^{1/n} \to 1$ imply that $\sum_n 1/\omega_n < \infty$. This can be proved in an elementary way, without recurring to the Gelfand theory.

Indeed, by (4.1), there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \frac{\omega_n}{\omega_j \omega_{n-j}} \leq C$$

for every $n \geq 1$. Fix a positive integer L. Then obviously, for every $n \geq L$,

$$\sum_{j=1}^{L} \frac{\omega_n}{\omega_j \omega_{n-j}} \leq C. \tag{5.5}$$

Let us see that

$$\limsup_{n\to\infty} \min_{1\leq j\leq L} \frac{\omega_n}{\omega_{n-j}} \geq 1. \tag{5.6}$$

Indeed, if (5.6) were false, then there would exist some $r < 1$ and a positive integer N such that

$$\min_{1\leq j\leq L} \frac{\omega_n}{\omega_{n-j}} \leq r \quad \text{for } n \geq N.$$

From this, it is easy to see that

$$\omega_n \leq r^{s_n} \left(\max_{0\leq k\leq N} \omega_k \right), \qquad s_n := \left[\frac{n-N}{L} \right] + 1,$$

where $[a]$ denotes the integer part of a. Since s_n behaves asymptotically as n/L, it follows that $\limsup_{n\to\infty} \omega_n^{1/n} \leq r^{1/L} < 1$, which contradicts the hypothesis that $\omega_n^{1/n} \to 1$. Therefore, (5.6) is true.

Now, using (5.5), it follows that

$$C \geq \sum_{j=1}^{L} \frac{\omega_n}{\omega_j \omega_{n-j}} \geq \left(\min_{1\leq j\leq L} \frac{\omega_n}{\omega_{n-j}} \right) \sum_{j=1}^{L} \frac{1}{\omega_j}.$$

Taking lim sup when $n \to \infty$, and using (5.6), we get that $\sum 1/\omega_j$ converges.

The following statement shows that in the subcritical case, the hypotheses of Theorem 1.5 imply that the radius of convergence of the series for α is equal to one.

Proposition 5.5. *If $\lim k_n^{1/n} = 1$ and α is of subcritical type, then α does not continue analytically to any disc $R\mathbb{D}$, where $R > 1$.*

Proof. Since $k(t)$ has nonnegative Taylor coefficients, we have $|k(t)| \leq k(1)$ for all $t \in \mathbb{D}$. Using that $k = 1/\alpha$, it follows that in the subcritical case, $|\alpha(t)| \geq \alpha(1) > 0$ for any $t \in \mathbb{D}$. So, α cannot continue analytically to any disc $R\mathbb{D}$, where $R > 1$, because in this case, the radius of convergence of the Taylor series for k would be greater than 1. □

6. Finite Defect

It is well-known that in the classical Sz.-Nagy-Foias model, the case of a finite rank (or Hilbert-Schmidt) defect operator is an important one, where much more tools and results are available. In this section, we derive some consequences of our model theorems for the case when an operator $T \in \mathcal{C}_\alpha^w$ is α-modelable and the defect operator $D = (\alpha(T^*, T))^{1/2}$ is of finite rank.

We will assume that the reproducing kernel Hilbert space $\mathcal{R}_k$ is a Banach algebra with respect to the multiplication of power series. By [60, Proposition 32], it suffices to assume that

$$\sup_n \sum_{j=0}^{n} \frac{k_j^2 k_{n-j}^2}{k_n^2} < \infty;$$

compare with the condition (4.1). Put

$$m_n = \inf_j \frac{k_j}{k_{n+j}}, \quad r_1 = \lim_{n\to\infty} m_n^{1/n}.$$

This limit exists, see [60, Proposition 12].

We will assume that

$$r_1 = \lim_{n\to\infty} k_n^{1/n} = 1. \tag{6.1}$$

Both equalities hold, in particular, if $\lim k_{n+1}/k_n = 1$. The same is true if, for instance, the last limit does not exist, but $0 < \sigma < k_n < C < \infty$ for all n and there is some $m \geq 2$ such that $\lim_n k_{n+m}/k_n = 1$. We also are assuming here that the isometric part S is not present in the model of T. Hence, T is unitarily equivalent to the restriction of the backward shift $B_k \otimes I_{\mathfrak{D}}$ on the space $\mathcal{H}_k \otimes \mathfrak{D}$ to an invariant subspace $\mathcal{L}$. More generally, this applies to similarity instead of the unitary equivalence (we bear in mind models of linear operators up to similarity, which are established in [1]).

Here we prove the following result.

Theorem 6.1. *Suppose that T is similar to a part of $B_k \otimes I_{\mathfrak{D}}$, acting on the space $\mathcal{H}_k \otimes \mathfrak{D}$, where $\mathcal{R}_k$ is a Banach algebra and $\mathfrak{D}$ is finite dimensional. If the spectrum $\sigma(T)$ does not cover the open disc $\mathbb{D}$, then $\sigma(T) \cap \mathbb{D}$ is contained in the zero set of a non-zero function in $\mathcal{R}_k$.*

Let us start with some preliminary remarks. Suppose that T is as in the above Theorem 6.1. That is, T is similar to $(B_k \otimes I_{\mathfrak{D}})|\mathcal{L}$, where $\mathcal{L} \subset \mathcal{H}_k \otimes \mathfrak{D}$ is an invariant subspace of $B_k \otimes I_{\mathfrak{D}}$. By fixing a basis in $\mathfrak{D}$, we may assume that $\mathfrak{D} = \mathbb{C}^d$, where $d = \dim \mathfrak{D}$. We will identify the space $\mathcal{H}_k \otimes \mathfrak{D}$ with $\mathcal{H}_k^d = \oplus_1^d \mathcal{H}_k$, whose elements are columns with entries in $\mathcal{H}_k$. The adjoint of B_k on the space $\mathcal{H}_k^d$ is the multiplication operator M_z on the space $\mathcal{R}_k^d$; this later space can be seen as a Banach module over the Banach algebra $\mathcal{R}_k$. Put

$$\mathcal{J} = \mathcal{L}^\perp \subset \mathcal{R}_k \otimes \mathfrak{D}.$$

Then $\mathcal{J}$ is M_z-invariant, and T^* is similar to the quotient operator

$$\mathcal{M}_z : \mathcal{R}_k^d/\mathcal{J} \to \mathcal{R}_k^d/\mathcal{J}, \qquad \mathcal{M}_z[f] = [zf].$$

Here $[f] \in \mathcal{R}_k^d/\mathcal{J}$ denotes the coset of a function f in $\mathcal{R}_k^d$. We adapt some ideas from Richter's work [53], which treated the case $d = 1$.

Definition 6.2. (see [53]). Let $\mathcal{J}$ be a subspace of $\mathcal{R}_k^d$, invariant under M_z.

(1) Given a point $\lambda \in \overline{\mathbb{D}}$, the space

$$\mathcal{F}(\lambda) = \mathcal{F}_{\mathcal{J}}(\lambda) := \{g(\lambda) \,:\, g \in \mathcal{J}\}$$

will be referred to as *the fiber of* $\mathcal{J}$ *over* λ. Note that $\mathcal{F}(\lambda)$ is a subspace of $\mathbb{C}^d$.

(2) By *the spectrum of* $\mathcal{J}$ we understand the set

$$\sigma(\mathcal{J}) := \{\lambda \in \overline{\mathbb{D}} : \quad \mathcal{F}_{\mathcal{J}}(\lambda) \neq \mathbb{C}^d\}.$$

It will be shown that Theorem 6.1 is an easy consequence of the following result.

Theorem 6.3. *Given any subspace* $\mathcal{J}$ *of* $\mathcal{R}_k^d$*, invariant under* M_z*, one has*

$$\sigma(\mathcal{J}) \cap \mathbb{D} = \sigma(\mathcal{M}_z) \cap \mathbb{D}.$$

In the proof, we will use the following lemma

Lemma 6.4. *If* $g \in \mathcal{R}_k$*,* $\lambda \in \mathbb{D}$ *and* $g(\lambda) = 0$*, then* $(z-\lambda)^{-1}g(z) \in \mathcal{R}_k$*.*

Proof. Assume that $|\lambda| < r_1 = 1$. By [60, Proposition 13], the operator $M_z - \lambda$ is bounded from below. Notice that $(z-\lambda)^{-1}g(z) \in \mathcal{R}_k$ if and only if g belongs to the closed subspace $\operatorname{Ran}(M_z - \lambda)$. This happens if and only if g is orthogonal to $\ker(M_z^* - \bar{\lambda})$. This kernel is one-dimensional and is generated by the antilinear evaluation functional $g \mapsto \overline{g(\lambda)}$, which implies our assertion. □

Proof of Theorem 6.3. First we observe that $\eta \cdot \mathcal{J} \subset \mathcal{J}$ for any η in the algebra $\mathcal{R}_k$, which is easy to get, approximating η by polynomials.

Assume first that $\lambda \in \mathbb{D}$, but $\lambda \notin \sigma(\mathcal{J})$. This means that $\mathcal{F}(\lambda) = \mathbb{C}^d$. Let us prove that $\lambda \notin \sigma(\mathcal{M}_z)$ (this will give the inclusion $\sigma(\mathcal{M}_z) \cap \mathbb{D} \subset \sigma(\mathcal{J}) \cap \mathbb{D}$). That is, we will see that $\mathcal{M}_z - \lambda$ is invertible in $\mathcal{R}_k^d/\mathcal{J}$.

Claim. *If* $h \in \mathcal{R}_k^d$ *and* $(z-\lambda)h \in \mathcal{J}$*, then* $h \in \mathcal{J}$*.*

Indeed, assume that h satisfies these assumptions. Since $\mathcal{F}(\lambda) = \mathbb{C}^d$, there exist functions $\varphi_1, \dots, \varphi_d$ in $\mathcal{J}$ such that $\varphi_j(\lambda) = e_j$ (where $\{e_j\}$ is the canonical base in $\mathbb{C}^d$). Consider the $d \times d$ matrix-valued function

$$\Phi := (\varphi_1 | \cdots | \varphi_d) \in \mathcal{R}_k^{d\times d}, \qquad \text{and set} \qquad \varphi := \det \Phi \in \mathcal{R}_k.$$

Note that $\Phi\gamma = \gamma_1\varphi_1 + \cdots + \gamma_d\varphi_d \in \mathcal{J}$ for every $\gamma \in \mathcal{R}_k^d$. Hence, $\varphi h = \Phi\Phi^{\mathrm{ad}} h \in \mathcal{J}$. We also observe that $(\varphi - 1)h$ belongs to $\mathcal{J}$. Indeed, since $\varphi(\lambda) = 1$, we have

$$(\varphi - 1)h = \frac{\varphi(z) - \varphi(\lambda)}{z - \lambda}\,(z - \lambda)h \in \mathcal{J},$$

because $(\varphi(z) - \varphi(\lambda))/(z - \lambda) \in \mathcal{R}_k$ by Lemma 6.4 and $(z - \lambda)h \in \mathcal{J}$. Therefore,

$$h = \varphi h - (\varphi - 1)h \in \mathcal{J},$$

which proves our claim.

To check that $\mathcal{M}_z - \lambda$ is invertible in $\mathcal{R}_k^d/\mathcal{J}$, take an arbitrary element f in $\mathcal{R}_k^d$, and let us study the solutions of the equation

$$(\mathcal{M}_z - \lambda)[h] = [f]$$

with respect to an unknown coclass $[h] \in \mathcal{R}_k^d/\mathcal{J}$. By the above Claim, there is no more than one solution. On the other hand, if we set

$$h(z) = (z - \lambda)^{-1}\big(f(z) - \Phi(z)f(\lambda)\big),$$

then by Lemma 6.4, $h \in \mathcal{R}_k^d$, so that $[h]$ is a solution of the above equation. Note that $\Phi(\lambda) = I$. It follows that the above formula defines a bounded map $[f] \mapsto [h]$, which proves that the inverse to $\mathcal{M}_z - \lambda$ exists and is bounded on $\mathcal{R}_k^d/\mathcal{J}$. This completes the proof of the inclusion $\sigma(\mathcal{M}_z) \cap \mathbb{D} \subset \sigma(\mathcal{J}) \cap \mathbb{D}$.

To prove the opposite inclusion, take a point λ in $\sigma(\mathcal{J}) \cap \mathbb{D}$ and let us see that λ belongs to $\sigma(\mathcal{M}_z)$. In other words, we wish to prove that $\mathcal{M}_z - \lambda$ is not invertible in $\mathcal{R}_k^d/\mathcal{J}$.

Since $\lambda \in \sigma(\mathcal{J})$, the fiber $\mathcal{F}(\lambda)$ is not all $\mathbb{C}^d$. Hence, there exists a nonzero antilinear functional Ψ on $\mathbb{C}^d$ such that $\Psi|\mathcal{F}(\lambda) \equiv 0$. It defines an antilinear functional on $\mathcal{R}_k^d$, given by

$$\widehat{\Psi}(f) = \Psi(f(\lambda)).$$

Note that $\widehat{\Psi} \neq 0$, but $\widehat{\Psi}|\mathcal{J} \equiv 0$. Hence, we obtain the antilinear functional $\widetilde{\Psi}$ on the quotient $\mathcal{R}_k^d/\mathcal{J}$, given by

$$\widetilde{\Psi} : \mathcal{R}_k^d/\mathcal{J} \to \mathbb{C}, \qquad \widetilde{\Psi}([f]) := \widehat{\Psi}(f).$$

For every $f \in \mathcal{R}_k^d$ we have

$$\Big\langle (\mathcal{M}_z - \lambda)^*\widetilde{\Psi}, [f] \Big\rangle = \widetilde{\Psi}((\mathcal{M}_z - \lambda)[f]) = \widehat{\Psi}((z - \lambda)f) = 0,$$

because $(z - \lambda)f(z)$ vanishes for $z = \lambda$. Hence, $(\mathcal{M}_z - \lambda)^*\widetilde{\Psi} = 0$. Since $\widetilde{\Psi} \neq 0$, we get that $(\mathcal{M}_z - \lambda)$ is not invertible in $\mathcal{R}_k^d/\mathcal{J}$, as we wanted to prove. □

We remark that the above Claim is very close to Corollary 3.8 in the Richter's paper [53], which can be stated as follows: for any (reasonable) Banach algebra of analytic functions on $\mathbb{D}$, continuable to $\overline{\mathbb{D}}$, any its invariant subspace (which is the same as an ideal) has index one. Richter only studies scalar-valued algebras $\mathcal{R}_k$, and the above Claim can be seen as an extension of Richter's result to the case of a vector-valued space $\mathcal{R}_k \otimes \mathcal{E}$, where $\dim \mathcal{E} < \infty$. The indices of invariant subspaces of vector-valued spaces of analytic functions have been studied by Carlsson in [19]. In [8, Section 10], one can find a review of the index phenomena, related to invariant subspaces of Bergman spaces.

Proof of Theorem 6.1. We conserve the notation of the above proof. Fix any point λ in $\mathbb{D} \setminus \sigma(T)$. Then, as above, there exist functions $\varphi_j \in \mathcal{J}$ such that $\varphi_j(\lambda) = e_j$, $j = 1, \dots d$. Define the $d \times d$ matrix function $\Phi(z)$ as above and put $\varphi(z) = \det \Phi(z)$, then φ belongs to $\mathcal{R}_k$. Observe that $\varphi \not\equiv 0$. Notice that the fiber $\mathcal{F}_{\mathcal{J}}(z)$ equals to $\mathbb{C}^d$ whenever $\Phi(z)$ is invertible, that is, whenever $\varphi(z) \neq 0$. Hence $\sigma(\mathcal{J})$ is contained in the zero set of φ. By Theorem 6.3, $\sigma(T) \cap \mathbb{D} = \sigma(\mathcal{J}) \cap \mathbb{D}$, and this implies the statement of the theorem. □

For the proof of Theorem 1.13 we need the following lemma.

Lemma 6.5. *Assume the hypothesis of Theorem 6.1 and also* (1.8). *Then, there exists a positive number s such that the functions in $\mathcal{R}_k$ are Hölder continuous of order s on $\overline{\mathbb{D}}$.*

Proof. It is easy to see that (1.8) implies that for a sufficiently small $s \in (0,1)$,

$$\sup_{0<t<1} t^{-s} \left(\sum_{t^{s-1}}^{\infty} k_n \right)^{1/2} < \infty. \tag{6.2}$$

Fix such s, and let f be a function in $\mathcal{R}_k$. To prove that f is Hölder continuous of order s on $\overline{\mathbb{D}}$, it is enough to show that f is Hölder continuous of order s in $\mathbb{T}$ (see [39]). By a rotation argument, we just need to prove that

$$\sup_{\substack{\theta \neq 0 \\ \theta \in [-\pi,\pi]}} |\theta|^{-s} |f(1) - f(e^{i\theta})| < \infty.$$

By the Cauchy-Schwarz inequality, it is enough to prove that

$$\sup_{\substack{\theta \neq 0 \\ \theta \in [-\pi,\pi]}} |\theta|^{-s} \left(\sum_{n=0}^{\infty} k_n |1 - e^{in\theta}|^2 \right)^{1/2} < \infty. \tag{6.3}$$

Note that $|1 - e^{in\theta}|^2 \leq n^2\theta^2$, hence $|1 - e^{in\theta}|^2 \leq |\theta|^{2s}$ if $n \leq |\theta|^{s-1}$. Therefore

$$|\theta|^{-s} \left(\sum_{n=0}^{\infty} k_n |1 - e^{in\theta}|^2 \right)^{1/2} \leq \left(\sum_{n=0}^{|\theta|^{s-1}} k_n \right)^{1/2} + 2|\theta|^{-s} \left(\sum_{|\theta|^{s-1}}^{\infty} k_n \right)^{1/2},$$

which is uniformly bounded because $\sum k_n$ converges and (6.2) holds. Hence (6.3) follows and the statement is proved. □

Proof of Theorem 1.13. By Theorem 6.1, we have that $\sigma(T) \cap \mathbb{D}$ is contained in the zero set of a non-zero function f in $\mathcal{R}_k$. By Lemma 6.5, f is Hölder continuous of order s for some

$s > 0$. Note that E is a set of uniqueness for f. Hence the statement follows using [18, Theorem 1]. □

We conjecture that the statements of Theorems 1.13 and 6.1 are valid for the whole spectrum $\sigma(T)$. Some of our arguments do not apply and should be changed in order to prove it.

7. Ergodic properties of a-contractions

In this section we focus only on functions α of the form

$$\alpha(t) = (1-t)^a \tag{7.1}$$

for some $a > 0$. Recall that if $(1-t)^a(T^*, T) \geq 0$ for some $T \in L(H)$, then we say that T is an a-contraction. Now

$$k(t) = (1-t)^{-a} = \sum_{n=0}^{\infty} k^a(n) t^n \qquad (|t| < 1). \tag{7.2}$$

Observe that $k^a(n) > 0$ for all $n \geq 0$. It follows that $\alpha(t) = (1-t)^a$ satisfies Hypotheses 1.1. Moreover, here we are in the critical case.

If $0 < a \leq 1$, then $\alpha_n \leq 0$ for $n > 0$, so that in this case we can apply Theorem 1.4 to obtain a model for a-contractions. This was singled out as an important particular case in [21]. With the help of this model, we will derive here some ergodic properties of a-contractions for these values of a. We refer to the book [34] for a treatment of ergodic theory in the context of the theory of linear operators.

Notice that

$$k^a(n) = (-1)^n \binom{a}{n} = \begin{cases} \dfrac{a(a+1)\cdots(a+n-1)}{n!} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

These numbers are called *Cesàro numbers*. See [64, Volume I, p. 77]. We will need the following well-known facts about their asymptotic behavior.

Proposition 7.1. *If $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, then*

$$k^a(n) = \frac{\Gamma(n+a)}{\Gamma(a)\Gamma(n+1)} = \binom{n+a-1}{a-1} \qquad \forall n \geq 0,$$

where Γ is Euler's Gamma function. Therefore

$$k^a(n) = \frac{n^{a-1}}{\Gamma(a)}(1 + O(1/n)) \qquad \text{as } n \to \infty. \tag{7.3}$$

Moreover, if $0 < a \leq 1$, then

$$\frac{(n+1)^{a-1}}{\Gamma(a)} \leq k^a(n) \leq \frac{n^{a-1}}{\Gamma(a)} \qquad \forall n \geq 1.$$

Proof. See [64, Volume I, p. 77, Equation (1.18)] and [63, Equation (1)]. The last inequality follows from the Gautschi inequality (see [33, Equation (7)]). □

Any contraction T on H is an a-contraction for any $a \in (0,1)$. Indeed, in this case, $\alpha_n \le 0$ for all $n \ge 1$. Hence for any $x \in H$, $\sum_{n\ge 1} \alpha_n \|T^n x\|^2 \ge (\sum_{n\ge 1} \alpha_n)\|x\|^2 = -\|x\|^2$, which implies that $\alpha(T^*, T) \ge 0$.

Recall that in order to emphasize the dependence on the exponent a in (7.1), we denote the space $\mathcal{H}_k$ by $\mathcal{H}_a$ (see Notation 2.10), and use the notation B_a and F_a (these two operators act on $\mathcal{H}_a$). In the same way, when $T \in \mathcal{C}^w_\alpha$ (that is, when T is an a-contraction), we will write $T \in \mathcal{C}^w_a$, and instead of Adm^w_α we will use the notation Adm^w_a.

The *weighted space of Bergman-Dirichlet type* $\mathcal{D}_a$, where a is a real parameter, consists of all the analytic functions f in $\mathbb{D}$ with finite norm

$$\|f\|_{\mathcal{D}_s} := \left(\sum_{n=0}^{\infty} (n+1)^a |f_n|^2\right)^{1/2}.$$

In fact, it is a Bergman-type space if $a < 0$, and is a Dirichlet-type space if $a > 0$. For $a = 0$, we get the Hardy space.

Theorem 1.4 yields that for $0 < a \le 1$, any a-contraction is modelable as a part of an operator $(B_a \otimes I_{\mathcal{E}}) \oplus S$, where S is an isometry. Recall that the adjoint to the operator $B_a = B_k$ on $\mathcal{H}_k$ is the operator $M_z g(z) = zg(z)$ on $\mathcal{R}_k$, which is a space with the weighted norm

$$\|g\|^2_{\mathcal{R}_k} = \sum_{n=0}^{\infty} k_n^{-1} |g_n|^2.$$

Hence the characterization of invariant subspaces of M_z on $\mathcal{R}_k$ becomes important. In many cases, this question is related to the description of what is called inner functions in $\mathcal{R}_k$. Since $k_n \asymp (n+1)^{a-1}$, the norm in $\mathcal{R}_k$ is equivalent to the norm in $\mathcal{D}_{-a+1}$, which is a Dirichlet-type space. One can find results in this direction in the thesis of Schillo [58], in [51] by Pau and Peláez, and in the papers [59] and [12] by Seco and coauthors; see also references therein.

As a consequence of Theorem 2.12 we obtain the following result.

Theorem 7.2. *Let a and s be positive numbers. Then the following is true.*

(i) $B_s \in \mathrm{Adm}^w_a$.
(ii) *B_s is an a-contraction if and only if $a \le s$.*

Proof. Using the notation of Theorem 2.12 we have $\varkappa(t) = (1-t)^{-s}$ and $\alpha(t) = (1-t)^a$. Hence $\beta(t) = p(t) \pm \alpha(t)$, where $p(t)$ is a polynomial, say $p(t) = p_0 + p_1 t + \cdots + p_N t^N$, with all the coefficients p_j positive. Then

$$\gamma(t) = (p(t) \pm \alpha(t))\varkappa(t) = p(t)\varkappa(t) \pm (1-t)^{a-s} =: \widetilde{\gamma}(t) + \widehat{\gamma}(t).$$

To prove (i), it is enough to show that

$$\sup_{m\geq 0} \frac{|\widetilde{\gamma}_m|}{\varkappa_m} < \infty, \qquad \text{and} \qquad \sup_{m\geq 0} \frac{|\widehat{\gamma}_m|}{\varkappa_m} < \infty.$$

On one hand, for $m \geq N$ we have

$$|\widetilde{\gamma}_m| \leq (p_0 + \cdots + p_N) \cdot \max\{\varkappa_m, \dots, \varkappa_{m-N}\} \lesssim \varkappa_m.$$

On the other hand,

$$|\widehat{\gamma}_m| \asymp \frac{1}{m^{a-s+1}} \leq \frac{1}{m^{-s+1}} \asymp \varkappa_m.$$

Therefore (i) follows. Observe that (ii) is an immediate consequence of Theorem 2.12 (ii), since $\alpha(t)\varkappa(t) = (1-t)^{a-s}$. □

Note that it is immediate that

$$\|B_s^m\|^2 = \sup_{n\geq 0} \frac{\varkappa_n}{\varkappa_{n+m}} = \begin{cases} 1 & \text{if } 1 \leq s \\ 1/\varkappa_m & \text{if } 0 < s < 1, \end{cases} \tag{7.4}$$

and

$$\|F_s^m\|^2 = \sup_{n\geq 0} \frac{\varkappa_{n+m}}{\varkappa_n} = \begin{cases} \varkappa_m & \text{if } 1 \leq s \\ 1 & \text{if } 0 < s < 1, \end{cases} \tag{7.5}$$

for every $m \geq 0$. Therefore

$$\|B_s^m\|^2 \asymp (m+1)^{\max\{1-s,0\}} \qquad \text{and} \qquad \|F_s^m\|^2 \asymp (m+1)^{\max\{s-1,0\}}. \tag{7.6}$$

As an easy consequence we obtain the following example, which shows two relevant facts: 1) there are a-contractions that are not similar to contractions, and 2) the importance of considering the strong operator topology in the convergence of $\sum \alpha_n T^{*n}T^n$.

Example 7.3. Taking $a = s \in (0,1)$ in Theorem 7.2, we get that B_a is an a-contraction. It is not similar to a contraction, since it is not power bounded. Moreover, if $a = s \leq 1/2$, then

$$\sum_{n=0}^{\infty} |\alpha_n|\, \|B_a^n\|^2 \asymp \sum_{n=0}^{\infty} (n+1)^{-1-a}\,(n+1)^{1-a} = \sum_{n=0}^{\infty} (n+1)^{-2a} = \infty,$$

and therefore the series $\sum \alpha_n B_a^{*n} B_a^n$ does not converge in the uniform operator topology in $L(H)$. Note that, obviously, the model of the a-contraction B_a is itself.

Let us study now some ergodic properties of a-contractions.

Definition 7.4. Let $a \geq 0$. For any bounded linear operator T on a Banach space X, we call the operators $\{M_T^a(n)\}_{n\geq 0}$ given by

$$M_T^a(n) := \frac{1}{k^{a+1}(n)} \sum_{j=0}^{n} k^a(n-j) T^j,$$

the *Cesàro means of order a of T*. When this family of operators is uniformly bounded, that is,

$$\sup_{n\geq 0} \|M_T^a(n)\| < \infty,$$

we say that T is *(C,a)-bounded*.

Remarks 7.5.

(i) Note that $\sum_{j=0}^{n} k^a(j) = k^{a+1}(n)$ for any $a \geq 0$. Also, if $a \geq 0$, then $k^a(j) \geq 0$ for every $j \geq 0$.

(ii) If $a = 0$, then $M_T^0(n) = T^n$. Hence $(C,0)$-boundedness is just power boundedness.

(iii) If $a = 1$, then $M_T^1(n) = (n+1)^{-1}\sum_{j=0}^{n} T^j$. Hence $(C,1)$-boundedness is just Cesàro boundedness.

(iv) It is well-known that if $0 \leq a < b$, then (C,a)-boundedness implies (C,b)-boundedness. The converse is not true in general. For example, the Assani matrix

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

is $(C,1)$-bounded, but since

$$T^n = \begin{pmatrix} (-1)^n & (-1)^{n+1}2n \\ 0 & (-1)^n \end{pmatrix}$$

it is not power bounded (see [30, Section 4.7]).

Definition 7.6. If the sequence of operators $\{M_T^a(n)\}_{n\geq 0}$ given in Definition 7.4 converges in the strong operator topology, we say that T is *(C,a)-mean ergodic*.

If T is $(C,1)$-mean ergodic, it is conventional just to say that T is *mean ergodic*.

There is a well established literature on (C,a)-bounded operators, which explores quite a number of properties and their interplays. Properties, characterization through functional calculus and ergodic results for (C,a)-bounded operators can be found in [4, 9, 25, 27, 28, 30, 41] and references therein. The connection of these operators and ergodicity dates back to the fourties of last century, see [24] and [40]. In the latter paper, E. Hille studies (C,a)-mean ergodicity in terms of Abel convergence (that is, via the resolvent operator). As application, the well known mean ergodic von Neumann's theorem for unitary groups on Hilbert spaces is extended to (C,a)-mean ergodicity for every $a > 0$ [40, p. 255]. Also, the (C,a)-ergodicity on $L_1(0,1)$ of fractional (Riemann-Liouville) integrals is elucidated in [40, Theorem 11]. In particular, if V is the Volterra operator then $T_V := I - V$, as operator on $L_1(0,1)$, is not power-bounded, and it is (C,a)-mean ergodic if and only if $a > 1/2$ [40, Theorem 11]. This result can be extended to T_V acting on $L_p(0,1)$, $1 < p < \infty$, using estimates given in [44], see [3, Section 10].

In [42], Luo and Hou introduced a new notion of boundedness: a bounded linear operator T on a Banach space X is said to be *absolutely Cesàro bounded* if

$$\sup_{n\geq 0}\frac{1}{n+1}\sum_{j=0}^{n}\left\|T^j x\right\| \lesssim \|x\|$$

for every $x\in X$. In [14], the authors study the ergodic behaviour for this class of operators. The above definition has been extended recently by Abadias and Bonilla in [2]: T is said to be *absolutely* (C,a)*-Cesàro bounded* for some $a>0$ if

$$\sup_{n\geq 0}\frac{1}{k^{a+1}(n)}\sum_{j=0}^{n}k^a(n-j)\left\|T^j x\right\| \lesssim \|x\|$$

for every $x\in X$. Note that for $a=1$ the definition of Luo and Hou is recovered.

Remark 7.7. It is well-known that the following implications hold:

$$\begin{aligned}&\text{Power bounded} \;\Rightarrow\; \text{Absolutely } (C,a)\text{-bounded}\\ &\Rightarrow\; (C,a)\text{-bounded} \;\Rightarrow\; \|T^n\| = O(n^a).\end{aligned}$$

The first two implications are straightforward. For the sake of completeness, we give a proof of the last one. Suppose T is (C,a)-bounded for some $a\geq 0$. We denote by $[a]$ the integer part of a. Then, for $n>[a]$, we have

$$\begin{aligned}\|T^n\| &= \left\|\sum_{j=0}^{n}k^{-a}(j)\sum_{m=0}^{n-j}k^a(n-j-m)T^m\right\|\\ &\lesssim \sum_{j=0}^{n}|k^{-a}(j)|k^{a+1}(n-j)\\ &= \sum_{j=0}^{[a]}(-1)^j k^{-a}(j)k^{a+1}(n-j) + \sum_{j=[a]+1}^{n}(-1)^{[a]+1}k^{-a}(j)k^{a+1}(n-j)\\ &= \sum_{j=0}^{[a]}\left((-1)^j+(-1)^{[a]}\right)k^{-a}(j)k^{a+1}(n-j) + (-1)^{[a]+1}\sum_{j=0}^{n}k^{-a}(j)k^{a+1}(n-j)\\ &\lesssim \sum_{j=0}^{[a]}|k^{-a}(j)|k^{a+1}(n-j) + k^1(n) \lesssim k^{a+1}(n) \asymp (n+1)^a\,.\end{aligned}$$

The following extension of the above definitions will be important for us.

Definition 7.8. Let $a>0$ and $p\geq 1$. We say that a bounded linear operator T on a Banach space X is (C,a,p)*-bounded* if

$$\sup_{n\geq 0}\frac{1}{k^{a+1}(n)}\sum_{j=0}^{n}k^a(n-j)\|T^j x\|^p \lesssim \|x\|^p,$$

for all $x \in X$.

Note that for $p = 1$ this definition is just the absolute (C, a)-boundedness. The case $a = 1$ has been recently considered in [23]. We will use the term *quadratically* (C, a)*-bounded* instead of $(C, a, 2)$-bounded.

Using the asymptotics $k^a(n) \asymp (n+1)^{a-1}$ given in (7.3), it is easy to see that T is (C, a, p)-bounded if and only if

$$\sup_{n\geq 0} \frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \lesssim \|x\|^p \qquad (\forall x \in X). \tag{7.7}$$

The following observation will be essential for the proof of Theorem 1.14.

Lemma 7.9. *The following holds.*

(i) *If T is (C, a, p)-bounded, then any part of T is also (C, a, p)-bounded.*
(ii) *If T_1 and T_2 are (C, a, p)-bounded, then any direct sum $T_1 \dot{+} T_2$ is also (C, a, p)-bounded.*
(iii) *Let T be a bounded linear operator on a Hilbert space. If T is quadratically (C, a)-bounded, then $T \otimes I_{\mathcal{E}}$ is also quadratically (C, a)-bounded, where $I_{\mathcal{E}}$ is the identity operator on some Hilbert space $\mathcal{E}$.*

Proof. (i) and (ii) are immediate. For (iii) note that if $d = \dim \mathcal{E} \leq \infty$, then the orthogonal sum of d copies of T is clearly quadratically (C, a)-bounded (by the Pythagoras Theorem). $\square$

The following result is very useful. Its proof is simple, and we omit it.

Lemma 7.10. *Let $0 \leq a < b$. Then (C, a, p)-boundedness implies (C, b, p)-boundedness.*

This lemma shows an inclusion of classes of operators. By [2, Corollaries 2.2 and 2.3], if T is $(C, a, 1)$-bounded then $\|T^n\| = o(n^a)$ for $0 < a \leq 1$ and $\|T^n\| = O(n)$ for $a > 1$. The following result explains why the case $a = 1$ is special.

Theorem 7.11. *If $a > 1$ and $p \geq 1$, then (C, a, p)-boundedness is equivalent to $(C, 1, p)$-boundedness.*

Proof. Fix $a > 1$ and $p \geq 1$. By the above Lemma, we only need to prove that any (C, a, p)-bounded operator T is $(C, 1, p)$-bounded. Let T is (C, a, p)-bounded. Then

$$\frac{1}{k^{a+1}(2n)} \sum_{j=0}^{2n} k^a(2n-j) \left\|T^j x\right\|^p \lesssim \|x\|^p, \tag{7.8}$$

for every $n \geq 0$, and every $x \in X$. Since $a > 1$, $k^a(m)$ is an increasing function of m. In particular, $k^a(n) \leq k^a(2n-j)$ for $j = 0, \ldots, n$. Hence

$$k^a(n) \sum_{j=0}^{n} \left\|T^j x\right\|^p \leq \sum_{j=0}^{2n} k^a(2n-j) \left\|T^j x\right\|^p, \tag{7.9}$$

By (7.9) and (7.8),

$$\sum_{j=0}^{n} \left\|T^j x\right\|^p \lesssim \frac{k^{a+1}(2n)}{k^a(n)} \|x\|^p \lesssim (n+1)\|x\|^p,$$

which means that T is $(C, 1, p)$-bounded. □

Theorem 7.12. *Let $a > 0$ and $1 \leq q < p$. If T is (C, a, p)-bounded, then it is also (C, b, q)-bounded for each $b > qa/p$. In particular, (C, a, p)-boundedness implies (C, a, q)-boundedness.*

Proof. Let us first recall that if $r > -1$, then

$$\sum_{j=1}^{m} j^r \lesssim m^{r+1} \qquad (\forall m \geq 1). \tag{7.10}$$

Let T be (C, a, p)-bounded and let $b > qa/p$. Suppose first that $b \neq 1$, and put

$$s := \frac{p}{p-q}, \qquad s' := \frac{p}{q}, \qquad \gamma := \frac{q(a-1)}{p(b-1)}.$$

Note that s and s' are positive and satisfy $1/s + 1/s' = 1$. Since

$$(b-1)(1-\gamma)s = \frac{pb - qa}{p-q} - 1 > -1 \qquad \text{and} \qquad (b-1)\gamma s' = a - 1,$$

using Hölder's inequality and (7.10) it follows that

$$\begin{aligned}
&\frac{1}{(n+1)^b} \sum_{j=0}^{n} (n+1-j)^{b-1} \left\|T^j x\right\|^q \\
&\leq \frac{1}{(n+1)^b} \left(\sum_{j=0}^{n} (n+1-j)^{(b-1)(1-\gamma)s} \right)^{1/s} \left(\sum_{j=0}^{n} (n+1-j)^{(b-1)\gamma s'} \left\|T^j x\right\|^{qs'} \right)^{1/s'} \\
&\lesssim (n+1)^{-qa/p} \left(\sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \right)^{q/p} \\
&= \left(\frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|T^j x\right\|^p \right)^{q/p}
\end{aligned}$$

for every $x \in X$ and every non-negative integer n. Hence the statement follows using (7.7).

Now suppose that $b = 1$. Take any $b' \in (qa/p, 1)$. We have already proved that T is (C, b', p)-bounded. Then, by Lemma 7.10, it follows that T is $(C, 1, p)$-bounded. This completes the proof. □

Lemma 7.13. *Let $a > 0$ and $p \geq 1$. Then every isometry S is (C, a, p)-bounded.*

Proof. This is immediate, since indeed

$$\frac{1}{k^{a+1}(n)}\sum_{j=0}^{n}k^a(n-j)\|S^jx\|^p=\frac{1}{k^{a+1}(n)}\left(\sum_{j=0}^{n}k^a(n-j)\right)\|x\|^p=\|x\|^p \tag{7.11}$$

for every $x\in X$. □

Lemma 7.14. *Let $0<s<1$ and let $a>0$. Then B_s is quadratically (C,a)-bounded if and only if $1-s<a$. Moreover, for $1-s<a$ we have*

$$\lim_{n\to\infty}\frac{1}{k^{a+1}(n)}\sum_{j=0}^{n}k^a(n-j)\|B_s^jx\|^2=0 \qquad (\forall x\in\mathcal{H}_s). \tag{7.12}$$

Proof. Recall the notation $e_n=t^n\in\mathcal{H}_k=\mathcal{H}_s$, where $k(t)=(1-t)^{-s}$. Suppose that $a=1-s$. Then

$$\begin{aligned}\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^je_n\right\|^2&\gtrsim\frac{1}{(n+1)^{1-s}}\sum_{j=0}^{n}(n+1-j)^{-s}(n+1-j)^{s-1}\\&=\frac{1}{(n+1)^{1-s}}\sum_{j=1}^{n+1}j^{-1}\gtrsim\log(n+2)\left\|e_n\right\|^2\end{aligned} \tag{7.13}$$

for every n. Therefore B_s is not quadratically $(C,1-s)$-bounded, and by Lemma 7.10 we obtain that B_s is not quadratically (C,a)-bounded for $a<1-s$.

Let us assume now that $1-s<a\le 1$ and fix $x\in\mathcal{H}_s$. Write x in the form $x=\sum x_me_m$, where $x_m\in\mathbb{C}$. Then

$$\left\|B_s^jx\right\|^2=\sum_{m=j}^{\infty}k^s(m-j)|x_m|^2\lesssim\sum_{m=j}^{\infty}(m+1-j)^{s-1}|x_m|^2,$$

for every $j\ge 0$. Hence

$$\begin{aligned}&\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^jx\right\|^2\\&\qquad\lesssim\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\sum_{m=j}^{\infty}(m+1-j)^{s-1}|x_m|^2\\&\qquad=\frac{1}{(n+1)^a}\sum_{m=0}^{n}|x_m|^2\sum_{j=0}^{m}(n+1-j)^{a-1}(m+1-j)^{s-1}\\&\qquad+\frac{1}{(n+1)^a}\sum_{m=n+1}^{2n}|x_m|^2\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\\&\qquad+\frac{1}{(n+1)^a}\sum_{m=2n+1}^{\infty}|x_m|^2\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\\&\qquad=:(I)+(II)+(III).\end{aligned}$$

In (I), note that since $1-s<a\le 1$, and $m\le n$, we have

$$\sum_{j=0}^{m}(n+1-j)^{a-1}(m+1-j)^{s-1}\le\sum_{j=0}^{m+1}(m+1-j)^{a+s-2}\lesssim(m+1)^{a+s-1}, \tag{7.14}$$

where in the last estimate we used (7.10). Therefore

$$\begin{aligned}(I)&\lesssim\frac{1}{(n+1)^a}\sum_{m=0}^{n}|x_m|^2(m+1)^{a+s-1}=\left\{\sum_{m=0}^{[\sqrt{n}]}+\sum_{m=[\sqrt{n}]+1}^{n}\right\}|x_m|^2\,\frac{(m+1)^{a+s-1}}{(n+1)^a}\\&\lesssim\frac{\|x\|^2}{\sqrt{n^a}}+\sum_{m=[\sqrt{n}]+1}^{n}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).\end{aligned}$$

In (II), using that $m>n$ and $s-1<0$, we have

$$\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\le\sum_{j=0}^{n}(n+1-j)^{a+s-2}\lesssim(n+1)^{a+s-1}.$$

Therefore

$$\begin{aligned}(II)&\lesssim\frac{1}{(n+1)^a}\sum_{m=n+1}^{2n}|x_m|^2(n+1)^{a+s-1}=(n+1)^{s-1}\sum_{m=n+1}^{2n}|x_m|^2\\&\lesssim\sum_{m=n+1}^{2n}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).\end{aligned}$$

Finally, in (III), since $m>2n$ we have that

$$\sum_{j=0}^{n}(n+1-j)^{a-1}(m+1-j)^{s-1}\lesssim(m+1)^{s-1}\sum_{j=0}^{n}(n+1-j)^{a-1}\lesssim(m+1)^{s-1}(n+1)^a.$$

Therefore

$$(III)\lesssim\sum_{m=2n+1}^{\infty}|x_m|^2(m+1)^{s-1}\longrightarrow 0\quad(\text{as } n\to\infty).$$

Hence (7.12) follows when $1-s<a\le 1$. Finally, suppose that $1<a$. Then

$$\frac{1}{(n+1)^a}\sum_{j=0}^{n}(n+1-j)^{a-1}\left\|B_s^jx\right\|^2\le\frac{1}{n+1}\sum_{j=0}^{n}\left\|B_s^jx\right\|^2\longrightarrow 0\quad(\text{as } n\to\infty),$$

since this is the case of $a=1$ in (7.12) (already proved). Note that (7.12) implies quadratical (C,a)-boundedness, so the proof is complete. □

This lemma allows us to prove the following more general result.

Theorem 7.15. *Let $0 < s < 1$ and $1 \le q \le 2$. Then B_s is (C, b, q)-bounded if and only if $b > q(1-s)/2$. Moreover, for $b > q(1-s)/2$ we have*

$$\lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\|B_s^j x\right\|^q = 0 \qquad (\forall x \in H). \tag{7.15}$$

Proof. Note that $q = 2$ is precisely Lemma 7.14. So we assume that $1 \le q < 2$. If $b = q(1-s)/2$, taking $x = e_n$, we get, as in (7.13), that

$$\frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \left\|B_s^j e_n\right\|^q \gtrsim \log(n+2) \left\|e_n\right\|^2$$

for every n. Therefore B_s is not $(C, q(1-s)/2, q)$-bounded, and by Lemma 7.10 we get that B_s is not (C, b, q)-bounded for $b < q(1-s)/2$.

Now suppose that $b > q(1-s)/2$. Then $b = qa/2$ for some $a > 1-s$. Using Hölder's inequality as in the proof of Theorem 7.12, we obtain

$$\frac{1}{(n+1)^b} \sum_{j=0}^{n} (n+1-j)^{b-1} \left\|B_s^j x\right\|^q \lesssim \left(\frac{1}{(n+1)^a} \sum_{j=0}^{n} (n+1-j)^{a-1} \left\|B_s^j x\right\|^2 \right)^{q/2} \xrightarrow[n\to\infty]{} 0,$$

by Lemma 7.14. Hence (7.15) follows. □

Proof of Theorem 1.14. Let $T \in \mathcal{C}_a^w$ with $0 < a < 1$ and let $b > 1-a$. By Theorem 1.4 and Theorem 1.12 (i), T is unitarily equivalent to a part of $(B_a \otimes I_{\mathfrak{D}}) \oplus S$. Hence, by Lemma 7.9 (i), it is enough to prove that $(B_a \otimes I_{\mathfrak{D}}) \oplus S$ is quadratically (C, b)-bounded. But this is immediate using Lemma 7.9 (ii) and (iii), and Lemmas 7.13 and 7.14. □

For the proof of Theorem 1.15 we need the following lemma, which is in the spirit of Lemma 7.9.

Lemma 7.16. *The following holds.*

(i) *If T satisfies (1.9), then any part of T also satisfies (1.9).*
(ii) *If T_1 and T_2 satisfy (1.9), then any direct sum $T_1 \dotplus T_2$ also satisfies (1.9).*
(iii) *Let T be a bounded linear operator on a Hilbert space. If T satisfies (1.9), then the operator $T \otimes I_{\mathcal{E}}$ also satisfies (1.9), where $I_{\mathcal{E}}$ is the identity operator on some Hilbert space $\mathcal{E}$.*

Proof. (i) and (ii) are immediate. For (iii) we use the same argument as in Lemma 7.9 (iii) and a simple application of Lebesgue's Dominated Convergence Theorem. □

Proof of Theorem 1.15. As in the proof of Theorem 1.14, we have that T is unitarily equivalent to

$$(B_a \otimes I_{\mathfrak{D}}) \oplus S \,|\, \mathcal{L},$$

where $\mathcal{L}$ is a subspace of $(\mathcal{H}_a \otimes \mathfrak{D}) \oplus \mathcal{W}$ invariant by $(B_a \otimes I_{\mathfrak{D}}) \oplus S$.

Let us prove the circle of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Suppose that (i) is true. That is, T is unitarily equivalent to

$$(B_a \otimes I_{\mathfrak{D}}) \mid \mathcal{L},$$

where $\mathcal{L}$ is a subspace of $\mathcal{H}_a \otimes \mathfrak{D}$ invariant by $B_a \otimes I_{\mathfrak{D}}$. Then (ii) follows using Lemmas 7.14 and 7.16.

Suppose now that

$$\liminf_{n\to\infty} \|T^n x\| > 0$$

for some $x \in H$. Then, obviously, $\|T^n x\| > \varepsilon > 0$ for every $n \geq 0$. Hence for this vector x (1.9) does not hold. Therefore we have proved that (ii) $\Rightarrow$ (iii).

Finally, suppose that the isometry S appears in the minimal model. Then for some vector $\ell = (\ell_1, \ell_2) \in \mathcal{L}$, its second component $\ell_2 \in \mathcal{W}$ is not 0. Therefore

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \big\| ((B_a \otimes I_{\mathfrak{D}}) \oplus S)^j \ell \big\|^2 \\ &= \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \big\| (B_a \otimes I_{\mathfrak{D}})^j \ell_1 \oplus S^j \ell_2 \big\|^2 \\ &= \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \big\| (B_a \otimes I_{\mathfrak{D}})^j \ell_1 \big\|^2 + \lim_{n\to\infty} \frac{1}{k^{b+1}(n)} \sum_{j=0}^{n} k^b(n-j) \big\| S^j \ell_2 \big\|^2. \end{aligned}$$

The second limit is $\|\ell_2\|^2 \neq 0$ because of (7.11). Hence we obtain that (iii) $\Rightarrow$ (i). ☐

Remark 7.17. In the same way, we get that if T is an a-contraction and $0 < a \leq 1$, then

$$\liminf_{n\to\infty} \|T^n x\| \leq \|x\|.$$

In particular, this lower limit is finite for any x.

Since $\mathcal{C}_1^w$ is just the set of all contractions on H, $T \in \mathcal{C}_1^w$ iff $T^* \in \mathcal{C}_1^w$. However, this is no longer true for $a \in (0,1)$.

Proposition 7.18. *If $a \in (0,1)$, then there is an operator $T \in \mathcal{C}_a^w$ such that $T^* \notin \mathcal{C}_a^w$.*

Proof. Note that B_a^* is a forward weighted shift such that $\|B_a^{*n} f_0\| \to \infty$ as n goes to ∞. So $B_a \in \mathcal{C}_a^w$, whereas its adjoint cannot belong to $\mathcal{C}_a^w$, because B_a^* is not quadratically (C,b)-bounded for any b (see Lemma 7.14). ☐

It is natural to pose the following question.

Question 7.19. For which functions α, satisfying Hypotheses 1.1, is it true that $T \in \mathcal{C}_\alpha^w$ implies $T^* \in \mathcal{C}_\alpha^w$?

It is so for $\alpha(t) = 1 - t$ and, more generally, for $\alpha(t) = 1 - t^n$, $n \geq 1$. The authors do not know other examples.

Remarks 7.20.

(i) If T is an operator in $\mathcal{C}_a^w$ with $0 < a < 1$, and $0 < q < 2$, then by Theorem 1.14 and Theorem 7.12, it follows that T is (C, b, q)-bounded for all $b > \frac{q(1-a)}{2}$.

(ii) An m-isometry T, which is not an isometry, cannot be (C, a, p)-bounded, because there are vectors x such that the norms $\|T^n x\|$ go to infinity. The possibility for these operators to have weaker ergodic properties, such as the Cesàro boundedness and weak ergodicity, have been studied in [13].

(iii) Let T be an operator in $\mathcal{C}_a^w$ with $0 < a < 1$. Using Theorem 1.14 (i) and Theorem 7.12 (with $p = 2$ and $q = 1$) we obtain that T is $(C, b, 1)$-bounded for every $b > (1 - a)/2$.

By [2, Corollary 3.1], we get that T is (C, b)-mean ergodic, that is, there exists

$$P_b x := \lim_{n\to\infty} M_T^b(n)x, \quad x \in H.$$

Therefore, by [3, Theorem 3.3], we have

$$H = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Ran}(I - T)}.$$

In fact,

$$\mathrm{Ker}(I - T) = \mathrm{Ran} P_b \ \text{ and } \ \overline{\mathrm{Ran}(I - T)} = \mathrm{Ker} P_b.$$

Also note that

$$M_T^b(n)x = x \text{ for } x \in \mathrm{Ker}(I - T), \text{ and } \lim_{n\to\infty} M_T^b(n)x = 0 \text{ for } x \in \overline{\mathrm{Ran}(I - T)}.$$

Let now $0 < \gamma < 1$, by [3, Proposition 4.8 and Remark 4.9], one can define a bounded operator $(I - T)^\gamma$ by means of a certain functional calculus, and

$$\mathrm{Ker}(I - T) = \mathrm{Ker}(I - T)^\gamma, \quad \overline{\mathrm{Ran}(I - T)} = \overline{\mathrm{Ran}(I - T)^\gamma},$$

with $\mathrm{Ran}(I - T) \subseteq \mathrm{Ran}(I - T)^\gamma$. Furthermore if $\gamma < 1 - b$, for $x \in \overline{\mathrm{Ran}(I - T)}$,

$$x \in \mathrm{Ran}(I - T)^\gamma \iff \sum_{n=1}^\infty \frac{1}{n^{1-\gamma}} T^n x \ \text{ converges},$$

see [3, Theorem 9.2].

(iv) By [2, Theorem 3.1], if T is an operator in $\mathcal{C}_a^w$ with $0 < a < 1$ and $b > (1 - a)/2$, then

$$\lim_{n\to\infty} \|M_T^b(n + 1) - M_T^b(n)\| = 0.$$

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